Quillen Cohomology of Operadic Algebras and Obstruction Theory

Michael A. Mandell

Indiana University

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Outline

1. Introduction to Quillen homology and cohomology of operadic algebras
2. Structure of Quillen homology
3. Postnikov towers and obstructions
4. Application: $BP$
Introduction to Quillen homology and cohomology of operadic algebras

Structure of Quillen homology

Postnikov towers and obstructions

Application: \(BP\)
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**General Context**

- A closed model category $\mathcal{M}$
- Assume that subcategory of abelian objects $\mathcal{A}$ also forms a model category
- With fibrations and weak equivalences as in $\mathcal{M}$
- And assume that the forgetful functor has a left adjoint $\text{Ab}$ ("abelianization")

**Definition**

Quillen homology is the left derived functor of abelianization. Quillen cohomology with coefficients in $N \in \mathcal{A}$ is $[\cdot, N]$. 
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Abelian Objects in Operadic Algebras

Let $\mathcal{O}$ be an operad in chain cxs of abelian groups or modules. Assume $\mathcal{O}(0) = 0$.

**Question**
What is an abelian object in the category of $\mathcal{O}$-algebras?

Must have structure map $\mathcal{O}(m) \otimes N^\otimes m \to N$ be zero for $m > 1$.
Just has the structure of an $R$-module for $R = \mathcal{O}(1)$.
Equivalently, the structure of an algebra over $\mathcal{I}_R$.

**Answer**
An $\mathcal{I}_R$-algebra = $R$-module.
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What is an abelian object in the category of \( \mathcal{O} \)-algebras?

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Just has the structure of an \( R \)-module for \( R = \mathcal{O}(1) \).

Equivalently, the structure of an algebra over \( \mathcal{I}_R \).

**Answer**

An \( \mathcal{I}_R \)-algebra = \( R \)-module.
Abelianization = Indecomposables

Forgetful functor from abelian $\mathcal{O}$-algebras to $\mathcal{O}$-algebras is “restriction of scalars” from $\mathcal{I}_R$-algebras to $\mathcal{O}$-algebras along $\mathcal{O} \to \mathcal{I}_R$.

Left adjoint is “extension of scalars”

$$\mathcal{I}_R \circ \mathcal{O} (-)$$

which is indecomposables:

$$\mathcal{I}_R \circ \mathcal{O} \circ A \longrightarrow \mathcal{I}_R \circ A \longrightarrow QA$$

$$R \otimes (\bigoplus \mathcal{O}(n) \otimes \Sigma_n A^\otimes n) \longrightarrow R \otimes A \longrightarrow QA$$
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$R \otimes R \otimes A \rightarrow R \otimes A \rightarrow A$
Quillen Homology = Koszul Dual Coalgebra

Quillen homology = Left derived functor of \( I_R \circ O (-) \)

We know how to do this left derived functor much more generally than we know that \( O \)-algebras are a closed model category.

Choose a flat right \( O \)-module approximation \( E \) of \( I_R \) and look at \( E \circ O (-) \).

If \( R \) is commutative and \( O \) nice, can take \( E = (DO \circ O, d) \).

Then \( E \circ O A = B_O A \) is the “bar dual” or “Koszul dual” \( DO \)-coalgebra of \( A \).
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Homotopical Origin of Coalgebra Structure

Now don’t assume that $R$ is commutative but do assume that $\mathcal{O}$-algebras form a closed model category.

Then we have a Quillen adjunction

$$Q : \mathcal{O}\text{-Alg} \leftrightarrow \mathcal{I}_R\text{-Alg} : Z$$

and a derived adjunction

$$Q^L : \text{Ho}(\mathcal{O}\text{-Alg}) \leftrightarrow \text{Ho}(\mathcal{I}_R\text{-Alg}) : Z^R$$

$Q^L A$ is a coalgebra over the comonad $Q^L Z^R$.

Goodwillie Calculus: $Q^L Z^R (X) = \bigoplus (D(n) \otimes X^\otimes n)_{h\Sigma_n}$
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Koszul Duality

($R$ is commutative, probably a field.)
($O$ is $\Sigma_*$ projective.)

