Problem A. Consider the first order system
\[
\begin{pmatrix}
    x'_1 \\
    x'_2
\end{pmatrix} = \begin{pmatrix}
    0 & 1 \\
    -2 & -i
\end{pmatrix}
\begin{pmatrix}
    x_1 \\
    x_2
\end{pmatrix}
\]
for two functions \(x_1, x_2 : \mathbb{R} \rightarrow \mathbb{C}\).

(i) Find the general solution by finding the eigenvalues and eigenvectors.
(ii) Are there any real solutions, i.e. \(x_1(t) \in \mathbb{R}\) and \(x_2(t) \in \mathbb{R}\) besides \(x(t) \equiv (0, 0)\)?
(iii) Writing \(x_1\) and \(x_2\) in terms of their real and imaginary parts, i.e. \(x_1 = y_1 + iy_2\) and \(x_2 = y_3 + iy_4\), find the equivalent real system of the form \(y' = By\) where \(y = (y_1, y_2, y_3, y_4)\) and \(B\) is a real \(4 \times 4\) matrix.

Reminder: No class on Tuesday, Sept. 2.
For the matrix $A$ given by
\[ A = \begin{pmatrix} 2 & 1 \\ 0 & -3 \end{pmatrix} \]
determine the stable subspace $E^s$, the unstable subspace $E^u$, and positive constants $C, K, \lambda$ and $\mu$ so that the following estimates hold:
\[ \|e^{tA}v\| \leq Ce^{-\lambda t} \|v\| \quad \text{for all} \quad v \in E^s \quad \text{and} \quad t > 0 \quad \text{and} \quad \|e^{tA}v\| \leq Ke^{\mu t} \|v\| \quad \text{for all} \quad v \in E^u \quad \text{and} \quad t < 0. \]
Here $\|\|$ denotes the usual Euclidean norm. Note: these constants are not unique; just find any such constants that work.

Determine the phase portrait for the system
\[ \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & k \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \]
You will need to make a separate portrait for each of the cases
\[ k < -2, \quad k = -2, \quad -2 < k < 0, \quad k = 0, \quad 0 < k < 2, \quad k = 2, \quad \text{and} \quad k > 2. \]

Assignment 3 Due in class on Thursday, Sept. 25

(i) Prove a counter-example to the following assertion:
If $\Omega \subset \mathbb{R}^n$ is an open, connected set and $f : \Omega \to \mathbb{R}^n$ is continuously differentiable on $\Omega$ and satisfies $\sup_{x \in \Omega} |Df(x)| \leq M$ for some $M > 0$ then $f$ is Lipschitz continuous in $\Omega$, i.e.
\[ \exists L > 0 \quad \text{such that} \quad |f(x_1) - f(x_2)| \leq L \|x_1 - x_2\| \quad \text{for all} \quad x_1, x_2 \in \Omega. \quad (\ast) \]
(Here $\| \cdot \|$ denotes Euclidean distance in $\mathbb{R}^n$.) Then prove that $(\ast)$ is true if one replaces “Lipschitz” by “locally Lipschitz”, meaning that for every compact set $K \subset \Omega$ there exists a number $L_K > 0$ such that $(\ast)$ holds for all $x_1, x_2 \in K$.
Note: For your counter-example you can take $f : \Omega \to \mathbb{R}^1$ if you wish. Also, if you describe the behavior of $f$ clearly in words, you do not need to necessarily find an explicit formula for it in your counter-example.

(ii) Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable. Letting $B_R$ denote the open ball in $\mathbb{R}^n$ centered at the origin of radius $R$, suppose that for some $R > 0$ one has that
\[ f(x) \cdot x < 0 \quad \text{for all} \quad x \in \partial B_R. \]
Prove that for any $x_0 \in B_R$ there exists a global solution (that is, valid for all $t > 0$) to the problem
\[ \dot{x} = f(x), \quad x(0) = x_0. \]

(iii) Show that the $n^{th}$ Picard iterate for the solution of the ODE $\dot{x} = tx, \ x(0) = 1$ is the sum of the first $n + 1$ terms of the power series expansion for $e^{t^2/2}$. 

