Out of BD

Exercises 2.5

4) The only critical point is \( y = 0 \) and it is unstable since \( f(y) \times y > 0 \) for \( y \neq 0 \).

9) The critical points are \( y = 0, 1, -1 \). The origin is semistable, \( y = 1 \) is unstable and \( y = -1 \) is asymptotically stable.

22) (a) \( f(y) = \alpha y(1 - y) \) with \( \alpha > 0 \), so the equilibrium points are \( y = 0 \) (unstable) and \( y = 1 \) (asymptotically stable).

(b) Rewrite the equation as

\[
\frac{y'}{y(1 - y)} = \alpha.
\]

Noting that

\[
\frac{1}{y(1 - y)} = \frac{1}{y} + \frac{1}{1 - y} = (\ln |y| - \ln |1 - y|)' \]

we get, upon integrating the ODE and using the initial condition \( y(0) = y_0 \),

\[
\frac{|y(t)|}{|1 - y(t)|} = \frac{|y_0|}{|1 - y_0|} e^{\alpha t}.
\]

By assumption we know that \( 0 < y_0 < 1 \), and the same is true for \( y(t) \) for \( t \geq 0 \), so we get

\[
y(t) = \frac{y_0 e^{\alpha t}}{y_0 e^{\alpha t} + 1 - y_0} = \frac{1}{1 + (1/y_0 - 1)e^{-\alpha t}}
\]

which converges to 1 as \( t \to \infty \).

23) \( y' = -\beta y, \quad x' = -\alpha xy, \quad x, y > 0 \).

(a) \( y(t) = y_0 e^{-\beta t} \).

(b)

\[ x' = -\alpha y_0 e^{-\beta t} x \]

so

\[
\ln \frac{|x(t)|}{|x_0|} = -\int_0^t \alpha y_0 e^{-\beta s} ds = \frac{\alpha y_0}{\beta} (e^{-\beta t} - 1)
\]

\[
x(t) = x_0 \exp \left[ -\frac{\alpha y_0}{\beta} (1 - e^{-\beta t}) \right]
\]

(c)

\[
\lim_{t \to \infty} x(t) = x_0 e^{-\alpha y_0/\beta}.
\]
Out of MG

1) The equilibrium points are the zeros of \( f(N) = (a - bN)N - h \), namely

\[
N = \frac{a}{2b} + \sqrt{\frac{a^2}{4b^2} - \frac{h}{b}}, \quad a = \frac{a^2}{2b} - \sqrt{\frac{a^2}{4b^2} - \frac{h}{b}}.
\]

The first one is asymptotically stable and the second one is unstable, provided they are real and distinct, that is

\[ h < \frac{a^2}{4b}. \]

If \( h = \frac{a^2}{4b} \) then \( N = \frac{a}{2b} \) is a semistable equilibrium, and if \( h > \frac{a^2}{4b} \) then the ODE has no equilibrium points and \( N(t) \to -\infty \) as \( t \to \infty \).

4f) Since \(|\sin x| \leq |x|\) for all \( x \), the curves \( y = 3x \) and \( y = \sin x \) can only intersect at \( x = 0 \). Furthermore, \( f(x) = 3x - \sin x > 0 \) for \( x > 0 \) and \( f(x) = -f(-x) < 0 \) for \( x < 0 \). Thus the only equilibrium point is \( x = 0 \), and it is unstable.

**Problem A:**

\[ f(P) = P(P - 1,000)(10,000 - P) \]

The equilibrium points are \( P_1 = 0, P_2 = 1,000 \) and \( P_3 = 10,000 \). \( P_1 \) and \( P_3 \) are asymptotically stable and \( P_2 \) is unstable.

**Problem B:**

The zeros of \( g \) are \( y_0 = 0 \) and \( y_n = \frac{1}{n\pi}, n \in \mathbb{Z} \setminus \{0\} \). For \( y \neq 0 \) we have

\[ g'(y) = 2y \sin(1/y) - \cos(1/y) \]

so that, for \( n \neq 0 \),

\[ g'(y_n) = -\cos(n\pi) \]

which is positive if \( n \) is odd and negative if \( n \) is even. Thus \( y_n \) is asymptotically stable for \( n \) even, and unstable for \( n \) odd.

To see that \( y_0 \) is stable, let \( \epsilon > 0 \) be given. Since \( y_n \to 0 \) as \( n \to \pm \infty \), there exists an odd positive integer \( n_0 \) such that \( \frac{1}{n_0\pi} < \epsilon \). Let \( \delta = \frac{1}{n_0\pi} \) and assume \(|y(0)| < \delta \). Since \( y_{n_0} \) and \( y_{-n_0} \) are unstable, \( y(t) \) will stay inside the interval \((-\delta, \delta)\) for all \( t \geq 0 \), and so it will lie in the interval \((-\epsilon, \epsilon)\) by construction. Hence \( y(t) \in (-\epsilon, \epsilon) \) for all \( t \geq 0 \), so \( y_0 = 0 \) is a stable equilibrium.

Furthermore, if \( y(0) \neq 0 \) then there exists an even integer \( n \) such that \( y_{n+2} \leq y(0) \leq y_n \), and so either \( y(0) = y_{n+1} \), in which case \( y(t) = y_{n+1} \) for all \( t \geq 0 \), or \( y(t) \) approaches \( y_n \) or \( y_{n+2} \) as \( t \to \infty \). Either way, \( \lim_{t \to \infty} y(t) \neq 0 = y_0 \), so the origin is not asymptotically stable.
Problem C:

\[ y' - 2y - e^{3t} \leq 0. \]

Multiplying by a positive function does not affect the inequality, so we have

\[ e^{-2t}y' - 2e^{-2t}y - e^t \leq 0 \]

namely,

\[ (e^{-2t}y - e^t)' \leq 0. \]

This means that the function \( t \mapsto e^{-2t}y(t) - e^t \) is non-decreasing. Thus, for \( t \geq 0 \),

\[ e^{-2t}y(t) - e^t \leq e^{-2\times 0}y(0) - e^0 = 1, \]

\[ y(t) \leq e^{3t} + e^{2t}, \quad t \geq 0. \]

Similarly, for \( t \leq 0 \) we have

\[ e^{-2t}y(t) - e^t \geq e^{-2\times 0}y(0) - e^0 = 1 \]

\[ y(t) \geq e^{3t} + e^{2t}, \quad t \leq 0. \]