If $A$ is connected, you can recover $A$ from $DO$ structure on $B_O A$
($= Q^L A$):

$$\varinjlim C_O B_O A \cong A$$

[Getzler-Jones], [Fresse]

$C_O$ is a cobar coalgebra for $DO$-coalgebras
That gives a $O$-alg.
rather than $DDO$-alg.
Eckmann-Hilton Duality

\[ \text{Cell } \text{alg} \]

\[ A_1 \rightarrow A_2 \rightarrow A_3 \rightarrow \cdots \rightarrow A \]

\[ K \text{O } [\text{S}^n] \rightarrow [\text{S}^n] \leftarrow B^n \]
Eckmann-Hilton Duality

\[
\begin{align*}
A_1 & \leftarrow A_2 \rightarrow A_3 \leftarrow \cdots \leftarrow A \\
\text{In analogy, } & S^n & \cong & \text{Cont.} \\
K & \xrightarrow{\iota} K\mathbb{B}^n \\
KA_1 & \rightarrow KA_2 \rightarrow KA_3 \rightarrow \cdots \rightarrow KA
\end{align*}
\]
Eckmann-Hilton Duality

\[ A_1 \leftarrow A_2 \leftarrow A_3 \leftarrow \cdots \leftarrow A \]

\[ KA_1 \rightarrow KA_2 \rightarrow KA_3 \rightarrow \cdots \rightarrow KA \]

\[ \Rightarrow \text{ Postnikov Tower} \]
Now possibly working in spectra with $\mathcal{O}$ an operad of spaces.

No longer assume $\mathcal{O}(0) = \ast$. Might want unit.
Postnikov Towers for Operadic Algebras

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For any $\mathcal{O}$-algebra $A$, tower of $\mathcal{O}$-modules

$$A \to \cdots \to A_{n+1} \to A_n \to \cdots \to A_0$$

with

- $\pi_iA \to \pi_iA_n$ iso for $i \leq n$
- $\pi_iA_n = 0$ for $i > n$

Problem

Build as a tower of principal fibrations of $\mathcal{O}$-modules.
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Problem

Build as a tower of principal fibrations of $\mathcal{O}$-modules.
Principal Fibrations

\[ A \rightarrow \cdots \rightarrow A_{n+1} \rightarrow A_n \rightarrow \cdots \rightarrow A_0 \]

(For simplicity, assume $O$ augments to comm. operad.)

Work in the category of $O$-algebras lying over $A_0 = H_{\pi_0}A$.

For an $A_0$-module $M$, have the square-zero $O$-algebra $A_0 \ltimes M = A_0 \vee M$.

We will construct the Postnikov tower with

\[ \begin{array}{c}
A_{n+1} \\
\downarrow \\
A_n \\
\downarrow \\
A_0 \ltimes \Sigma^{n+2} H_{\pi_{n+1}}A
\end{array} \]

homotopy fiber squares.
Postnikov Towers and Obstructions

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(For simplicity, assume \( O \) augments to comm. operad.)

Work in the category of \( O \)-algebras lying over \( A_0 = H_{\pi_0}A \).

For an \( A_0 \)-module \( M \), have the square-zero \( O \)-algebra \( A_0 \ltimes M = A_0 \lor M \).

We will construct the Postnikov tower with homotopy fiber squares.
Topological (André-)Quillen Cohomology

The map

$$A_n \to A_0 \times \Sigma^{n+2} H_{\pi n+1}A$$

is an element of topological Quillen cohomology

$$k_O^{n+1} \in D^{n+2}(A_n; H_{\pi n+1}A)$$

Step 1. Have extension of scalars isomorphism

$$\text{Ho}(\mathcal{O}\text{-Alg}/A_0)(C, A_0 \times M) \cong \text{Ho}(\mathcal{O}\text{-}A_0\text{-Alg}/A_0)(A_0 \wedge C, A_0 \times M)$$

Now in context of augmented algebras.
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Step 2. The augmented/non-unital equivalence

$$\text{Ho}(\mathcal{O}-A_0\text{-Alg}/A_0)(A_0 \land C, A_0 \ltimes M) \cong \text{Ho}(\tilde{\mathcal{O}}-A_0\text{-Alg})(I^R(A_0 \land C), ZM)$$

for $\tilde{\mathcal{O}}$ the non-unital version of $\mathcal{O}$,

$$\tilde{\mathcal{O}}(n) = \begin{cases} 
\mathcal{O}(n) & n > 0 \\
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\[ \cong \pi_0 F_{A_0}(Q^L I^R(A_0 \wedge C), M). \]
How do you construct the Postnikov tower?

**Theorem (Hurewicz Theorem)**

Suppose $A \rightarrow B$ is $n$-connected and $M$ is connected. Then $D_q(B, A; M) = 0$ for $q \leq n$ and $D_{n+1}(B, A; M) = H_{n+1}(B, A; M)$.

**Theorem (Universal Coefficient Theorem)**

There is a natural spectral sequence

$$E_2^{p,q} = \text{Ext}^{p,q}_{\pi_*A_0}(D_*(B, A; A_0), \pi_*M)$$

converging conditionally to $D^{p+q}(B, A; M)$. 
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Apply to the $(n+1)$-connected map $A \to A_{n+1}$.
How do you construct the Postnikov tower? (cont.)

