HETEROGENEOUS BELIEFS AND TESTS OF PRESENT VALUE MODELS

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Motivation

1. Present Value Asset Pricing Models Don’t Work
   - Variance Bounds
   - Return Predictability
   - Cross-Equation Restrictions
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1. Present Value Asset Pricing Models Don’t Work
   - Variance Bounds
   - Return Predictability
   - Cross-Equation Restrictions

2. Models With Time-Varying Risk Premia Don’t Work Either
   - Hansen-Jagannathan Bounds
Objectives and Results

Objectives

Incorporate Higher-Order Beliefs into a Standard Linear Present-Value Asset Pricing Model.

Higher-Order Beliefs = Beliefs About Other People’s Beliefs

Are Stock Markets Beauty Contests?

Results

Heterogeneous information doesn’t automatically translate into heterogeneous beliefs.

- Must prevent *the history* of prices from revealing other investors’ information.

2. Infinite Regress $\Rightarrow$ Infinite-Dimensional Fixed Point Problem

Solution

Guess and Verify in the Frequency Domain
Main Ingredients

1. Old-fashioned linear Present-Value Model
   ⇒ Constant discount rate
   ⇒ All the action is in belief dynamics
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3. Traders observe the sum and different components of the sum.
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4. Background “liquidity traders”. (No-Trade Theorem).
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1. Old-fashioned linear Present-Value Model
   ⇒ Constant discount rate
   ⇒ All the action is in belief dynamics

2. Fundamentals consist of the sum of orthogonal components.

3. Traders observe the sum and different components of the sum.

4. Background “liquidity traders”. (No-Trade Theorem).

5. Use a reverse engineering, frequency domain approach to solve a complex (infinite-dimensional) signal extraction problem. Futia (1981)
   ⇒ Posit Blaschke factors instead of solve Riccati equations.
The Model

- Market Equilibrium
  \[ p_t = \beta \int_0^1 E_t^i p_{t+1} di + f_t + u_t \]
THE MODEL

- Market Equilibrium
  \[ p_t = \beta \int_0^1 E_t p_{t+1} dt + f_t + u_t \]

- Observed Fundamentals Process
  \[ f_t = a_1(L) \varepsilon_{1t} + a_2(L) \varepsilon_{2t} \]
The Model

- Market Equilibrium
  \[ p_t = \beta \int_0^1 E_t^i \rho_{t+1} d\rho_t + f_t + u_t \]

- Observed Fundamentals Process
  \[ f_t = a_1(L)\varepsilon_{1t} + a_2(L)\varepsilon_{2t} \]

- Noise Process
  \[ u_t = b_1\varepsilon_{1t} + b_2\varepsilon_{2t} + v_t \]
The Model (cont)

- Equilibrium with Two Trader Types
  - Trader 1 observes \((p_t, f_t, \varepsilon_{1t})\)
  - Trader 2 observes \((p_t, f_t, \varepsilon_{2t})\)

\[
p_t = \beta \left\{ \kappa_1 E_t^1 p_{t+1} + \kappa_2 E_t^2 p_{t+1} \right\} + f_t + u_t
\]
THE MODEL (CONT)

- Equilibrium with Two Trader Types
  - Trader 1 observes \((p_t, f_t, \varepsilon_{1t})\)
  - Trader 2 observes \((p_t, f_t, \varepsilon_{2t})\)

\[
p_t = \beta \{ \kappa_1 E_t^1 p_{t+1} + \kappa_2 E_t^2 p_{t+1} \} + f_t + u_t
\]

- Moving Average Representation (for Trader 1)

\[
\begin{bmatrix}
\varepsilon_{1t} \\
f_t \\
p_t
\end{bmatrix}
= \begin{bmatrix}
1 & 0 & 0 \\
\alpha_1(L) & \alpha_2(L) & 0 \\
\pi_1(L) & \pi_2(L) & \pi_3(L)
\end{bmatrix}
\begin{bmatrix}
\varepsilon_{1t} \\
\varepsilon_{2t} \\
v_t
\end{bmatrix}
\]
THE MODEL (CONT)

