Out of BD

Exercises 2.2

2) \[ y' = \frac{x^2}{y(1 + x^3)} \]

Rewrite the equation as

\[ yy' = \frac{x^2}{1 + x^3}. \]

Note that \( yy' = \left(\frac{1}{2}y^2\right)' \) and \( \frac{x^2}{1 + x^3} = \left(\frac{1}{3} \ln |1 + x^3|\right)' \), thus, in implicit form,

\[ \frac{1}{2}y^2 = \frac{1}{3} \ln |1 + x^3| + C \]

or, in explicit form,

\[ y = \pm \sqrt{\ln(1 + x^3)^{2/3} + C}. \]

6) \[ xy' = (1 - y^2)^{1/2} \]

\[ \frac{y'}{(1 - y^2)^{1/2}} = \frac{1}{x} \]

(arcsin \( y \))' = (ln |x|)'

arcsin \( y \) = ln |x| + C

\[ y = \sin(\ln |x| + C) \]

Note that \( y \equiv 1 \) and \( y \equiv -1 \) are also solutions.

22) \[ y' = \frac{3x^2}{3y^2 - 4} \]

\( (3y^2 - 4)y' = 3x^2 \)

\( (y^3 - 4y)' = (x^3)' \)

\[ y^3 - 4y = x^3 + C \]

Setting \( y(1) = 0 \) we get

\[ (0)^3 - 4 \times (0) = 0 = (1)^3 + C \]

so \( C = -1. \)
Thus the solution is given implicitly by
\[ y^3 - 4y = x^3 - 1. \]

The interval of existence must contain \( x = 1 \) and be such that \( y' \) is defined, namely, \( 3y^2 - 4 \neq 0 \). Since when \( x = 1 \) we have \( 3 \times (0)^2 - 4 = -4 < 0 \), the latter condition becomes, by continuity, \( 3y^2 - 4 < 0 \), or \( -\frac{2}{\sqrt{3}} < y < \frac{2}{\sqrt{3}} \).

Substituting in the implicit formula for \( y(x) \) we obtain
\[
\left( \frac{2}{\sqrt{3}} \right)^3 - 4 \cdot \frac{2}{\sqrt{3}} = x^3 - 1
\]
and
\[
\left( -\frac{2}{\sqrt{3}} \right)^3 + 4 \cdot \frac{2}{\sqrt{3}} = x^3 - 1
\]
or
\[
x_l^3 = \frac{-16 + 3\sqrt{3}}{3\sqrt{3}}
\]
and
\[
x_r^3 = \frac{16 + 3\sqrt{3}}{3\sqrt{3}}
\]
so, since \( x_l < 1 < x_r \), the interval of existence is
\[
[x_l, x_r] = \left[ \frac{3\sqrt{3} - 16 + 3\sqrt{3}}{\sqrt{3}}, \frac{3\sqrt{3} + 16 + 3\sqrt{3}}{\sqrt{3}} \right].
\]

**Exercise 2.3**

2) Let \( Q(t) \) be the amount of salt in the tank in liters at time \( t \). Salt enters the mix at a rate of \( 2\gamma \) grams per minute, so we add a term \( \gamma \) to \( Q' \). A well stirred mixture has a concentration of \( Q/120 \) g/liter, and leaves at the same rate of 2, so we subtract \( 2 \times Q/120 \) from \( Q' \). The result is
\[
Q' = 2\gamma - \frac{1}{60}Q
\]
with the initial condition \( Q(0) = 0 \) (initially only water).

The solution is
\[
Q(t) = 120\gamma \left[ 1 - e^{-t/60} \right]
\]
so, in the limit \( t \to \infty \), the amount of salt in the tank is
\[
\lim_{t \to \infty} Q(t) = 120\gamma.
\]

4) Let \( S(t) \) and \( M(t) \) be the amount of salt in the tank in pounds and total mixture in gallons, respectively, at time \( t \). We know that \( M(0) = 200 \), with salt in unit concentration pouring in at a rate of 3 and mixture exiting at a rate of 2, so \( M' = 3 - 2 = 1 \), which implies that \( M(t) = 200 + t \) prior to the overflow, which happens when \( M(t) = 500 \) or \( t = 300 \).
Since salt is entering at a rate of 3 and mixture is leaving at a rate of 2, we have

\[ S' = 3 - 2 \frac{S}{M} = 3 - \frac{2S}{200 + t}. \]

The solution of this ODE with the initial condition \( S(0) = 100 \) is

\[ S(t) = 200 + t - \frac{100 \times (200)^2}{(200 + t)^2}. \]

The concentration of salt is \( c = \frac{S(t)}{M(t)} \) so that, on the point of overflowing \( t = 300 \) we have

\[ c = \frac{500 - \frac{100 \times (200)^2}{(500)^2}}{500} = \frac{121}{125} \text{ lb/gal}. \]