Applying Hurewicz to the \((n + 1)\)-connected map \(A \to A_{n+1}\), we get

\[
D_q(A_n, A; A_0) = 0, \quad q \leq n + 1
\]
\[
D_{n+2}(A_n, A; A_0) = H_{n+2}(A_n, A; A_0) = H_{n+2}(A_{n+1}, A; \pi_0 A)
\]
\[
= \pi_0 A \otimes \pi_{n+1} A
\]

Applying Universal Coefficient, we get

\[
D^{n+2}(A_n, A; H\pi_{n+1} A) = \text{Hom}_{\pi_0 A}(\pi_0 A \otimes \pi_{n+1} A, \pi_{n+1} A)
\]
\[
= \text{Hom}(\pi_{n+1} A, \pi_{n+1} A).
\]
Applying Hurewicz to the \((n + 1)\)-connected map \(A \to A_{n+1}\), we get

\[
D_q(A_{n+1}, A; A_0) = 0, \quad q \leq n + 1
\]

\[
D_{n+2}(A_{n+1}, A; A_0) = H_{n+2}(A_{n+1}, A; A_0) = H_{n+2}(A_{n+1}, A; \pi_0 A)
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\[
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How do you construct the Postnikov tower? (cont.)

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$$D^{n+2}(A_n, A; H\pi_{n+1}A) = \text{Hom}(\pi_{n+1}A, \pi_{n+1}A).$$

Choosing identity element, we get a (homotopy class of) diagram

\[
\begin{array}{ccc}
A & \rightarrow & A_0 \\
\downarrow & & \downarrow \\
A_n & \rightarrow & A_0 \times \Sigma^{n+2}H\pi_{n+1}A \\
\end{array}
\]

Construct $A_{n+1}$ as homotopy pullback. Get $A \rightarrow A_{n+1} \rightarrow A_n$. 
How do you construct the Postnikov tower? (cont.)

\[ D^{n+2}(A_n, A; H\pi_{n+1}A) = \text{Hom}(\pi_{n+1}A, \pi_{n+1}A). \]

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\[
\begin{array}{c}
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\downarrow \\
A_{n+1} \\
\downarrow \\
A_n \\
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Construct \( A_{n+1} \) as homotopy pullback.
Get \( A \rightarrow A_{n+1} \rightarrow A_n \).
A map of $O$-algebras $f : B \to A_n$ lifts (in the homotopy category) to a map of $O$-algebras $B \to A_{n+1}$ if and only if $f^* k^{n+1}_O = 0$ in $D^{n+2}(B; \pi_{n+1}A)$. When a lift exists, the set of lifts has a free transitive action of $D^{n+1}(B; \pi_{n+1}A)$.
Obfuscation Theory

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Theorem

An $O$-algebra structure on $A_n$ lifts to an $O$-algebra structure on $A_{n+1}$ if and only if the spectrum-level $k$-invariant $k^{n+1} \in H^{n+2}(A_n, \pi_{n+1} A)$ lifts to an element of $D^{n+2}(A_n, \pi_{n+1} A)$.
Application: \( BP \)

**Theorem (Basterra-Mandell)**

\( BP \) has an \( E_4 \) ring spectrum structure. It is unique up to automorphism in the homotopy category of \( E_4 \) ring spectra.

**Existence:**
- Compute topological Quillen (co)homology in a range for Postnikov section \( BP_n \).
- Play off of \( MU \)

**Uniqueness:**
- Compute topological Quillen (co)homology.
- Obstructions for constructing an \( E_4 \) map \( BP \to BP' \) are zero.
- Any map of spectra \( BP \to BP \) is either zero on an equivalence.
Theorem (Basterra-Mandell)

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Application: \textit{BP}

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Computing topological Quillen homology of $BP$

Need some facts:

- $H_* BP = \mathbb{Z}_p[\xi_1, \xi_2, \ldots]$
- For augmented/non-unital $E_n H\mathbb{Z}_p$-algebras, topological Quillen homology can be computed as an iterated bar construction.

$$D^*(BP; H\mathbb{Z}_p) \cong \pi_{*+4}B^4(H\mathbb{Z}_p \wedge BP)$$

We compute that this is free and concentrated in even degrees.

It follows that $D^*(BP; H\mathbb{Z}_p)$ is concentrated in even degrees.

Obstruction for lifting a map $BP \to BP'_n$ to $BP'_{n+1}$ is an element of $D^{n+2}(BP; H\pi_{n+1}BP)$. 
Computing topological Quillen homology of $BP$

Need some facts:

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Computing topological Quillen homology of $BP$

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$$D_*(BP; H\mathbb{Z}_p) \cong \pi_{*+4} \tilde{B}^4 (H\mathbb{Z}_p \land BP)$$

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Computing topological Quillen homology of $BP$

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\[
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Computing topological Quillen homology of $BP$

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Obstruction for lifting a map $BP \rightarrow BP'_n$ to $BP'_{n+1}$ is an element of $D^{n+2}(BP; H\pi_{n+1}BP)$. 
The Computation

Want to show $\pi_* B^4(H\mathbb{Z}_p \wedge BP)$ is free and concentrated in even degrees.