- Equilibrium with Two Trader Types
  - Trader 1 observes \((p_t, f_t, \varepsilon_{1t})\)
  - Trader 2 observes \((p_t, f_t, \varepsilon_{2t})\)

\[
p_t = \beta \left\{ \kappa_1 E^1_t p_{t+1} + \kappa_2 E^2_t p_{t+1} \right\} + f_t + u_t
\]

- Moving Average Representation (for Trader 1)

\[
\begin{bmatrix}
\varepsilon_{1t} \\
f_t \\
p_t
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 \\
\alpha_1(L) & \alpha_2(L) & 0 \\
\pi_1(L) & \pi_2(L) & \pi_3(L)
\end{bmatrix}
\begin{bmatrix}
\varepsilon_{1t} \\
\varepsilon_{2t} \\
v_t
\end{bmatrix}
\]

- Noninvertibility Condition

**Assumption 3.1**: The analytic functions \(a_i(z)\) have a single, identical root inside the unit circle. This common root is shared by the equilibrium pricing functions \(\pi_i(z)\). The pricing function, \(\pi_3(z)\) has no zeros inside the unit circle.
Traditional Solution Strategies

1. “Guess and Verify”
   - What is the state?

2. Forward Iteration + Hansen-Sargent Pred Formula
   - Breakdown of Law of Iterated Expectations
Why the Frequency Domain?

Define

\[ \bar{E}_t^0 f_{t+1} = \int_0^1 E[f_{t+1}|\Omega_t^i] \, di \]

and

\[ \bar{E}_t^k f_{t+k+1} = \bar{E}_t \bar{E}_{t+1} \cdots \bar{E}_{t+k} f_{t+k+1} \]

Then we can write

\[ p_t = f_t + u_t + \beta \sum_{k=0}^{\infty} \beta^k \bar{E}_t^k[f_{t+k+1} + u_{t+k+1}] \]

Evaluation of \( \bar{E}^k \) is a difficult fixed point problem. \( \bar{E}^k \) depends on the info conveyed by \( p_t \), but \( p_t \) depends on the entire infinite sequence of \( \bar{E}^k \).

Usual approach: Truncation. \( \bar{E}^k = \bar{E}^0 \) for \( k > \bar{k} \).
**Signal Extraction**

**Wold Representation for Trader 1**

\[
\begin{bmatrix}
\varepsilon_{1t} \\
\epsilon_t \\
p_t
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 \\
(L - \lambda)\tilde{a}_1(L) & (1 - \lambda L)\tilde{a}_2(L) & 0 \\
(L - \lambda)\tilde{\pi}_1(L) & (1 - \lambda L)\tilde{\pi}_2(L) & \pi_3(L)
\end{bmatrix}
\begin{bmatrix}
\varepsilon_{1t} \\
e_{2t} \\
v_t
\end{bmatrix} = M_1^*(L)\varepsilon_{1t}^*
\]

where

\[
\varepsilon_{2t} = \frac{L - \lambda}{1 - \lambda L} \varepsilon_{2t} \quad \text{(Blaschke Factor)}
\]

**Wiener-Kolmogorov Prediction Formula**

\[
E_1^{t}x_{1,t+1} = \left[ \frac{M_1^*(L)}{L} \right]_{+} \varepsilon_{1t}^* = L^{-1}[M_1^*(L) - M_1(0)^*]\varepsilon_{1t}^*
\]

\[
E_1^{t}p_{t+1} = L^{-1} \{[(L - \lambda)\tilde{\pi}_1(L) + \lambda\tilde{\pi}_1(0)]\varepsilon_{1t} + [(1 - \lambda L)\tilde{\pi}_2(L) - \tilde{\pi}_2(0)]\epsilon_{2t} + [\pi_3(L) - \pi_3(0)]\varepsilon_{3t} \}
In our model, traders just try to forecast future prices.

There is no direct attempt here to ‘forecast the forecasts of others’.

In our Walrasian environment, there is no need to, since nothing you do can influence the expectations of others.

However, since traders use endogenously generated prices as an input to their own forecasts, and since prices depend on other agents’ forecasts, there is an indirect sense in which each trader’s forecast embodies a forecast of other traders’ forecasts.
Heterogeneous Beliefs

\[ E^1_t p_{t+1} - E^2_t p_{t+1} = \frac{1 - \lambda^2}{1 - \lambda L} (\rho_1 \varepsilon_{1t} - \rho_2 \varepsilon_{2t}) \]

\( \rho_i < 0 \) when \( b_i > 0 \)

- Info about other trader’s signal only obtainable by observing prices and aggregate fundamentals.