In an infinite tank the limit concentration would be

\[ \lim_{t \to \infty} \frac{S(t)}{M(t)} = \lim_{t \to \infty} 1 - \frac{100 \times (200)^2}{(200 + t)^3} = 1 \text{ lb/gal}. \]

10) The equation for the amount of money borrowed \( S(t) \) at year \( t \) is

\[ S' = \frac{6}{100}S - 1500 \times 12 \]

so

\[ S(t) = e^{\frac{6}{100}t} \left( S_0 - \frac{1500 \times 12 \times 100}{6} \right) + \frac{1500 \times 12 \times 100}{6}. \]

(a) The debt would be paid in full by time \( t^* \) if \( S(t^*) = 0 \), that is,

\[ S_0 = \frac{1500 \times 12 \times 100}{6} \left( 1 - e^{-\frac{6}{100}t^*} \right) \]

Setting \( t^* = 20 \) we get

\[ S_0 = 209,641.74 \]

and setting \( t^* = 30 \) we get

\[ S_0 = 250,410.33. \]

(b) The total interest paid is \( 1500 \times 12 \times \text{years} - S_0 \). For 20 years we get

\[ 1500 \times 12 \times 20 - 209,641.74 = 150,358.26 \]

and for 30 years

\[ 1500 \times 12 \times 30 - 250,410.33 = 289,589.67. \]
16) Let $T$ be the temperature of the cup of coffee. Since the temperature of the surroundings is 70F, Newton’s law says that

\[ T' = k(T - 70) \]

for some constant $k$. The solution of this ODE with initial condition $T(0) = 200F$ is

\[ T(t) = 200e^{kt} - 70ke^{kt} - \frac{1}{k} = 130e^{kt} + 70 \]

When $t = 1$ we have $T = 190$, so

\[ 190 = 130e^k + 70 \]

thus

\[ e^k = \frac{120}{130} = \frac{12}{13} \]

which implies that

\[ T(t) = 130 \left( \frac{12}{13} \right)^t + 70 \]

so that $T = 150$ when $t$ satisfies

\[ 150 = 130 \left( \frac{12}{13} \right)^t + 70 \]

namely

\[ t \ln \left( \frac{12}{13} \right) = \ln \left( \frac{8}{13} \right) \]

or

\[ t = \frac{\ln(13/8)}{\ln(13/12)} \approx 6.0656095 \text{ minutes} \approx 6 \text{ min, 4 sec.} \]

Out of MG

27a)

\[ mx'' + cx' = mg \sin \alpha. \]

Integrating once gives

\[ mx' + cx = mg \sin(\alpha)t + C. \]

Using the intial conditions $x(0) = 0$ and $x'(0) = 0$ gives

\[ 0 = C \]

so that

\[ mx' + cx = mg \sin(\alpha)t. \]

Dividing by $m$ and multiplying by $e^{\frac{x}{m}t}$ we get

\[ (e^{\frac{x}{m}t}x)' = g \sin(\alpha)te^{\frac{x}{m}t} = \frac{m^2g}{c^2} \sin \alpha \left( \frac{c}{m}te^{\frac{x}{m}t} - e^{\frac{x}{m}t} \right)'. \]
Integrating between $t = 0$ and $t$, and using that $x(0) = 0$ we obtain

$$e^{\frac{c}{m}t}x(t) = \frac{m^2 c}{c^2} \sin \alpha \left( \frac{c}{m} t e^{\frac{c}{m}t} - e^{\frac{c}{m}t} + 1 \right)$$

so that

$$x(t) = \frac{m^2 c}{c^2} \sin \alpha \left( \frac{c}{m} t - 1 + e^{-\frac{c}{m}t} \right).$$

Problem A: Let $y = g(t)$ be a solution of the ODE $y' = f(y)$.

Let $z(t) = g(t - c)$. Then

$$z'(t) = g'(t - c) = y'(t - c) = f(y(t - c)) = f(g(t - c)) = f(z(t))$$

that is, $z$ satisfies

$$z' = f(z)$$

as required.

Problem B: For $c > 0$ let

$$z_c(t) = \begin{cases} \frac{1}{4} (t - c)^2 & \text{if } t > c, \\ 0 & \text{if } t \leq c. \end{cases}$$

We then have

$$z'_c(t) = \begin{cases} \frac{1}{2} (t - c) & \text{if } t > c, \\ 0 & \text{if } t \leq c. \end{cases}$$

which is a continuous function on $\mathbb{R}$, and, since $t - c > 0$ for $t > c$,

$$z'_c(t) = \begin{cases} \sqrt{\frac{1}{4} (t - c)^2} & \text{if } t > c, \\ \sqrt{0} & \text{if } t \leq c. \end{cases} = \sqrt{z_c(t)}$$

that is, $z_c$ solves the ODE

$$z'_c = \sqrt{z_c}.$$

Since for $t \in [0, c]$ we have $z_c(t) = 0$, in particular $z_c(0) = 0$. Thus the ODE has infinitely many distinct solutions.