Suffices to compute homotopy groups of

$$B^4(H\mathbb{Z}_p \wedge BP) \wedge_{H\mathbb{Z}_p} H\mathbb{Z}/p \cong B^4(H\mathbb{Z}/p \wedge BP)$$

Start with

$$\pi_*(H\mathbb{Z}/p \wedge BP) = H_*(BP; \mathbb{Z}/p) = \mathbb{Z}/p[\xi_1, \xi_2, \ldots], \quad |\xi_i| = 2p^i - 2.$$ 

Spectral sequence to compute $\pi_*$ of $B(H\mathbb{Z}/p \wedge BP)$ collapses at $E_2$ and you get $\pi_* B(\mathbb{Z}/p \wedge BP)$ is exterior on odd degree classes $\sigma \xi_i$ in degree $2p^i - 1$. 
Want to show $\pi_* B^4(\mathbb{H}\mathbb{Z}(p) \wedge BP)$ is free and concentrated in even degrees.

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The Computation

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The Computation

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$$B^4(H\mathbb{Z}_p \wedge BP) \wedge H\mathbb{Z}/p \approx B^4(H\mathbb{Z}/p \wedge BP)$$

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Spectral sequence to compute $\pi_\ast$ of $B^2(H\mathbb{Z}/p \wedge BP)$ collapses at $E_2$ and you get $E_\infty$-term is truncated polynomial on classes $\gamma_j\sigma^2\xi_i$ in degree $2p^i+j$, for $i = 1, 2, \ldots$, and $j = 0, 1, \ldots$

Need to figure out multiplicative extensions in order to get the full computation for $\pi_\ast B^2(\mathbb{Z}/p \wedge BP)$

Map $BP \to H\mathbb{Z}/p$ is an $E_4$ map, so can read of the Dyer-Lashof operations (that exist on the homology of $E_4$ ring spectra) on $H_\ast BP$ from the Dyer-Lashof operations for $H_\ast$. The Dyer-Lashof operation

$$Q^{p^i} \xi_i = \xi_{i+1} + \text{decomposables}$$

implies that

$$Q^{p^i+j} \gamma_j\sigma^2\xi_i = \gamma_j\sigma^2\xi_{i+1}.$$ 

But this is now the $p$-th power operation.
Spectral sequence to compute $\pi_\ast$ of $B^2(H\mathbb{Z}/p \wedge BP)$ collapses at $E_2$ and you get $E_\infty$-term is truncated polynomial on classes $\gamma_j \sigma^2 \xi_i$ in degree $2p^i + j$, for $i = 1, 2, \ldots$, and $j = 0, 1, \ldots$

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$$(\gamma_j\sigma^2\xi_i)^p = Q^{p^{i+j}} \gamma_j\sigma^2\xi_i = \gamma_j\sigma^2\xi_{i+1}.$$

But this is now the $p$-th power operation.
Associated graded was truncated poly on $\gamma_j \sigma^2 \xi_i$; now know $(\gamma_j \sigma^2 \xi_i)^p = \gamma_j \sigma^2 x_{i+1}$. So we get

$$\pi_* B^2(H\mathbb{Z}/p \wedge BP) = \mathbb{Z}/p[\gamma_0 \sigma^2 x_1, \gamma_1 \sigma^2 x_1, \gamma_2 \sigma^2 x_1, \ldots]$$

polynomial on classes in degrees $2p^{j+1}$ for $j = 0, 1, \ldots$

Looking at the spectral sequence, we get $\pi_* B^3(H\mathbb{Z}/p \wedge BP)$ is exterior on odd degree classes in degrees $2p^{j+1} + 1$.

Looking at the spectral sequence, we get $\pi_* B^4(H\mathbb{Z}/p \wedge BP)$ is concentrated in even degrees.
AssOCIATED GRADED WAS TRUNCATED POLY ON $\gamma_j \sigma^2 \xi_i$; NOW KNOW $(\gamma_j \sigma^2 \xi_i)^p = \gamma_j \sigma^2 x_{i+1}$. SO WE GET

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POLYNOMIAL ON CLASSES IN DEGREES $2p^{j+1}$ FOR $j = 0, 1, \ldots$

LOOKING AT THE SPECTRAL SEQUENCE, WE GET $\pi_* B^3(\mathbb{H}\mathbb{Z}/p \wedge BP)$ IS EXTERIOR ON ODD DEGREE CLASSES IN DEGREES $2p^{j+1} + 1$.

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