- ‘Over-reaction’ to public signals.
Conjectured Equilibrium

\[ \pi_i(L) = (L - \lambda)[\rho_i + Lg_i(L)] \quad |\lambda| < 1 \]
EQUILIBRIUM

- Conjectured Equilibrium

\[ \pi_i(L) = (L - \lambda)[\rho_i + Lg_i(L)] \quad |\lambda| < 1 \]

- Equilibrium Pricing Functions:

**Proposition 3.4:** Under Assumption 3.3 (given below), there exists a unique heterogeneous beliefs Rational Expectations pricing function with \( z \)-transforms given by:

\[
\begin{align*}
\pi_1(z) &= (z - \lambda) \left[ \tilde{a}_1(\beta) + \frac{\kappa_2 b_1}{\kappa_1} \frac{\lambda}{1 - \lambda \beta} + \frac{z}{z - \beta} \left\{ \tilde{a}_1(z) - \tilde{a}_1(\beta) + \frac{\kappa_2 b_1}{\kappa_1} \lambda \left( \frac{1}{1 - \lambda z} - \frac{1}{1 - \lambda \beta} \right) \right\} \right] \\
\pi_2(z) &= (z - \lambda) \left[ \tilde{a}_2(\beta) + \frac{\kappa_1 b_2}{\kappa_2} \frac{\lambda}{1 - \lambda \beta} + \frac{z}{z - \beta} \left\{ \tilde{a}_2(z) - \tilde{a}_2(\beta) + \frac{\kappa_1 b_2}{\kappa_2} \lambda \left( \frac{1}{1 - \lambda z} - \frac{1}{1 - \lambda \beta} \right) \right\} \right]
\end{align*}
\]
Assumption 3.3: For any given $|\lambda| < 1$ the components of observed fundamentals satisfy the restrictions,

$$\beta\tilde{a}_i(\beta) + b_i \frac{\kappa_i^{-1} - \lambda\beta}{1 - \lambda\beta} = 0 \quad i = 1, 2$$

Comment: This can be interpreted as imposing a scaling, or relative variance restriction on the $f_t$ process.
Proposition 3.5: Given Assumption 3.3, the heterogeneous beliefs pricing functions can be decomposed into the sum of a conventional homogeneous expectations pricing function and an autoregressive component:

\[ \pi_i(z) = \pi_i^s(z) + \frac{b_i(\kappa_i^{-1} - 1)}{1 - \lambda \beta} \left( \frac{1 - \lambda^2}{1 - \lambda z} \right) \]

where the \( \pi_i^s(z) \) are given by the Hansen-Sargent prediction formula.
Proposition 3.5: Given Assumption 3.3, the heterogeneous beliefs pricing functions can be decomposed into the sum of a conventional homogeneous expectations pricing function and an autoregressive component:

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where the \( \pi_i^s(z) \) are given by the Hansen-Sargent prediction formula.

Corollary 3.6: Higher-order beliefs amplify the initial response of asset prices to innovations in fundamentals

\[ \pi_i(0) = \pi_i^s(0) + \frac{b_i(\kappa_i^{-1} - 1)(1 - \lambda^2)}{1 - \lambda \beta} > \pi_i^s(0) \]
Question: Suppose the world is described by a heterogeneous expectations equilibrium, but an outside econometrician interprets the data as if it were generated from a homogeneous expectations equilibrium. What kind of inferential errors could result?

We focus on 3 possibilities:

1. Violations of variance bound inequalities.
2. Excess return predictability.
3. Rejections of cross-equation restrictions.
Suppose $\sigma^2_v \approx 0$. Then econometrician’s Wold rep becomes,

$$
\begin{bmatrix}
    f_t \\
    p_t
\end{bmatrix} =
\begin{bmatrix}
    (1 - \lambda L)\tilde{a}_1(L) & (1 - \lambda L)\tilde{a}_2(L) \\
    (1 - \lambda L)\tilde{\pi}_1(L) & (1 - \lambda L)\tilde{\pi}_2(L)
\end{bmatrix}
\begin{bmatrix}
    \epsilon_{1t} \\
    \epsilon_{2t}
\end{bmatrix}
$$

where $\epsilon_{i,t} = \left( \frac{L - \lambda}{1 - \lambda L} \right) \epsilon_{i,t}$ for $i = 1, 2$.

