Quiz ♦ 1

Due date: Thursday, January 29

1. Calculate the four-point correlation function

\[ K_4(x_1, x_2, x_3, x_4) = \lim_{n \to \infty} \sum_{\sigma} \sigma(x_1)\sigma(x_2)\sigma(x_3)\sigma(x_4)\mu_n(\sigma) \]

in the 1D Ising model with zero magnetic field.

**Solution.** For \( h = 0 \), we can get the eigenvalues of transfer matrix are

\[ \lambda_{1,2} = e^{\beta J} \pm e^{-\beta J}, \]

and the corresponding eigenvectors are

\[ e_1 = \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \\ 0 \end{pmatrix}. \]

Therefore, we can obtain that

\[ S e_1 = e_2, \quad S e_2 = e_1, \]

where \( S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \). Denote

\[ Z_n(x_1, x_2, x_3, x_4) = \sum_{\sigma} \sigma(x_1)\sigma(x_2)\sigma(x_3)\sigma(x_4)e^{-\beta H_n(\sigma)}, \]

then we will have

\[ Z_n(x_1, x_2, x_3, x_4) = (t_1, T^{x_1-1}ST^{x_2}ST^{x_3}ST^{x_4}ST^{n-x_4}t_2). \] \hspace{1cm} (1)

If we set

\[ t_1 = \alpha_1 e_1 + \beta_1 e_2, \quad t_2 = \alpha_2 e_1 + \beta_2 e_2, \]

then by (1), one has

\[ Z_n(x_1, x_2, x_3, x_4) = (t_1, \alpha_2 \lambda_1^{n-x_4+x_3-x_2+x_1-1} \lambda_2^{x_4-x_3+x_2-x_1} e_1) + (t_1, \beta_2 \lambda_2^{n-x_4+x_3-x_2+x_1-1} \lambda_1^{x_4-x_3+x_2-x_1} e_2) \]

\[ = \alpha_1 \alpha_2 \lambda_1^{n-x_4+x_3-x_2+x_1-1} \lambda_2^{x_4-x_3+x_2-x_1} e_1 + \beta_1 \beta_2 \lambda_2^{n-x_4+x_3-x_2+x_1-1} \lambda_1^{x_4-x_3+x_2-x_1} e_1 \]

since \( (e_1, e_1) = 1, (e_2, e_2) = 1 \) and \( (e_1, e_2) = 0 \). Moreover, it is trivial that

\[ \alpha_1 = (t_1, e_1) > 0, \quad \alpha_2 = (t_2, e_1) > 0. \]
2. Calculate the free energy and the two-point correlation function $K_2(x_1, x_2)$ in the 1D Potts model with 3 states and zero magnetic field.

**Solution.** I will use the notation in '1D Potts model with external magnetic field'. Since $h = 0$, we have

$$t_1 = \begin{pmatrix} e^{\beta J(b_1, \omega_0)} \\ e^{\beta J(b_1, \omega_1)} \\ e^{\beta J(b_1, \omega_2)} \end{pmatrix}, \quad t_2 = \begin{pmatrix} e^{\beta J(b_2, \omega_0)} \\ e^{\beta J(b_2, \omega_1)} \\ e^{\beta J(b_2, \omega_2)} \end{pmatrix},$$

and the transfer matrix

$$T = \begin{pmatrix} e^{\beta J(\omega_0, \omega_0)} & e^{\beta J(\omega_0, \omega_1)} & e^{\beta J(\omega_0, \omega_2)} \\ e^{\beta J(\omega_1, \omega_0)} & e^{\beta J(\omega_1, \omega_1)} & e^{\beta J(\omega_1, \omega_2)} \\ e^{\beta J(\omega_2, \omega_0)} & e^{\beta J(\omega_2, \omega_1)} & e^{\beta J(\omega_2, \omega_2)} \end{pmatrix} = \begin{pmatrix} e^{\beta J} & e^{-\frac{1}{2} \beta J} & e^{-\frac{1}{2} \beta J} \\ e^{-\frac{1}{2} \beta J} & e^{\beta J} & e^{-\frac{1}{2} \beta J} \\ e^{-\frac{1}{2} \beta J} & e^{-\frac{1}{2} \beta J} & e^{\beta J} \end{pmatrix}. $$

