Some preparation for problem 3 below and for the $T(1)$ theorem.

Recall the conditional expectation with respect to the $\sigma$-algebra $E_j$ consisting of dyadic intervals of length $2^{-j}$,

$$E_j f(x) := \mathbb{E}(f|E_j) = \sum_{|I|=2^{-j}} \langle f, \frac{1}{|I|} 1_I \rangle 1_I(x)$$

and let $\Delta_j = E_j - E_{j-1}$ be the martingale difference. It is immediate to check that

$$\Delta_j f(x) = \sum_{|I|=2^{-j+1}} \langle f, h_I \rangle h_I(x).$$

Note that by the Lebesgue differentiation theorem, we have that if $f \in L^1$

$$\lim_{j \to \infty} E_j f(x) = f(x), \ a.e. x$$

Note also that

$$\lim_{j \to -\infty} E_j f(x) = 0, \ a.e. x$$

Thus

$$f = \sum_{j=-\infty}^{\infty} \Delta_j f(x) = \sum_{I \in D} \langle f, h_I \rangle h_I.$$ 

This justifies a fact that we used a few times, namely that the Haar functions $h_I$ form an (orthonormal) basis on $\mathbb{R}$.

The same reasoning shows that for each dyadic $I$, $1_I$ together with the haar functions $h_J$, $J \subseteq I$ form a basis on $I$.

The first 3 problems below are concerned with proving a very interesting generalization of Marcienkiewicz’s multiplier theorem, due to Ronald Coifman, Rubio de Francia, Stephen Semmes, 1988.

1. Let $1 \leq q < \infty$ and let $m : \mathbb{R} \to \mathbb{R}$ have bounded $q$ variation, that is

$$\|m\|_{V_q} := \sup_{x_1 < x_2 < \ldots < x_N : N \geq 1} \left\{ \left( \sum_{i} |m(x_{i+1}) - m(x_i)|^q \right)^{1/q} \right\} < \infty.$$ 

Define

$$h(x) := \sup_{x_1 < x_2 < \ldots < x_N \leq x : N \geq 1} \left\{ \sum_{i} |m(x_{i+1}) - m(x_i)|^q \right\}$$

Prove that there exists $f : [0, \|m\|_{V_q}^q] \to \mathbb{R}$ such that

$$m(x) = f(h(x))$$
and such that \( f \) is \( 1/q \)-Holder

\[
|f(x) - f(y)| \leq |x - y|^{1/q}
\]

Hint: Define first \( f \) on the range \( R(h) \) of \( h \), check that it is \( 1/q \)-Holder there, and then define the extension

\[
f(x) := \inf_{y \in R(h)} \{ f(y) + |x - y|^{1/q} \},
\]

for the remaining \( x \).

Comment: the extension above can be done on the whole \( \mathbb{R} \).

2. Let \( \Omega \) be a collection of \( N \) pairwise disjoint intervals, and let \( a_\omega \) be some weights. Prove that for each \( 2 \leq p < \infty \) and each \( \epsilon > 0 \)

\[
\| \sum_{\omega \in \Omega} a_\omega S\omega f \|_p \lesssim_{p, \epsilon} \| a \|_{L^\infty} N^{\frac{1}{2} + \frac{\epsilon}{p}} \| f \|_p
\]

where \((S_\omega f)^* = \hat{f} \chi_{\omega} \).

Hint: Use P4/hw 11.

Comment: Problem 3 below actually immediately implies that the \( \epsilon \) can be removed.

3. (Ronald Coifman, Rubio de Francia, Stephen Semmes, 1988). Let \( 2 < q < \infty \) and let \( m : \mathbb{R} \to \mathbb{R} \) be a multiplier such that \( \|m\|_\infty + \|m\|_{V_q} < \infty \). Prove that \( m \in M_p \) for each

\[
|\frac{1}{p} - 1| \leq \frac{1}{q}
\]

more precisely show that

\[
\|T_m f \|_p \lesssim_{p, q} (\|m\|_\infty + \|m\|_{V_q}) \| f \|_p
\]

where \((T_m f)^* = m \hat{f} \).