### TABLE 1

<table>
<thead>
<tr>
<th>ARMA(1,1) ESTIMATES: ANNUAL DATA (1871-2006)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
</tr>
<tr>
<td>$d_t$</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>$p_t$</td>
</tr>
<tr>
<td></td>
</tr>
</tbody>
</table>

Notes: (1) Estimates pertain to the model, $x_t = \frac{1 - \lambda L}{1 - \gamma L} e_t$.
The Data

Detrended Real Stock Prices

Detrended Real Dividends
SHILLER BOUND

- Orthogonal Decomposition

\[ P_t^* = E_t P_t^* + u_t \]

\[ u_t = \text{forecast error} \]
**Shiller Bound**

- **Orthogonal Decomposition**

\[ P_t^* = E_t P_t^* + u_t \]

\( u_t = \) forecast error

- From orthogonality,

\[ \text{var}(P_t^*) = \text{var}(E_t P_t^*) + \text{var}(u_t) \]
Shiller Bound

- Orthogonal Decomposition

\[ P_t^* = E_t P_t^* + u_t \quad u_t = \text{forecast error} \]

- From orthogonality,

\[ \text{var}(P_t^*) = \text{var}(E_t P_t^*) + \text{var}(u_t) \]

- Therefore,

\[ \text{var}(P_t) = \text{var}(P_t^*) - \text{var}(u_t) < \text{var}(P_t^*) \]
**Shiller Bound**

- Orthogonal Decomposition

\[ P_t^* = E_t P_t^* + u_t \quad u_t = \text{forecast error} \]

- From orthogonality,

\[ \text{var}(P_t^*) = \text{var}(E_t P_t^*) + \text{var}(u_t) \]

- Therefore,

\[ \text{var}(P_t) = \text{var}(P_t^*) - \text{var}(u_t) < \text{var}(P_t^*) \]

- In words, prices should be less volatile than ex post realized future fundamentals.
SHILLER’S PLOT

FIGURE 1

Note: Real Standard and Poor’s Composite Stock Price Index (solid line \( p \)) and \textit{ex post} rational price (dotted line \( p^* \)), 1871–1979, both detrended by dividing a long-run exponential growth factor. The variable \( p^* \) is the present value of actual subsequent real detrended dividends, subject to an assumption about the present value in 1979 of dividends thereafter. Data are from Data Set 1, Appendix.
Conventional Explanations

1. Statistical Biases in the Construction of the Bound
   - Flavin (1981), Kleidon (1986)

2. Behavioral Finance
   - Fads and Market Psychology

3. Time-Varying Risk Premia
   - Hansen-Jagannathan Bounds

4. Incomplete Markets/Nondiversifiable Labor Income
Proposition 4.1: As \( \text{var}(\varepsilon_{2t}) \to 0 \), a necessary and sufficient condition for asset prices to violate Shiller’s variance bound is

\[
\beta < \frac{2\lambda \kappa_1}{1 + \lambda^2 \kappa_1^2}
\]

Comments:

1. Violations more likely when a high variance shock is observed by a large fraction of traders.

2. With private information, agents must effectively forecast other agents’ forecasts. These forecasts play the role of unobserved fundamentals. Since other agents’ forecasts are never directly revealed \textit{ex post}, there is no guarantee that errors in forecasting them are uncorrelated with prices and fundamentals.
SHILLER BOUNDS

$\kappa_1 = 0.99 \quad \beta = 0.90$

![Graph 1: Var(p)/Var(p^f): $\lambda = -0.2$](image1)

![Graph 2: Var(p)/Var(p^f): $\lambda = 0.8$](image2)
SIMULATION

\[ \kappa_1 = 0.99 \quad \lambda = 0.8 \quad \gamma = 0.98 \quad \beta = 0.90 \]
**RETURN PREDICTABILITY**