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(iv) Prove that for any positive number \( r < \pi/2 \) the Picard iterates

\[
\phi_{n+1} := \int_0^t \left(1 + (\phi_n(s))^2\right) \, ds, \quad \phi_0 \equiv 0
\]

converge uniformly on the closed interval \(|t| \leq r\). What is the ODE and initial condition that this iteration procedure corresponds to? What is the corresponding solution to this ODE?

(v) Suppose \( f : \mathbb{R} \to \mathbb{R} \) is continuous and suppose there exist two different solutions to the problem

\[
\dot{x} = f(x), \quad x(0) = x_0 \quad \text{for } 0 \leq t \leq 1.
\]

Prove that there exist infinitely many solutions to this problem valid in an interval of the form \([0, a]\) for some \( a > 0\).

Assignment 4 Due in class on Thursday, Oct. 9

Out of Sideris: Read the statement and proof of Thm. 4.1 and Cor. 4.1 on pages 54-55. Work exercise 4.1 on page 71.

(i) For any fixed \( \varepsilon \in (0, 1] \), consider the O.D.E.

\[
\dot{x} = -3x + \varepsilon x^2 \ln x, \quad x(0) = \frac{1}{2} + \frac{1}{2} \sin \varepsilon.
\]

Determine that a solution \( \phi = \phi(t, \varepsilon) \) exists for all \( t \geq 0 \). Then, invoking our smooth dependence on parameters theorem from class, as well as Taylor’s theorem, note that

\[
\phi(t, \varepsilon) = \phi(t, 0) + \varepsilon \frac{\partial \phi}{\partial \varepsilon}(t, 0) + o(\varepsilon)
\]

on compact time intervals. Solve to explicitly find the approximate solution

\[
\phi(t, 0) + \varepsilon \frac{\partial \phi}{\partial \varepsilon}(t, 0).
\]

(ii) The motion of a projectile against air resistance (assumed proportional to the speed of the projectile) is described by the differential equations

\[
y_1'' = -\lambda y_1', \quad y_2'' = -g - \lambda y_2'
\]

where \( y_1 \) and \( y_2 \) represent horizontal and vertical distances respectively, the independent variable is time \( t \), \( g \) is the gravitational constant and \( \lambda \) is the drag constant. (When \( \lambda = 0 \) one is ignoring air resistance.) Consider this system subject to the initial conditions

\[
y_1 = y_2 = 0 \quad \text{and} \quad y_1' = v_0 \cos \theta_0, \quad y_2' = v_0 \sin \theta_0 \quad \text{at} \ t = 0,
\]
where \( v_0 \) is the initial speed and \( \theta_0 \) gives the initial direction.

Use the first order Taylor approximation in \( \lambda \) to \( y_1(t, \lambda), y_2(t, \lambda) \) to (approximately) determine the effect of the air resistance on the range of the projectile (the value of \( y_1 \) for \( y_2 = 0 \)) and the time of flight (the value of \( t \) when \( y_2 = 0 \)).

(iii) Suppose \( p : \mathbb{R} \to \mathbb{R} \) and \( q : \mathbb{R} \to \mathbb{R} \) are continuous and suppose \( u_1 \) and \( u_2 \) are two linearly independent solutions to the scalar O.D.E.:

\[
(*) \quad u'' + p(t)u' + q(t)u = 0 \quad \text{on} \quad \mathbb{R}.
\]

a. By rephrasing problem (\( * \)) as a system of first order O.D.E.’s, explain why it cannot be the case that either \( u_1(t_0) = 0 = u_2(t_0) \) or that \( u_1'(t_0) = 0 = u_2'(t_0) \) for any \( t_0 \in \mathbb{R} \).

b. Prove that between any two consecutive zeros of \( u_1 \) there exists exactly one zero of \( u_2 \).