Then, we will have

$$Z_n = (t_1, T^{n-1}t_2). \quad (2)$$

On the other hand, the eigenvalues of $T$ are $\lambda_1 = e^{\beta J} + 2e^{-\frac{1}{2} \beta J}$ and $\lambda_2 = \lambda_3 = e^{\beta J} - e^{-\frac{1}{2} \beta J}$. Let $e_1, e_2$ and $e_3$ be the corresponding eigenvectors respectively such that they are an orthonormal basis of $\mathbb{R}^3$ and $e_1$ has positive components. Moreover, denote

$$t_1 = \alpha_1 e_1 + \beta_1 e_2 + \gamma_1 e_3, \quad t_2 = \alpha_2 e_1 + \beta_2 e_2 + \gamma_2 e_3.$$

In fact $e_1 = (\sqrt{3}/3, \sqrt{3}/3, \sqrt{3}/3)^T$, hence $\alpha_1 = (t_1, e_1) > 0$ and $\alpha_2 = (t_2, e_1) > 0$. Then by (2), we obtain

$$Z_n = \alpha_1 \alpha_2 \lambda_1^{n-1} + \beta_1 \beta_2 \lambda_2^{n-1} + \gamma_1 \gamma_2 \lambda_3^{n-1}. \quad (3)$$

Note that $\lambda_1 > |\lambda_2| = |\lambda_3|$, it follows that

$$F = -\lim_{n \to \infty} \frac{1}{n} \ln Z_n = -\ln \lambda_1 = -\ln(e^{\beta J} + 2e^{-\frac{1}{2} \beta J}).$$
Now, let us consider the two-point correlation function \( K_2(x_1, x_2) \), which should be defined as

\[
K_2(x_1, x_2) = \sum_{\sigma} \langle \sigma(x_1), \sigma(x_2) \rangle \mu(\sigma).
\]

Note that

\[
K_2(x_1, x_2) = \sum_{\sigma} \sigma(x_1) \langle \sigma(x_2) \rangle \mu(\sigma) + \sum_{\sigma} \sigma(x_1) \sigma(x_2) \mu(\sigma).
\]

If we denote

\[
S_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}, \quad S_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & -\frac{\sqrt{2}}{2} \end{pmatrix}.
\]

Then, we have

\[
K_2(x_1, x_2) = \lim_{n \to \infty} \frac{(t_1, T^{x_1-1} S_1 T^{x_2-x_1} S_1 T^{n-x_2} t_2) + (t_1, T^{x_1-1} S_2 T^{x_2-x_1} S_2 T^{n-x_2} t_2)}{Z_n}.
\] (4)

Actually, we can choose \( e_2 = (\sqrt{2}/\sqrt{3}, -1/\sqrt{6}, -1/\sqrt{6})^T \) and \( e_3 = (0, \sqrt{2}/2, -\sqrt{2}/2)^T \). From direct calculation, we can get

\[
S_1 e_1 = \frac{\sqrt{3}}{2} e_2, S_1 e_2 = \frac{\sqrt{3}}{2} e_1 + \frac{1}{2} e_2, S_1 e_3 = -\frac{1}{2} e_3,
\]

and

\[
S_2 e_1 = \frac{\sqrt{3}}{2} e_3, S_2 e_2 = -\frac{1}{2} e_3, S_2 e_3 = \frac{\sqrt{2}}{2} e_1 - \frac{1}{2} e_2.
\]

Now it is easy to see,

\[
(t_1, T^{x_1-1} S_1 T^{x_2-x_1} S_1 T^{n-x_2} t_2) = \frac{1}{2} \alpha_1 \alpha_2 \lambda_1^{n-x_2+x_1} \lambda_2^{x_2-x_1} + \frac{\sqrt{3}}{4} \alpha_1 \beta_1 \lambda_1^{n-x_2} \lambda_2^{x_2-1} + \ldots
\] (5)

and

\[
(t_1, T^{x_1-1} S_2 T^{x_2-x_1} S_2 T^{n-x_2} t_2) = \frac{1}{2} \alpha_1 \alpha_2 \lambda_1^{n-x_2+x_1} \lambda_2^{x_2-x_1} - \frac{\sqrt{3}}{4} \alpha_1 \beta_1 \lambda_1^{n-x_2} \lambda_2^{x_2-1} + \ldots.
\] (6)