**Proposition 4.2:** Define the time-$t + 1$ excess return as $R_{t+1} = \beta p_{t+1} + d_t - p_t$. Then as $\text{var}(\varepsilon_2) \to 0$ the Wold representation generates the following projection of $R_{t+1}$ onto the (econometrician’s) time-$t$ information set

$$R_{t+1} = b_1 \lambda \left( \frac{\tilde{a}_1^{-1}(L)}{1 - \lambda L} \right) d_t.$$

**TABLE 2**

**RETURN PREDICTABILITY: ANNUAL DATA (1871-2006)**

<table>
<thead>
<tr>
<th></th>
<th>$R_{t-1}$</th>
<th>$R_{t-2}$</th>
<th>$D_{t-1}$</th>
<th>$D_{t-2}$</th>
<th>$\bar{R}^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_t =$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>.344</td>
<td>(0.195)</td>
<td></td>
<td></td>
<td></td>
<td>0.111</td>
</tr>
<tr>
<td>.398</td>
<td>(0.166)</td>
<td>-.160</td>
<td>(0.141)</td>
<td></td>
<td>0.127</td>
</tr>
<tr>
<td>3.19</td>
<td>(1.29)</td>
<td></td>
<td></td>
<td></td>
<td>0.034</td>
</tr>
<tr>
<td>12.9</td>
<td>(4.58)</td>
<td>-10.2</td>
<td>(5.10)</td>
<td></td>
<td>0.046</td>
</tr>
</tbody>
</table>
# Return Predictability: Simulation

## Table 2A

**Return Predictability: Simulated Data**

<table>
<thead>
<tr>
<th></th>
<th>$R_{t-1}$</th>
<th>$R_{t-2}$</th>
<th>$D_{t-1}$</th>
<th>$D_{t-2}$</th>
<th>$\bar{R}^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_t =$</td>
<td>$-0.504$</td>
<td>$-0.354$</td>
<td></td>
<td></td>
<td>$0.246$</td>
</tr>
<tr>
<td></td>
<td>($0.075$)</td>
<td>($0.082$)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$R_t =$</td>
<td>$-0.681$</td>
<td></td>
<td></td>
<td></td>
<td>$0.338$</td>
</tr>
<tr>
<td></td>
<td>($0.081$)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$R_t =$</td>
<td>$2.81$</td>
<td></td>
<td></td>
<td></td>
<td>$0.214$</td>
</tr>
<tr>
<td></td>
<td>($0.459$)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$R_t =$</td>
<td>$-3.72$</td>
<td>$1.90$</td>
<td></td>
<td></td>
<td>$0.291$</td>
</tr>
<tr>
<td></td>
<td>($0.493$)</td>
<td>($0.495$)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Write Wold Representation as:

\[
\begin{bmatrix}
  f_t \\
  p_t \\
\end{bmatrix} =
\begin{bmatrix}
  (1 - \lambda L)\tilde{a}_1(L) & (1 - \lambda L)\tilde{a}_2(L) \\
  K(\tilde{a}_1(L)) & K(\tilde{a}_2(L)) \\
\end{bmatrix}
\begin{bmatrix}
  e_{1,t} \\
  e_{2,t} \\
\end{bmatrix}
\]

where \( K(\tilde{a}(L)) = (1 - \lambda L)\tilde{\pi}(L) \) and \( K = K^s + K^h \)

Proposition 4.3: Standard cross-equation restriction tests, which falsely presume a common information set, can produce spurious rejections.

Proof: Standard test based on the misspecification \( K = K^s \).
**SIMULATION**

\[ \kappa_1 = .99 \quad \lambda = .80 \quad \gamma = .98 \quad \beta = .90 \]

### TABLE 3

**CROSS-EQUATION RESTRICTIONS: ANNUAL DATA (1871-2006)**

<table>
<thead>
<tr>
<th>Data</th>
<th>( D_{t-1} )</th>
<th>( P_{t-1} )</th>
<th>( \chi^2(2) )</th>
<th>( \text{var}(P)/\text{var}(\hat{P}) )</th>
<th>( \text{corr}(P, \hat{P}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( D_t = )</td>
<td>.912 (.036)</td>
<td>.001 (.001)</td>
<td>11.8</td>
<td>38.8</td>
<td>.47</td>
</tr>
<tr>
<td>( P_t = )</td>
<td>-1.28 (1.16)</td>
<td>.927 (.040)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Simulation</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( D_t = )</td>
<td>.425 (.100)</td>
<td>.004 (.006)</td>
<td>91.4</td>
<td>451.4</td>
<td>.72</td>
</tr>
<tr>
<td>( P_t = )</td>
<td>3.85 (1.65)</td>
<td>.128 (.107)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Actual vs. Predicted (Data)

Actual vs. Predicted (Simulation)

KASA, WALKER, WHITEMAN
HETEROGENEOUS BELIEFS AND TESTS OF PRESENT VALUE MODEL
Conclusions

1. Interpreting stock market fluctuations as revisions of a single, “representative”, investor’s expectations is a mistake.