Hint: Work with the function \( u_1 / u_2 \).

(iv) Let \( A(t) \) be the defined by

\[
A(t) := \begin{pmatrix}
\cos t & 3 - \sin^2 t \\
\cos t \sin t & \sin t + a
\end{pmatrix}.
\]

Prove that if the real number \( a \) is positive, then the origin is unstable for the system \( x' = A(t)x \) while if \( a \) is negative then there exists (at least) a one parameter family of solutions to this ODE that asymptotically approach zero as \( t \to \infty \).

Assignment 5: Due Tuesday, October 28

Out of Chicone: Exercise 2.69 on page 180, 2.73 on page 181

(i) Consider the system

\[
\begin{align*}
x &= -2x - y^4 \\
y &= -y - x^2.
\end{align*}
\]

(a) Argue that \((0, 0)\) is an asymptotically stable equilibrium using the linear stability criterion (i.e. looking at the eigenvalues of the linearized equation).

(b) Now find a strict Lyapunov function for the equilibrium \((0, 0)\) and then find \( R > 0 \) as large as possible so that initial data in the ball \( B_R \) of radius \( R \) centered at \((0, 0)\) asymptotically approaches \((0, 0)\) as \( t \to \infty \) using this Lyapunov function.

(ii) In class you saw a proof of the statement that linear instability (i.e. at least one eigenvalue of \( Df(0) \) lies in right half-plane) implies nonlinear instability of the origin for \( x' = f(x) \) for the case where \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) and where \( Df(0) \) has real eigenvalues of opposite sign. Generalize this proof to the case \( f : \mathbb{R}^n \to \mathbb{R}^n \), \( n \geq 2 \) and where you only assume that at least one eigenvalue has positive real part. Hint: Again think about decomposing \( \mathbb{R}^n \) into two subspaces, each of which is invariant under \( Df(0) \).

(iii) Let \( V : \mathbb{R}^2 \to \mathbb{R} \) be given by

\[
V(x, y) = \frac{1}{4}(x^2 - 1)^2 + \frac{1}{2}y^2.
\]
Find all critical points of $V$, i.e. all points such that $\nabla V(x, y) = (0, 0)$. Make a (perhaps rough) sketch of the geometry of the level sets of $V$. Pay particular attention to the level set $\{(x, y) : V(x, y) = \frac{1}{4}\}$. a. Use your work above and any other methods discussed in class to analyze the stability of each equilibrium for the flow

$$
\dot{z} = -\nabla V(z) \quad \text{where } z = (x, y).
$$

Based on this work, make a sketch of the phase portrait for this flow in the plane. In particular, try to identify the basins of attraction (if they exist) for each equilibrium point.

b. (Corrected version) For any vector $(a, b) \in \mathbb{R}^2$, define $(a, b)^\perp := (-b, a)$ and now consider the flow

$$
\dot{z} = (\nabla V(z))^\perp.
$$

(Such a system is an example of a Hamiltonian flow.) What does the linear stability approach tell you at each equilibrium with regard to nonlinear stability? Now try using the Lyapunov approach. Use the geometry of the level sets of your Lyapunov function to decide which initial conditions lie on periodic orbits. You should also be able to identify two homoclinic orbits.

Assignment 6: Due Tuesday, November 11
Out of Chicone: Exercise 1.183 on page 99, 1.184 on page 184, 1.198 on page 104

(i) Prove that the system

$$
\begin{align*}
\dot{x} &= x - y - x^3 \\
\dot{y} &= x + y - y^3
\end{align*}
$$

has a unique limit cycle that attracts all initial data except $(x(0), y(0)) = (0, 0)$.

