Out of BD

Exercises 3.1

21) \( y'' - 2y' - 2 = 0, \quad y(0) = \alpha, y'(0) = 2. \)

Characteristic equation:
\[
\lambda^2 - \lambda - 2 = 0 \implies \lambda = -1, 2.
\]

General solution:
\[
y(t) = C_1 e^{2t} + C_2 e^{-t}.
\]

Since \( \lim_{t \to \infty} e^{2t} = \infty \) and \( \lim_{t \to \infty} e^{-t} = 0 \), a necessary and sufficient condition for \( \lim_{t \to \infty} y(t) = 0 \) is that \( C_1 = 0 \).

Initial conditions:
\[
C_1 + C_2 = \alpha, \quad 2C_1 - C_2 = 2
\]
\[\implies C_1 = \frac{\alpha + 2}{3}, \quad C_2 = \frac{2\alpha - 2}{3}.\]

Thus the value of \( \alpha \) that we want is
\[
\alpha = -2.
\]

24) \( y'' + (3 - \alpha)y' - 2(\alpha - 1)y = 0. \)

Characteristic equation:
\[
\lambda^2 + (3 - \alpha)\lambda - 2(\alpha - 1) = 0.
\]
\[\implies \lambda_{\pm} = \frac{1}{2}(\alpha - 3 \pm (\alpha + 1)) = \alpha - 1, -2.
\]

All solutions tend to zero if and only if all the roots of the characteristic equation have negative real parts, which happens if
\[
\alpha < 1.
\]

Since \( e^{-2t} \) is always a solution, there is no value of \( \alpha \) for which all non-zero solutions are unbounded.
Exercises 3.3

25) \( y'' + 2y' + 6y = 0, \quad y(0) = 2, y'(0) = \alpha \geq 0. \)

(a) \( y(t) = 2e^{-t} \cos(\sqrt{5}t) + \frac{\alpha + 2}{\sqrt{5}} e^{-t} \sin(\sqrt{5}t). \)

(b) \( y(1) = 2e^{-1} \cos(\sqrt{5}) + \frac{\alpha + 2}{\sqrt{5}} e^{-1} \sin(\sqrt{5}) = 0 \)
\[ \Rightarrow \alpha = -2 \left( 1 + \sqrt{5} \cot(\sqrt{5}) \right) \approx 1.50878 \]

(c) \( 2e^{-t} \cos(\sqrt{5}t) + \frac{\alpha + 2}{\sqrt{5}} e^{-t} \sin(\sqrt{5}t) = 0 \)
\[ \iff \tan(\sqrt{5}t) = - \frac{2\sqrt{5}}{\alpha + 2} \]

The smallest solution \( t \) is
\[ t(\alpha) = \frac{\pi - \arctan\left( \frac{2\sqrt{5}}{\alpha + 2} \right)}{\sqrt{5}}. \]

(d) \( \lim_{\alpha \to \infty} t(\alpha) = \frac{\pi}{\sqrt{5}} \approx 1.40496 \)

Out of MG

Exercises 2.3

8b) \( y'' + 2y' + 2y = 0 \)

Roots of the characteristic equation:
\[ 1 + i, 1 - i \]

General solution
\[ y(t) = C_1 e^{-t} \cos(t) + C_2 e^{-t} \sin(t). \]

8f) \( y'' - y' + 7y = 0 \)

Roots of the characteristic equation:
\[ \frac{1 + 3\sqrt{3}}{2}, \frac{1 - 3\sqrt{3}}{2} \]

General solution
\[ y(t) = C_1 e^{t/2} \cos\left( \frac{3\sqrt{3}}{2} t \right) + C_2 e^{t/2} \sin\left( \frac{3\sqrt{3}}{2} t \right). \]
8j) \[ 4y'' + y = 0 \]

Roots of the characteristic equation:
\[ \frac{i}{2}, \frac{-i}{2} \]

General solution:
\[ y(t) = C_1 \cos(t/2) + C_2 \sin(t/2) \]

9b) \[ y(\pi) = 0, y'(\pi) = 10 \]
\[ y(t) = -10e^{\pi-t} \sin t \]

9j) \[ y(0) = 4, y'(0) = 0. \]
\[ y(t) = 4 \cos(t/2). \]

12b) \[ y'' - 2y' + 2y = 0 \]

12j) None; \( 1 + i, 1 - 3i \) are not the roots of a second degree polynomial with real coefficients:
\[ (\lambda - (1 + i))(\lambda - (1 - 3i)) = \lambda^2 + 2(i - 1)\lambda + 4 - 2i \]

**Problem A**

\[ u'' - 81u = 0. \]

Roots of the characteristic equation
\[ 9, -9 \]

General solution
\[ u(t) = C_1 \cosh(9t) + C_2 \sinh(9t) \]

**Problem B**

(ii) For any (possibly complex valued) function \( u \) one has
\[
L[u^*] = (p_2(t)u'') + p_1(t)u' + p_0(t)u^* = p_2(t)(u'')^* + p_1(t)(u')^* + p_0(t)u^*
\]
\[ = p_2(t)(u^*)'' + p_1(t)(u^*)' + p_0(t)u^* = L[u^*]. \]

Thus, if \( L[U] = 0 \), then
\[ L[U^*] = L[U]^* = 0^* = 0. \]

(i) Let \( U \) be a solution of the ODE \( L[u] = 0. \) Then, using the linearity of \( L \) and (ii) we obtain
\[
L[\text{Re} U] = L \left[ \frac{U + U^*}{2} \right] = \frac{1}{2} (L[U] + L[U^*])
\]
\[ = \frac{1}{2} (L[U] + L[U]^*) = \frac{1}{2} (0 + 0^*) = 0. \]

For \( \text{Im} U \) is the same idea.
Problem C

\[ \frac{1}{4} u'' + (1 + 2i)u' + (1 + 4i)u = 0. \]

The roots of the characteristic equation are

\[ -2, \quad -2 - 8i \]

The general solution is

\[ u(t) = C_1 e^{-2t} + C_2 e^{-2t - 8it} \]

A real solution is \( e^{-2t} \) and a complex one is \( e^{-2t - 8it} \). The real and imaginary parts of the complex solution are

\[ u_r(t) = e^{-2t} \cos(8t), \quad u_i(t) = e^{-2t} \sin(8t), \]

respectively.

Neither is a solution.

There is no contradiction with Prop (i) because neither \( p_0(t) = 1 + 4i \) nor \( p_1(t) = 1 + 2i \) is real-valued, which is a hypothesis of the Proposition.