1. All $L^p$ based sizes are essentially the same

Let $\mathcal{I}$ be a finite collection of dyadic intervals, and let $(a_I)_{I \in \mathcal{I}}$ be some complex numbers. Define the maximal size of $\mathcal{I}$ by

$$\text{size}^*(\mathcal{I}) := \sup_{J \text{ interval}} \left( \frac{1}{|J|} \sum_{I \subset J} |a_I|^2 \right)^{1/2} = \sup_{J \text{ interval}} \frac{1}{|J|^{1/2}} \| \left( \sum_{I \subset J} |a_I|^2 \frac{1}{|I|} \right)^{1/2} \|_{L^2}. $$

Prove that

$$\text{size}^*(\mathcal{I}) \sim \sup_{J \text{ interval}} \frac{1}{|J|} \| \left( \sum_{I \subset J} |a_I|^2 \frac{1}{|I|} \right)^{1/2} \|_{L^1 \rightarrow \infty}. $$

Comment: This is some version of John-Nirenberg’s inequality. The fact that $\text{RHS} \lesssim \text{LHS}$ follows directly from Holder, since $\left( \sum_{I \subset J} |a_I|^2 \frac{1}{|I|} \right)^{1/2}$ is supported on $J$.

Hint: Denote by $A$ the RHS. Let $J$ be such that

$$\alpha^2 := \text{size}^*(\mathcal{I})^2 |J| = \sum_{I \subset J} |a_I|^2. $$

Note that

$$|B := \{ x : \left( \sum_{I \subset J} |a_I|^2 \frac{1}{|I|} \right)^{1/2} > 2A \}| \leq \frac{1}{2} |J|$$

Note that $B$ is the disjoint union of dyadic intervals $I_1, \ldots, I_N$. Estimate

$$\sum_{I \subset J} |a_I|^2 = \sum_i \sum_{I \subset I_i} |a_I|^2 + \sum_{I \notin \mathcal{I}_0} |a_I|^2,$$

from above, for some appropriate collection $\mathcal{I}_0$. Find a pointwise upper bound for

$$\left( \sum_{I \in \mathcal{I}_0} |a_I|^2 \frac{1}{|I|} \right)^{1/2}. $$

Conclude that something like $\alpha^2 \lesssim \frac{1}{2} \alpha^2 + 4A^2 |J|$ holds.

2. Single tree estimate: $M_1$ controls the size.

The notation is as in Problem 1 above. Let $f : \mathbb{R} \rightarrow \mathbb{C}$, and let $\phi_I$ be functions $L^2$ adapted to $I$ (of sufficiently large order, this should not be an issue) whose Fourier transform is supported away from the origin. Let $a_I := \langle f, \phi_I \rangle$. Prove that

$$\text{size}^*(\mathcal{I}) \lesssim \max_{I \in \mathcal{I}} \inf_{x \in I} M_1 f(x),$$

where $M_1 f(x) = M f(x)$ is the usual HL maximal function.

Comment: $\phi_I$ having mean zero would suffice. One would have to apply the principle behind Problem 3/Hw 7, and decompose each $\phi_I$ is LP pieces supported away from the origin (the mean zero condition will ensure extra decay for the pieces).
This is the $M_1$ version of Cor/160 (and it strengthens it, since, by Holder, $M_1 f(x) \leq M_2 f(x)$), and the proof follows the same lines. First, note that by problem 1 above, it suffices to work with $$\frac{1}{|J|} \left( \sum_{I \in \mathcal{I}} |\langle f, \phi_I \rangle|^2 \frac{1}{|I|} \right)^{1/2} \| L^{1, \infty}.$$ Recall that (this is essentially Problem 1/hw8) whenever $\psi_I$ are $L^2$ adapted to $I$ and have mean zero, the square function maps $L^1$ to $L^\infty$, that is $$\| \left( \sum_{I \in \mathcal{D}} |\langle g, \psi_I \rangle|^2 \frac{1}{|I|} \right)^{1/2} \|_{L^{1, \infty}} \lesssim \| g \|_1.$$ Finally, use the localization trick, (Prop/151). Most of the details for this problem are in the notes.

3. Prove that $$\| \sum_{k=1}^N a_k e^{2\pi i x 2^k} \|_{BMO_{\Delta}(\mathbb{R})} \lesssim \| a_k \|_{l^2}$$ with the implicit constant independent of $N$.

Hint: The proof is very similar to the proof of P3/hw9.

4. Use the result of P3 above and the John Nirenberg inequality to prove that $$\| \sum_{k=1}^N a_k e^{2\pi i x 2^k} \|_{L^p([0,1])} \lesssim_p \| a_k \|_{l^2},$$ for each $\infty > p > 2$.

Comment: The same is true for $p < 2$, simply by Holder. This problem provides another proof to P4/hw 7. The moral is that, in general, when various $L^p$ norms are equivalent, there is a BMO type estimate behind it. Note also that the result is false for $p = \infty$.

Hint: Use P3 above and Obs. 2/142.