Context: This is a strengthening of the Marcienkiewicz multiplier theorem (page 96 in the notes), which asserts that if \( m \) satisfies the above with \( q = 1 \), then \( m \in M_p \) for all \( 1 < p < \infty \). Indeed, note that \( \|m\|_{V_q} \leq \|m\|_{V_1} \) for \( q > 1 \).

Hint: Since \( T_m \lambda = \lambda T_m \), and since \( \|\lambda m\|_\infty + \|\lambda m\|_{V_q} = \lambda (\|m\|_\infty + \|m\|_{V_q}) \) you can assume by re-scaling that \( \|m\|_\infty + \|m\|_{V_q} = 1 \). Since \( \|T_m\|_{p' \to p'} = \|T_m\|_{p' \to p'} \) if \( 1/p + 1/p' = 1 \) (\( T_m \) is self dual!), it suffices to assume \( p \geq 2 \). Use P1 to create \( f : [0, 1] \to \mathbb{R} \) such that

\[
m(x) = f(h(x))
\]

and such that \( f \) is \( 1/q \)-Holder. Write

\[
f(x) = \int_0^1 f + \sum_{2^{J} \leq 1} \sum_{I \in \mathcal{D} : |I| = 2^J} \langle f, h_I \rangle h_I(x), \quad x \in [0, 1].
\]

Use the fact that \( f \) is \( 1/q \) Holder to estimate the coefficients \( |\langle f, h_I \rangle| \) from above. Next, note that for example,

\[
\sum_{I \in \mathcal{D} : |I| = 2^J} \langle f, h_I \rangle h_I(x) = \sum_{J \in \mathcal{J}_J} a_J 1_J(x),
\]

where \( \mathcal{J}_J \) is a finite partition of \( \mathbb{R} \) in disjoint intervals. Estimate from above the number of such intervals (this is immediate, since \( h_I \) is constant on two intervals), and use the estimate for \( |\langle f, h_I \rangle| \) to estimate from above \( |a_J| \). Then use P2 for each \( J \).
Comment: The restriction $|\frac{1}{p} - \frac{1}{2}| < \frac{1}{q}$ is sharp. Assume for example $p < 2$. Let $\epsilon \in \{1, -1\}^N$, $m_\epsilon(\xi) = \sum_{n=1}^N \epsilon_n 1_{[n,n+1]}(\xi)$. Let $\hat{f} = 1_{[1,N+1]}$. The Rademacher trick immediately shows that there is a choice of $\epsilon$ such that $\|T m_\epsilon f\|_p \sim N^{1/2}$. On the other hand, $\|f\|_p \sim N^{1 - \frac{1}{p}}$. Thus, $\|T m_\epsilon\|_{p \to p} \gtrsim N^{1 - \frac{1}{p}}/p^{1/2}$. Note that, trivially, $\|m_\epsilon\|_{\infty} + \|m_\epsilon\|_{V_q} \lesssim N^{1/q}$. This makes the point that $|\frac{1}{p} - \frac{1}{2}| \lesssim \frac{1}{q}$ is needed. It is not clear to me if $<$ is needed.

4. (Hilbert transform along the parabola) Prove that

$$p.v. \int F(x + t, y + t^2) dt/t$$

maps $L^2(\mathbb{R}^2) \cap \mathcal{S}$ to $L^2(\mathbb{R}^2)$. Hint: Note that this is equivalent to proving that

$$\sup_{A, B \in \mathbb{R}} |p.v. \int_{\mathbb{R}} e^{i(Ax^2 + Bx)} \frac{dx}{x}| < \infty$$

There are 3 distinct regimes: near the singularity 0, that is on interval $|x| \leq |B|^{-1}$ where you exploit cancelation from the kernel $1/x$, near the stationary point of the phase function $-B/2A$ (no oscillation), and the remaining region, where the phase produces a lot of cancelations, and you perform integrations by parts.