2. Much of the action in financial markets represents investors’ attempts to forecast other investors’ forecasts.

3. Markets with heterogeneous expectations and higher-order belief dynamics may appear to be subject to fads and market psychology.

4. Frequency domain methods are useful for modeling heterogeneous belief dynamics.
Relaxing Assumption 3.1

- Suppose we relax Assumption 3.1, and don’t impose common root on $\pi_i(z)$ functions.
- Also, now suppose noise is uncorrelated with fundamentals ($u_t = v_t$).
- New RE Fixed Point condition

$$\pi_1(z) = \beta z^{-1}[\pi_1(z) - \pi_1(0)] \left( \kappa_1 + \kappa_2 \frac{z - \lambda}{1 - \lambda z} \right) + (z - \lambda) \tilde{a}_1(z)$$

Analyticity $\Rightarrow \pi_1(0) = \frac{(\beta - \lambda) \tilde{a}_1(\beta)}{\kappa_1 + \kappa_2} \frac{\beta - \lambda}{1 - \lambda \beta}$

- New Equilibrium pricing function

$$\pi_1(z) = \frac{z(z - \lambda) \tilde{a}_1(z) - \beta(\beta - \lambda) \tilde{a}_1(\beta)}{z - \beta} \left( \frac{\kappa_1 + \kappa_2 \frac{z - \lambda}{1 - \lambda z}}{\kappa_1 + \kappa_2} \frac{\beta - \lambda}{1 - \lambda \beta} \right)$$
**Relaxing Assumption 3.1 (Cont.)**

- **HOB Decomposition**

\[
\pi_1(z) = \pi_1^s(z) - \left[ \frac{\beta(\beta - \lambda)\tilde{a}_1(\beta)(1 - \kappa_1(1 - \lambda^2))}{\kappa_1(1 - \lambda \beta) + \kappa_2(\beta - \lambda)} \right] \left( \frac{1}{1 - \lambda z} \right)
\]

- **Shiller Bound**

\[
\pi = \pi^{pf} - \beta(\beta - \lambda)\tilde{a}(\beta) \left[ \frac{1}{z - \beta} + \frac{1 - \kappa_1(1 - \lambda^2)}{\kappa_1(1 - \lambda \beta) + \kappa_2(\beta - \lambda)} \left( \frac{1}{1 - \lambda z} \right) \right]
\]

\[
= \pi^{pf} - \Phi \frac{1 - \eta z}{(1 - \lambda z)(z - \beta)}
\]

\[
\Phi = \beta(\beta - \lambda)\tilde{a}(\beta)(1 - \beta \delta) \quad \eta = \frac{\lambda - \delta}{1 - \beta \delta} \quad \delta = \frac{1 - \kappa_1(1 - \lambda^2)}{\kappa_1(1 - \lambda \beta) + \kappa_2(\beta - \lambda)}
\]
**Relaxing Assumption 3.1 (Cont.)**

- **Variances**

\[
\text{var}(\pi) = \text{var}(\pi^{pf}) + \beta^2 \tilde{a}^2 (\beta - \lambda) \left\{ -2\delta + (\beta - \lambda) \left[ \frac{\delta^2}{1 - \lambda^2} - \frac{1}{1 - \beta^2} \right] \right\}
\]

- **Sufficient Conditions for Bound Violation**

1. \( \kappa_1 \approx \kappa_2 \)
2. \( \beta > \lambda \)
3. \( \frac{(1+\lambda^2)^2}{1+\beta} > (1 - \lambda)^2(1 + \lambda) \left[ 2(1 + \lambda^2) + \frac{1-\lambda}{1-\beta} \right] \)