Out of BD

Exercises 3.2

10) \( y'' + \cos(t)y' + 3 \ln |t|y = 0, \quad y(2) = 3, y'(2) = 1. \)

The coefficients are continuous on \( \mathbb{R} \setminus \{0\} \), and the longest interval contained in this set that contains \( t = 2 \) is \((0, \infty)\).

15) \( y'' + p(t)y' + q(t)y = g(t). \)

Suppose \( \phi \) is a solution. Then \( y(t) = c\phi(t) \) satisfies
\[
y'' + p(t)y' + q(t) = c (\phi'' + p(t)\phi' + q(t)\phi) = cg(t).
\]

So if \( y \) is a solution we must have \( g(t) = cg(t) \), or \((1 - c)g(t) = 0\). Since \( c \neq 1 \) this implies that \( g(t) = 0 \) for all \( t \), a contradiction.

This equation is not homogeneous so it does not contradict Theorem 3.2.2.

16) Let \( y(t) = \sin t^2 \). Then
\[
y'(t) = 2t \cos t^2, \quad y''(t) = 2 \cos t^2 - 4t^2 \sin t^2.
\]

It follows that \( y(0) = y'(0) = 0 \) and \( y''(0) = 2 \).

If \( y \) solves the equation
\[
y'' + p(t)y' + q(t)y = 0
\]
on an interval containing \( t = 0 \) we would have
\[
y''(0) = 2 = -p(0)y'(0) - q(0)y(0) = 0,
\]
a contraction.

38) If there exists \( t_0 \in I \) such that \( y_1(t_0) = y_2(t_0) = 0 \) then
\[
W(y_1, y_2)(t_0) = y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0) = 0
\]
which by Abel’s Theorem implies that \( W(y_1, y_2)(t) = 0 \) for all \( t \in I \).

The result follows, since a fundamental set of solutions must have a nonzero Wronskian.
Exercises 3.4

2) \[9y'' + 6y' + y = 0.\]

The characteristic equation is \[9r^2 + 6r + 1 = 0 \implies r = -\frac{1}{3}.\]

The general solution is \[y(t) = c_1e^{-t/3} + c_2te^{-t/3}.\]

27) \[xy'' - y' + 4x^3y = 0, \quad y_1(x) = \sin x^2, \quad x > 0.\]

\[y(x) = v(x)y_1(x)\] is a solution of the ODE if \[xy_1(x)v'' + (2xy_1'(x) - y_1(x))v' = 0\]

namely \[v'' + \frac{4x^2 \cos x^2 - \sin x^2}{x \sin x^2}v' = 0.\]

Note that \[\frac{4x^2 \cos x^2 - \sin x^2}{x \sin x^2} = (\ln |\sin x^2|^2 - \ln |x|)' = \ln \left(\frac{\sin x^2}{x}\right)\]

since \(x > 0\), and thus \[v'(x) = c_1 \frac{x}{(\sin x^2)^2}\]

which implies that \[v(x) = c_2 + c_1 \int \frac{x}{(\sin x^2)^2}dx = c_2 + \frac{c_1}{2} \int \frac{1}{(\sin x^2)^2}d(x^2) = c_2 - \frac{c_1}{2} \cot x^2\]

So a second solution (found setting \(c_2 = 0\) and \(c_1 = -2\)) is \[y_2(x) = \cos x^2.\]

30) \[x^2y'' + xy' + (x^2 - \frac{1}{4})y = 0, \quad y_1(x) = x^{-1/2} \sin x, \quad x > 0.\]

Setting \(y(x) = v(x)y_1(x)\) we find that \(v\) must solve \[x^2y_1(x)v'' + (2x^2y_1'(x) + xy_1(x))v' = 0\]

namely \[(\sin x)v'' + 2(\cos x)v' = 0\]

thus \[(\sin x)^2v' = c_1 \implies v(x) = -c_1 \cot x + c_2\]

So a second solution is \[y_2(x) = x^{-1/2} \cos x.\]
Out of MG

Exercises 2.4

11) 
\[ 0 = y_2(y_1'' + p_1 y_1' + p_2 y_1) - y_1(y_2'' + p_1 y_2' + p_2 y_2) \]
\[ = y_2 y_1'' - y_1 y_2'' + p_1 (y_2 y_1' - y_1 y_2') + p_2 (y_2 y_1 - y_1 y_2) \]

which upon simplifying becomes exactly

\[ W' + p_1 W = 0 \]

and this is solved with the integrating factor \( e^{\int p_1} \) to give

\[ W = C e^{-\int p_1}. \]

Exercises 2.6

4d) 
\[ x^2 y'' - 9xy' + 25y = 0, \quad y(2) = 32, \ y'(2) = 0. \]
The \( r \) equation is

\[ r^2 - 10r + 25 = 0 \implies r = 5 \]

So the general solution is

\[ y(x) = (C_1 + C_2 \ln x) x^5 \]

Imposing the initial conditions we obtain

\[ y(x) = \left(1 - \ln \frac{x^5}{32}\right) x^5. \]

The interval of existence is \((0, \infty)\).

4n) 
\[ x^2 y'' - xy' + 10y = 0, \quad y(1) = 0, \ y'(1) = 6 \]
The \( r \) equation is

\[ r^2 - 2r + 10 = 0 \implies r = 1 \pm 3i. \]
The general solution is

\[ y(x) = (C_1 \cos(3 \ln x) + C_2 \sin(3 \ln x)) x \]

Imposing the initial conditions we obtain

\[ y(x) = 9x \cos(3 \ln x) - x \sin(3 \ln x). \]

The interval of existence is \((0, \infty)\).
Problem A

\[ u_1(t) = t^2, \quad u_2(t) = t|t|. \]

\[ W[u_1, u_2](t) = t^2 \cdot 2|t| - 2t \cdot t|t| = 0 \]

Suppose \( au_1(t) + bu_2(t) = 0 \) for all \( t \in [-1, 1] \). In particular this holds for \( t = -1 \) and \( t = 1 \), which gives the equations

\[ a - b = 0 \quad \text{and} \quad a + b = 0 \]

respectively.

The only solution of this system is \( a = b = 0 \), so \( u_1 \) and \( u_2 \) are linearly independent over \([-1, 1]\).

A non-vanishing Wronskian is a sufficient condition for linear independence, but not necessary.

Problem B

Since \( \{u_1, u_2\} \) is a basis of solutions on \( \mathbb{R} \), we must have

\[ W = W[u_1, u_2] \neq 0 \] on every subinterval. In particular, \( W \) does not change sign on \([a, b]\).

Suppose \( u_2(c) \neq 0 \) for every \( c \in (a, b) \). Then \( v(t) = \frac{u_1(t)}{u_2(t)} \) is well defined on \((a, b)\).

Furthermore, by Problem 38 above we also know that \( u_2(a) \neq 0 \neq u_2(b) \). Thus \( v \) is defined on \([a, b]\).

We have

\[ v' = \frac{u_1'u_2 - u_1u_2'}{u_2^2} = -\frac{1}{u_2}W \]

from where we obtain a contradiction:

\[ 0 = v(b) - v(a) = \int_a^b v'(t)dt = -\int_a^b \frac{W(t)}{u_2(t)^2} dt \neq 0 \]

where the last inequality holds since \( W \) cannot change sign or vanish identically on \([a, b]\).