RESEARCH STATEMENT

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1. Brief Introduction

My research interest lies in discrete probability with an emphasis on problems related to stochastic processes on graphs. More specifically I am fond of various percolation-related problems and I have worked on problems in Bernoulli percolation, uniform spanning forests and first-passage percolation.

Percolation theory is a rich area which has attracted much attention from both mathematicians and physicists. A percolation process on a connected, locally finite, infinite graph \( G = (V, E) \) is a random subgraph of \( G \). Bernoulli bond percolation and uniform spanning forests are classical examples of percolation processes. These models are not only interesting in themselves but also have many connections with statistical physics, combinatorics and other probability models such as loop-erased random walks, electrical networks and Abelian sandpiles.

Most of my work focuses on Bernoulli percolation and uniform spanning forests on nonunimodular transitive graphs. By a nonunimodular transitive graph we mean a graph whose automorphism group is nonunimodular and acts transitively on the graph. The mass-transport principle is a key tool for studying these objects.

Although named with percolation, first-passage percolation is a quite different model. First-passage percolation on the integer lattices \( \mathbb{Z}^d \) studies the random metric induced by the optimal paths on \( \mathbb{Z}^d \) with independent, identically distributed (i.i.d.) nonnegative weights on the edges. Part of my work is about the length of geodesics for critical first-passage percolation on \( \mathbb{Z}^2 \).

I want to continue the study of these objects in the near future and I list a few specific questions that interest me in Section 3. In percolation theory there are a lot of questions that are easy to state but hard to solve. I prefer concrete questions with explicit examples to abstract ones. I also want to learn and work on other areas such as random matrix theory and interacting particle systems.

2. Previous work

Bernoulli percolation was introduced by Broadbent and Hammersley in [2] as a model of porous media. Consider the integer lattices \( \mathbb{Z}^d = (V, E) \), where \( V = \{v_1, \ldots, v_d\} \) is the set of integer points and \( E := \{e = (x, y) : |x - y| = 1, x, y \in V\} \) is the set of nearest-neighbor edges. For each edge \( e \in E \), we keep it with probability \( p \) (call it an open edge) and delete it with probability \( 1 - p \) (call a closed edge), and we do this procedure for all the edges independently. The random subgraph obtained in this way is called Bernoulli\((p)\) (bond) percolation on \( \mathbb{Z}^d \) and its law is denoted by \( \mathbb{P}_p \). Bernoulli\((p)\) (bond) percolation on general graphs can be defined similarly. The connected components in the random subgraph are called open clusters. It is easy to show that there is a phase transition phenomenon for \( d \geq 2 \): there exists a constant \( p_c(d) \in (0, 1) \), called the critical probability such that for \( p < p_c(d) \), there is no infinite open cluster \( \mathbb{P}_p \) almost surely; for \( p > p_c(d) \), there is an infinite open cluster \( \mathbb{P}_p \) almost surely. One of the most famous open problems in probability is to show that at \( p = p_c(d) \), \( d \geq 2 \), there is no infinite open cluster.

Percolation on general transitive graphs also attracted a lot of attentions, especially since [5]. One important theorem in this field is the so-called indistinguishability theorem from [19], which roughly states that given any ‘reasonable’ property, either all infinite clusters satisfy it or none of them does. This was proved for unimodular nonamenable transitive graphs in [19]. It simply does not hold for
nonunimodular transitive graphs due to ‘light clusters’. It has been an open problem to prove that heavy clusters cannot be distinguished by any reasonable property, and this is what I proved in [22]. I also solved two other open problems that go back to 1999, namely the transience of infinite clusters for the simple random walk and connectivity decay in the intermediate regime. Both were known to hold in the nonamenable unimodular case [19], and I solved the nonunimodular case in [22].

For a finite connected graph $G$, it has finitely many spanning trees. The uniform spanning tree is just the uniform measure on the set of spanning trees of $G$. The study of uniform spanning trees has a long history—it even dates back to [16]. The Aldous/Broder algorithm [1, 6] provides a way of sampling uniform spanning trees for finite connected graphs using simple random walks. Namely, start a simple random walk on a connected graph and stop at the first time it visits all the vertices. The tree formed by all the first-entrance edges has the law of a uniform spanning tree. We showed that the tree formed by all the last-exit edges also has the law of a uniform spanning tree [14]. This answered a question of Tom Hayes and Cris Moore from 2010.

The wired uniform spanning forest and free uniform spanning forest of a connected, locally finite, infinite graph $G$ are the weak limits of uniform spanning trees on an exhaustion of $G$ with wired and free boundary conditions respectively. On $\mathbb{Z}^d$, the wired and free uniform spanning forests are the same and one interesting fact is that the uniform spanning forest is connected if and only if $d \leq 4$ [20]. Later Benjamini, Lyons, Peres and Schramm gave a systematic study of uniform spanning forests on general graphs in [4]. My work focuses on uniform spanning forests on nonunimodular transitive graphs [23]. I showed that almost surely each tree of the wired uniform spanning forest is light. The tilted volumes for the trees in the wired uniform spanning forest were also studied. For a nonunimodular transitive graph on which the wired uniform spanning forest is not the same as the free one, I conjectured that each tree of the free uniform spanning forest is heavy and has branching number bigger than one. I proved this conjecture for grandparent graphs and free products of nonunimodular transitive graphs with one edge.

