Homework 3 (M540)

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1 Evans Chapter 2 Problem 10

a) The function $v$ is certainly $C^2$ and harmonic away from the line segment $\{x_n = 0\} \cap U$ since $u$ is $C^2$ and harmonic in $U^+$. So if one can show $v$ is $C^2(U)$ then by continuity necessarily $\Delta v = 0$ on this line segment as well.

To show $v$ is smooth across this line segment one must compare one-sided derivatives from above and below. Namely your proof should include

$$\lim_{h \to 0^+} \frac{v(x + he_n) - v(x)}{h} = \lim_{h \to 0^+} \frac{u(x + he_n)}{h}$$
$$= \lim_{h \to 0^+} \frac{-v(x - he_n)}{h} = \lim_{h \to 0^+} \frac{v(x - he_n) - v(x)}{-h}$$
$$= \lim_{h \to 0^-} \frac{v(x + he_n) - v(x)}{h}$$
for all $x \in \partial U^+ \cap \{x_n = 0\}$.

Next, you should proceed to check higher partial derivative. In addition, $C^2$ means two-times continuously differentiable, meaning that second derivative is continuous. At any interior points of $U^+$ and $U^-$ you can say $v$ is $C^2$ because of smooth change of coordinates. But again you should check all $D^\alpha v(x), |\alpha| = 2$ are continuous for $x \in \partial U^+ \cap \{x_n = 0\}$. 

b) Define $w$ as the solution to

$$\begin{cases} 
\Delta w = 0 & x \in U \\
w = v & x \in \partial U 
\end{cases}$$

Because $v \in C(\partial U)$, the solution $w$ can be expressed by Poisson integral formula for a ball (Theorem 15). Then calculate to show $w = 0$ on $\partial U^+ \cap \{x_n = 0\}$. $w$ then agrees with $v$ on $\partial U^+$ and also on $\partial U^-$. By the maximum principle then necessarily $w = v$ everywhere in $U$.

## 2 EVANS CHAPTER 2 PROBLEM 12

$$v = \frac{\partial}{\partial \lambda} u|_{\lambda=1}$$

$$(\frac{\partial}{\partial t} - \Delta)v = (\frac{\partial}{\partial t} - \Delta)\frac{\partial}{\partial \lambda} u|_{\lambda=1} = \frac{\partial}{\partial \lambda} (\frac{\partial}{\partial t} - \Delta)u|_{\lambda=1} = 0$$

## 3 EVANS CHAPTER 2 PROBLEM 14

If one defines $v = e^{ct}u$, then $v$ solves

$$\begin{cases} 
v_t - \Delta v = f e^{ct} & \text{in } R^n \times (0, \infty) \\
v = g & \text{on } R^n \times \{t = 0\}
\end{cases}$$

Then use equation (17) to solve for $v$.

## 4 PROBLEM 15

Define

$$v(x, t) := \begin{cases} 
u(x, t) - g(t) & \text{for } x \geq 0 \\
u(-x, t) + g(t) & \text{for } x < 0
\end{cases}$$

Then we find

$$\begin{cases} 
v_t - v_{xx} = -g_t & \text{in } R_+ \times (0, \infty) \\
v_t - v_{xx} = g_t & \text{in } R_- \times (0, \infty) \\
v = 0 & \text{on } \{x = 0\} \times [0, \infty)
\end{cases}$$
By equation (17) on Page 51

\[ v(x, t) = \int_0^t \int_{-\infty}^0 \phi(x - y, t - s) g'(s) dy ds - \int_0^t \int_{-\infty}^\infty \phi(x - y, t - s) g'(s) dy ds \]

\[ = 2\int_0^t \int_{-\infty}^0 \phi(x - y, t - s) g'(s) dy ds - \int_0^t \int_{-\infty}^\infty \phi(x - y, t - s) g'(s) dy ds \]

\[ = 2\int_{-\infty}^t \int_0^t \phi(x - y, t - s) g'(s) dy ds - \int_0^t g'(s) ds \]

\[ = 2\int_{-\infty}^t \int_0^t \phi(x - y, t - s) g(s) dy ds - 2\int_0^t \int_{-\infty}^t \frac{d}{ds} \phi(x - y, t - s) g(s) dy ds - g(t) \]

\[ = -2\int_0^t \int_{-\infty}^t \frac{d}{ds} \phi(x - y, t - s) g(s) dy ds - g(t) \]

\[ = \frac{x}{\sqrt{4\pi}} \int_0^t \frac{1}{(t - s)^{3/2}} e^{\frac{-x^2}{4(t-s)}} g(s) ds - g(t) \]

So

\[ u(x, t) = v(x, t) + g(t) = \frac{x}{\sqrt{4\pi}} \int_0^t \frac{1}{(t - s)^{3/2}} e^{\frac{-x^2}{4(t-s)}} g(s) ds \]

5 Evans Chapter 2 Problem 16

If \( u_c \) attains its maximum at \((x_0, t_0) \in U_T = U \times (0, T)\), then \((u_c)_t \geq 0\). Actually \( > \) could happen when \( t_0 = T \). And in space, \( \Delta u_c \leq 0 \) so \(-\epsilon = (u_c)_t - \Delta u_c \geq 0\), contradiction. So the maximum of \( u_c \) can only happen on \( \Gamma_T \). And we have

\[ \max u - \epsilon T \leq \max_{\Gamma_T} (u - \epsilon t) \leq \max_{\Gamma_T} (u - \epsilon t) \leq \max_{\Gamma_T} u \]

let \( \epsilon \to 0 \), we get

\[ \max_{\Gamma_T} u \leq \max_{\Gamma_T} u \]

6 Problem A

Careful with your notation. For example do not write \((ix)^2 \hat{u}(x, y)\). You need to introduce a new variable, such as \( k \), and write \((ik)^2 \hat{u}(k, y)\).

7 Problem C

\[ |u(x, t)| = \left| \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} f(y) e^{-\frac{|x-y|^2}{4t}} dy \right| \leq \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} |f(y)| dy \leq \frac{1}{(4\pi t)^{\frac{n}{2}}} \|f\|_{L^1} \]

Since \( C_c(\mathbb{R}^n) \subset L^1(\mathbb{R}^n) \)