Consequently, by (3), (4), (5) and (6), we conclude that

\[
K_2(x_1, x_2) = \left( \frac{\lambda_2}{\lambda_1} \right)^{x_2-x_1} = \left( e^{\beta J} - e^{-\frac{1}{2} \beta J} \right)^{x_2-x_1}.
\]

3. Calculate the free energy in the classical 1D XYZ-model.

**Solution.** We will use the spherical coordinate on \( S^2 \). Then it is similar to the classical XY-model. We have

\[
Z_n = \int_{(S^2)^n} t_1(\sigma_{-n}) \prod_{j=n_1}^{n_2-1} t(\sigma_j, \sigma_{j+1}) t_2(\sigma_{n_2}) d\sigma_{-n_1} \cdots d\sigma_{n_2}.
\]
Consider the transfer operator
\[ T f(\theta_1, \phi_1) = \frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi K(\theta_1 \phi_1, \theta_2 \phi_2) f(\theta_2, \phi_2) d\theta_2 d\phi_2, \]  
(7)
where
\[ K(\theta_1 \phi_1, \theta_2 \phi_2) = \exp(K \cos \Theta), \]  
(8)
and \( K = \beta J \). Then \( Z_n = (t_1, T^n t_2) \). In particular, for classical XYZ-model, we have \( Z_n = (1, T^n 1) \).

We are going to find the eigenvalues and eigenfunctions of \( T \). Note that the kernel (8) is real and symmetric and is therefore of the Hilbert-Schmidt type. In this case, it can be shown that \( T \) possesses a complete set of mutually orthogonal eigenfunctions and that all eigenvalues are real.

The correct set of eigenfunctions of \( T \) are the spherical harmonics \((4\pi)^{1/2} Y_l^m(\theta, \phi)\), which can be expressed in terms of associated Legendre functions as follows:
\[ Y_l^m(\theta, \phi) = (-1)^m \left[ \frac{(2l + 1)(l - m)!}{4\pi(l + m)!} \right]^{1/2} P_l^m(\cos \theta) \exp(im\phi), \]  
(9)
with
\[ Y_l^{-m}(\theta, \phi) = (-1)^m Y_l^m(\theta, \phi). \]

To verify this statement, we evaluate the right-hand side of (7) using the expansion
\[ \exp(K \cos \Theta) = \left( \frac{\pi}{2K} \right)^{1/2} \sum_{l=0}^{\infty} (2l + 1) I_{l+\frac{1}{2}}(K) P_l(\cos \Theta) \]  
(where \( I_{l+\frac{1}{2}}(x) \) are modified Bessel functions of the first kind) and the addition theorem for spherical harmonics
\[ P_l(\cos \Theta) = 4\pi(2l + 1)^{-1} \sum_{m=-l}^{l} Y_l^{m*}(\theta_2, \phi_2) Y_l^m(\theta_1, \phi_1). \]

The integrations over \((\theta_2, \phi_2)\) can now be easily performed using the standard result
\[ \int_0^{2\pi} \int_0^\pi Y_l^{m*}(\theta, \phi) Y_l'^{*}(\theta, \phi) d\theta d\phi = \delta_{l1} \delta_{mm}. \]

It is found that \((4\pi)^{1/2} Y_l^m(\theta, \phi)\) is an eigenfunction of \( T \) with a corresponding eigenvalue
\[ \lambda_l(K) = \left( \frac{\pi}{2K} \right)^{1/2} I_{l+\frac{1}{2}}(K). \]

For free energy, we just need find the largest eigenvalue of \( T \), which is
\[ \lambda_0 = \left( \frac{\pi}{2K} \right)^{1/2} I_{1/2}(K) = \left( \frac{\pi}{2K} \right)^{1/2} \left( \frac{2}{\pi K} \right)^{1/2} \sinh K = \frac{\sinh K}{K}. \]

Hence the free energy is \( -\frac{1}{\beta} \ln \frac{\sinh(\beta J)}{\beta J} \). ■