(ii) Suppose for some smooth $f : \mathbb{R}^2 \to \mathbb{R}^2$, there exists a point $p \in \mathbb{R}^2$ such that $\phi_t(p)$ exists for all $t \geq 0$ and $p \in \omega(p)$, where $\phi_t(p)$ as usual is the solution to $\dot{x} = f(x)$, $x(0) = p$. Argue that either $p$ is an equilibrium or else $p$ lies on a periodic orbit. Then give an example of an $f : \mathbb{R}^3 \to \mathbb{R}^3$ such that this claim is false. Not so easy to construct such a counter-example!

(iii) a. Suppose $f : \mathbb{R}^2 \to \mathbb{R}^2$ is smooth. Suppose that the solution $\phi_t(p)$ to the flow $\dot{x} = f(x)$, $x(0) = p$ exists for all $t \geq 0$ for some $p \in \mathbb{R}^2$ and satisfies the condition that for all $t \geq 0$ one has $V(\phi_t(p)) = V(p)$ for some smooth function $V : \mathbb{R}^2 \to \mathbb{R}$. Assume furthermore that the level set $\Gamma_p := \{x : V(x) = V(p)\}$ is a smooth simple closed curve in the plane that contains no equilibria of $f$. Prove that $\phi_t(p)$ must be a periodic orbit, indeed that $\phi_t(p) = \Gamma_p$.

b. Consider the system

$$
\begin{align*}
\dot{x} &= -y^3 \\
\dot{y} &= x^5
\end{align*}
$$

Prove that for any initial condition $(x(0), y(0))$ other than $(0, 0)$, the orbit is periodic. Is the equilibrium point $(0, 0)$ stable? asymptotically stable? unstable? Are the periodic orbits you have found limit cycles or not?
Assignment 7: Due Tuesday, December 9

(i) Consider the following system

\[ \dot{x} = \left(1 - \frac{1}{2}x - \frac{1}{2}y\right)x, \quad \dot{y} = \left(-\frac{1}{4} + \frac{1}{2}x\right)y. \]

(a) Suppose we view this as a model for two co-existing species. Give an interpretation (in words, that is) of this system from the perspective of one species being a predator and the other being a prey. That is, explain the significance of each term on the right-hand sides of the two ODE’s and discuss how it differs from the predator-prey model we analyzed in class.

(b) Now analyze the system by finding all equilibria, looking at linear stability, direction fields, etc in order to get a description of the phase plane in the first quadrant. Will all orbits (starting in the first quadrant) exist for all positive time? If so, where do they tend to end up as \(t \to \infty\)?

(ii) Suppose you were going to come up with a model for two competing species, that is, two species that say are fighting for the same land and/or food in order to survive. Starting from the basic model

\[ \dot{x} = M(x, y)x, \quad \dot{y} = N(x, y)y \]

explain what properties \(M\) and \(N\) should possess (think in particular about the signs of certain partial derivatives) in order for this system to capture such competition.

(iii) As a particularly simple-minded competition model, consider the system

\[ \dot{x} = (a - by)x, \quad \dot{y} = (c - dx)y \]

where \(a, b, c\) and \(d\) are positive constants. Note that this model predicts exponential growth for one species in the absence of the other species. Analyze this model by finding all equilibria and applying a linear stability analysis. Look at the direction field and try to figure out the phase plane. In the process, prove that all solutions starting out in the first quadrant exist for all time. Do you think there are any heteroclinic or homoclinic orbits? (You may have to speculate here a bit for this last part; that’s all right, but explain your speculation.)

(iv) Consider the system

\[ \dot{x} = xy, \quad \dot{y} = -y - x^2. \]

Determine the eigenvalues and eigenvectors for the linearized system. Since this analysis leaves the stability of the origin in question, carry out a center manifold analysis to determine the stability/instability of the origin.

(v) Do the same analysis for the system

\[ \dot{x} = -y + xz, \quad \dot{y} = x + yz, \quad \dot{z} = -z - x^2 - y^2 + z^2 \]

to determine the stability/instability of the origin.