First-passage percolation is a quite different probability model from the above-mentioned percolation processes on graphs. Consider the integer lattices $\mathbb{Z}^d = (V,E)$. Take a set of i.i.d. nonnegative random variables $\{X_e : e \in E\}$. For each edge $e$ of $\mathbb{Z}^d$, assign a random passage time $X_e$. First-passage percolation studies the random metric induced by the optimal paths on $\mathbb{Z}^d$ equipped with these random passage times $X_e$. The ‘critical’ first-passage percolation refers to the special case that $\mathbb{P}(X_e = 0) = p_c(d)$. The critical first-passage percolation has a close relation to critical Bernoulli percolation. We studied the length of geodesics from the origin $(0,0)$ to $(n,0)$ on $\mathbb{Z}^2$ in [9] and showed that the length grows superlinearly in the critical case.

3. Future plans

3.1. Bernoulli percolation on nonunimodular transitive graphs. The book [10] is a good reference for Bernoulli percolation on $\mathbb{Z}^d$. For percolation on general graphs, see [17, Chapter 7 and 8]. Many interesting questions remain open in percolation theory. I am interested in the following ones in particular.

(1) In [15] Häggström, Peres and Schonmann made a conjecture that if there are infinitely many infinite open clusters for a Bernoulli percolation on a nonamenable transitive graph, then almost surely any two infinite clusters have finitely many neighboring pairs. Later Timár [24] proved this conjecture for the unimodular case. I am interested in the nonunimodular case. A possible strategy is to consider the indistinguishability of pairs of infinite clusters by a particular property. Another way is to study the expected weighted chemical distance. This way combining with the results from [12] yields that the finite neighboring conjecture holds for $p \in (p_c, p_t)$, where $p_t$ is the tilted probability introduced in [12].

(2) Is the expected chemical distance for neighbors finite in supercritical Bernoulli percolation on transitive graphs? For the $\mathbb{Z}^d(d \geq 2)$ case, the answer is yes. For example, it is a direct
3.2. Uniform spanning forests on nonunimodular transitive graphs. For the wired uniform spanning forest, there is a nice algorithm called Wilson’s method rooted at infinity [4] to sample it. Also Hutchcroft [13] gave another way of sampling it using the interlacement process and the Aldous/Broder algorithm. These algorithms provide useful tools for studying the wired uniform spanning forest. Much less is known about the free uniform spanning forest. A consequence of [10, Theorem 6.10]. I am interested in studying this question for general transitive graphs beyond $\mathbb{Z}^d$ [21, Question 12.35].

I am interested in the following questions about uniform spanning forests on general nonunimodular transitive graphs.

1. Limit behavior of the intersection of a component in WUSF with lower levels on general nonunimodular transitive graphs. The case of a toy model in [23] relates this problem to the growth of the generation sizes of a critical branching process with immigration under certain moment conditions.
2. Linear growth rate conjecture for the resistance in the induced graphs of the wired uniform spanning forest on general nonunimodular transitive graphs [18].
3. Heaviness of trees in FUSF on general nonunimodular transitive graphs.
4. Branching number of trees in FUSF on general nonunimodular transitive graphs.
5. Indistinguishability of trees in FUSF on general nonunimodular transitive graphs.
6. Number of trees in FUSF on general nonunimodular transitive graphs.

3.3. First-passage percolation. In a recent MRC conference on spatial stochastic models, Michael Damron proposed several questions in first-passage percolation, of which I am most interested in the following two.

1. Limit behavior of the passage time $T(0, \partial B(n))$ from the origin to the boundary of a ball centered at the origin on subgraphs of $\mathbb{Z}^d$. The paper [8] dealt with the $\mathbb{Z}^2$ case and gave a criterion for the almost sure finiteness of $\lim_{n \to \infty} T(0, \partial B(n))$. Together with Wai-Kit Lam and Michael Damron, we want to establish a similar criterion for $T(0, \partial B(n))$ on subgraphs of $\mathbb{Z}^d$ with the form $G = \{ (x, y) : x \geq 0, 0 \leq y \leq f(x) \}$ where $f(x) = a \log x + b \log \log(x)$ for some $a > 0, b \in \mathbb{R}$.
2. The next step is to consider the limit behavior of $T(0, \partial B(n))$ on high-dimensional lattices like $\mathbb{Z}^3$. Invasion percolation is a key tool for studying these questions [8].

3.4. Invasion percolation. Invasion percolation itself is also an interesting model which exhibits self-organized criticality without any parameter [7]. It also has a close connection with the wired minimal spanning forest [17, Chapter 11]. Let’s recall the invasion basin of $x$ on a connected, locally finite, infinite graph $G = (V, E)$. Take i.i.d. random variables $U(e) \in [0, 1]$ for $e \in E$ such that each of them has the uniform distribution on $[0, 1]$. Then almost surely, $U(e) \neq U(e')$ whenever $e \neq e'$. Let $I_0(x) = \{x\}$. For each $n \geq 1$, let $I_n(x) = I_{n-1}(x) \cup \{e\}$, where $e$ is the unique edge with lowest label $U(e)$ among the edges that are not in $I_{n-1}(x)$ but adjacent to $I_{n-1}(x)$. Finally we define the invasion basin of $x$ to be $I(x) := \bigcup_n I_n(x)$. Invasion percolation studies the properties of $I(x)$. I am interested in studying invasion percolation on the following two types of graphs and using the invasion percolation to study the wired minimal spanning forest.

1. Invasion percolation on nonunimodular transitive graphs. For example, it is easy to show that the invasion basin $I(x)$ is almost surely light on a nonunimodular transitive graph $G$. What is the asymptotic behavior of the probability that $I(x)$ intersects with a high level above $x$? What is the typical size of the intersection of $I(x)$ with a low level below $x$? Answering these questions would be a first step of studying the behavior of $I(x)$.
2. Invasion percolation on $\mathbb{Z}^d$ for large $d$. A lot of things about high-dimensional invasion percolation remain unknown, like the two-point function and the scaling limits [11, Section 16.1].
References


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