Some introduction:

Let $S$ be a collection of generalized tiles $s$, that is $s = I_s \times \omega_s$ where $I_s, \omega_s$ are (not necessarily dyadic) intervals and $|I_s||\omega_s| \sim 1$.

Let also $(\phi_s)_{s \in S}$ be a collection of functions (wave packets) time-frequency adapted to $s = I_s \times \omega_s$, that is, $\text{Mod}_{c(\omega_s)} \phi_s$ is $L^2 O(1)$-adapted to $I_s$. Recall that (see the discussion on pages 105-107) this is equivalent to the fact that $\text{Mod}_{c(I_s)} \hat{\phi_s}$ is $L^2 O(1)$-adapted to $\omega_s$. (s was called a Heisenberg box for $\phi_s$). Recall also, that time-frequency adaptability is a flexible concept: once $\phi_s$ is time-frequency adapted to a generalized tile $s$, it is also time-frequency adapted (with potentially a slightly larger adaptation constant) to any generalized tile $s'$ such that $s$ and $s'$ intersect and, their scales are similar $|I_s| \sim |I_{s'}|$. Recall that a quick way to generate wave packets is to start with some Schwartz generating function $\phi$, and define $\phi_s(x) := |I_s|^{-1/2} \phi(\frac{x-c(I_s)}{|I_s|})e^{2\pi i c(\omega_s)x}$. However, in the following, we will not assume that the wave packets are generated by a unique $\phi$, but simply that they are time-frequency adapted to $s = I_s \times \omega_s$.

We say that a collection $S$ of generalized tiles is good for the strong Bessel inequality if for each collection $(\phi_s)_{s \in S}$ of wave packets time-frequency adapted to $s \in S$, which in addition have the property that $\hat{\phi_s}$ is supported in $\omega_s$, it follows that $(\phi_s)_{s \in S}$ are almost orthogonal, that is, the strong Bessel inequality holds

$$\sum_{s \in S} |\langle f, \phi_s \rangle|^2 \lesssim \|f\|^2_2,$$

for each $f \in L^2(\mathbb{R})$ (with the implicit bound independent of $f$.)

We have already seen in class an example of an infinite collection of tiles which is good for the strong Bessel inequality: The LP (Littlewood-Paley) tiles, that is, the tiles of the form $[2^k, (l+1)2^k] \times [2^{-k}, 2^{-k+1}]$, $l, k \in \mathbb{Z}$. See Prop/113. Problems 1, 2, 3 below are concerned with other examples and counterexamples.

1. Let $S$ be a collection of generalized tiles, such that the intervals $(I_s)_{s \in S}$ are pairwise disjoint. Prove that $S$ is good for the strong Bessel inequality.
   
   Hint: Use Thm/16.

2. Let $\Omega$ be a collection of disjoint frequency intervals, and let

$$S := \{I \times \omega : \omega \in \Omega, I \text{ dyadic}, 1 \leq |I||\omega| < 2\}$$

Prove that $S$ is good for the strong Bessel inequality.

3. Let

$$S = \{[2^j, 2^{j+1}] \times [(n - \frac{1}{2})2^{-j}, (n + \frac{1}{2})2^{-j}] : j, n \in \mathbb{Z}\}.$$

Prove that $S$ is NOT good for the strong Bessel inequality.

Comment: This example shows that the fact that the tiles are disjoint does not guarantee that the associated wave packets are truly almost orthogonal. We will see however that there exists a satisfactory substitute for the strong Bessel, called...
the weak Bessel inequality, which holds not only for pairwise disjoint tiles, but even for strongly disjoint families of trees.

Hint: Let \( \phi \) be a real valued Schwartz function with \( \phi(-3/2) > 0 \) and \( \hat{\phi} \) supported on \([-1/2, 1/2]\). Define

\[
\phi_s(x) := |I_s|^{-1/2} \phi\left(\frac{x - c(I_s)}{|I_s|}\right)e^{2\pi i \chi(x)}x,
\]

and \( f = 1_{[-1,0]} \).

4. (Square function for arbitrary intervals; due to Rubio de Francia) Let \( \Omega \) be a collection of frequency intervals such that the intervals \( 3\omega \) are pairwise disjoint. Let \( \hat{S}_{\omega}f := \hat{f}1_{\omega} \) be the Fourier restriction to the interval \( \omega \). Prove that

\[
\|S_{\Omega}f := (\sum_{\omega \in \Omega} |S_{\omega}f|^2)^{1/2}\|_p \lesssim_p \|f\|_p
\]

for each \( \infty > p \geq 2 \).

Comments: The requirement that \( 3\omega \) are pairwise disjoint can be relaxed to the weaker requirement that \( \omega \) are pairwise disjoint, using an extra trick (which is beyond the purpose of this hw). There is nothing special about 3, your argument applies equally well to the case \( c\omega \), for any \( c > 1 \), but the constant will depend on \( c \).

The result is known to be false for \( p < 2 \), for some collections \( \Omega \), even in the case \( \Omega = \{[n,n+1] : n \in \mathbb{Z}\} \). For example, note that if \( \hat{f}_L = 1_{[0,L]} \), then

\[
\|f_L\|_{L^p(\mathbb{R})} \sim L^{1-1/p},
\]

\[
\|\left(\sum_{L=0}^{L-1} |\int \hat{f}_L(\xi)1_{[L,L+1]}(\xi)e^{2\pi i \xi x}d\xi|^2\right)^{1/2}\|_{L^p(\mathbb{R})} \sim L^{1/2}.
\]

Of course, when \( \Omega = \{[2^k,2^{k+1}] : k \in \mathbb{Z}\} \), \( S_{\Omega} \) is the classical Littlewood Paley square function, which is known to be bounded for all \( 1 < p < \infty \).

Hint: One can give a more direct (but technically more complicated) argument, and prove right away that \( S_{\Omega} \) maps \( L^\infty \) to \( \text{BMO} \), then use interpolation with \( L^2 \). The argument I suggest is technically simpler, and relies on discretization.

First, use problem 1/hw 6 to reduce matters to proving that

\[
\|T_{\phi_\omega}f\|_{L^p(\mathbb{R})} \lesssim_p \|f\|_p
\]

where \( \phi_\omega \) are \( L^\infty \) adapted to and supported on, say, \( \frac{3}{2}\omega \) (i.e replace rough with smooth cutoffs). Here

\[
\hat{T}_{\phi_\omega}f = \hat{f}\phi_\omega
\]

Use Thm/111 to further reduce to proving the boundedness of a discrete square function like

\[
\|\left(\sum_{s \in S_n} |\langle f, \psi_s \rangle|^2 / |I_s|^{-1L_s}\right)^{1/2}\|_p \lesssim_p \|f\|_p,
\]

where \( S_n \) is an appropriate collection of tiles, \( \psi_s \) is time-frequency adapted to \( s \), and \( \hat{\psi}_s \) is supported on \( 3\omega_s \).

Finally prove that the above inequality holds (see Thm/154 and Problem 2 above)