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part b)

\[ u(x, t) = \frac{1}{2} (g(x + t) + g(x - t)) + \frac{1}{2} \int_{x-t}^{x+t} h(s) \, ds \]

Differentiate against t and x we get

\[ u_x = \frac{1}{2} (g'(x + t) + g'(x - t)) + \frac{1}{2} (h(x + t) - h(x - t)) \]

\[ u_t = \frac{1}{2} (g'(x + t) - g'(x - t)) + \frac{1}{2} (h(x + t) + h(x - t)) \]

We then see \( u_x^2 - u_t^2 = (u_x + u_t)(u_x - u_t) = (g'(x + t) + h(x + t))(g'(x - t) - h(x - t)) \). Notice that distance between point \( x + t \) and \( x - t \) is \( 2t \), for all large \( t \) and so any \( x \in \mathbb{R} \) one of \( (g'(x + t) + h(x + t)) \) and \( (g'(x - t) - h(x - t)) \) must vanish. We are then just integrating a zero function.

2 Problem A

For the second part, because \( r \geq 0 \), so you need to use D’Alembert’s formula for half line. The final answer should be

\[
   u(x, t) = \begin{cases} 
   \frac{1}{2|x|} \left( (|x| + ct)f(|x| + ct) + (|x| - ct)f(|x| - ct) \right) + \frac{1}{2c|x|} \int_{|x|+ct}^{x} s g(s) \, ds, & |x| \geq ct \\
   \frac{1}{2|x|} \left( (|x| + ct)f(|x| + ct) - (ct - |x|)f(ct - |x|) \right) + \frac{1}{2c|x|} \int_{ct-|x|}^{x} s g(s) \, ds, & |x| < ct 
   \end{cases}
\]
3 Problem C

For point \((x, t) \in \mathbb{R}^3 \times [0, \infty)\), solution of 3D wave equation depends only on \(\partial B(x, ct)\), while solution for a 2D wave equation depends on \(B(x, ct)\).

For 3D case

1. \(0 \leq ct < \rho_1\), The vanishing set is \(B(0, \rho_1 - ct) \cup \{R^3 \setminus B(0, \rho_2 + ct)\}\)
2. \(\rho_1 < ct < \rho_2\), The vanishing set is \(\{R^3 \setminus B(0, \rho_2 + ct)\}\)
3. \(\rho_1 < ct < \rho_2\), The vanishing set is \(\{x : ct - \rho_2 < |x| < ct + \rho_2\}\)

For 2D case

1. \(0 \leq ct < \rho_1\), The vanishing set is \(B(0, \rho_1 - ct) \cup \{R^3 \setminus B(0, \rho_2 + ct)\}\)
2. \(\rho_1 < ct\), The vanishing set is \(\{R^3 \setminus B(0, \rho_2 + ct)\}\)

4 Problem D

We see \(u_{tt} - c^2 \Delta u = \cos\left(\frac{b}{c} x_2\right)\left(w_{tt} - c^2 w_{x_1 x_1} + b^2 w\right) = 0\). So \(u\) satisfies the 2D wave equation.

We have

\[
u(x_1, x_2, t) = \frac{1}{2\pi c^2 t^2} \int_{B(x_1, ct)} \psi(y_1) \cos\left(\frac{b}{c} y_2\right) \frac{c^2 t^2}{\sqrt{c^2 t^2 - |y - x|^2}} dy_1 dy_2
\]

Then since \(w(x_1, t) = u(x_1, 0, t)\), we can use the formula above to write

\[
w(x_1, t) = \frac{1}{2\pi} \int_{(y_1 - x_1)^2 + y_2^2 \leq c^2 t^2} \psi(y_1) \cos\left(\frac{b}{c} y_2\right) \frac{1}{\sqrt{c^2 t^2 - (y_1 - x_1)^2 - y_2^2}} dy_1 dy_2
\]

We see it is a double integral.