M540 Homework Spring 2017

Homework is due in class on the due date. Extensions may occasionally be permitted, but only with instructor permission. All problems from Evans refer to the 2nd edition. All problems from Jost refer to the 2nd edition of PDE by Jurgen Jost, which is available on-line for free. Please make sure you are not taking the exercises from the first edition of either book!

Assignment 1. Due Thursday, January 19

Out of Evans Chapter 1: Exercise 1 on page 12. (See Definitions on page 2.)
Out of Evans Chapter 2: Exercises 1,2,4,5 on page 85.
Out of Jost: Exercise 1.8 on page 31.

Assignment 2. Due Tuesday, January 31

Out of Evans Chapter 2: 6 on page 86.
Out of Jost: Exercise 1.10 on page 31.

Problem A. Let $\Omega \subset \mathbb{R}^n$ be open and suppose $u_k : \Omega \to \mathbb{R}^1$, $k = 1, 2, \ldots$ is a sequence of harmonic functions converging uniformly on $\Omega$ to a function $u$. Prove that $u$ is harmonic.

Problem B. Let $\Omega \subset \mathbb{R}^n$ be a bounded, connected open set containing the origin. Let $\{u_k\}$ denote a sequence of positive harmonic functions defined on $\Omega$ that all satisfy the condition $u_k(0) = 1$. Prove that there exists a subsequence $\{u_{k_j}\}$ that converges uniformly to a harmonic function $u$ on compact subsets of $\Omega$.

Problem C. Let $\Omega \subset \mathbb{R}^n$ be a bounded, connected open set with smooth boundary. Let $\alpha : \partial \Omega \to \mathbb{R}$ be a non-negative function that is not identically zero. Prove that for any given functions $g : \partial \Omega \to \mathbb{R}$ and $F : \Omega \to \mathbb{R}$ there exists at most one solution $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ solving the Poisson problem with Robin boundary conditions:

$$
\Delta u = F \text{ in } \Omega, \quad \nabla u \cdot \nu + \alpha(x)u = g \text{ for all } x \in \partial \Omega
$$

where $\nu$ denotes the outer unit normal to $\partial \Omega$. Then construct a counter-example to uniqueness when $\alpha$ is allowed to be negative. *Hint for counter-example: Work in 1d and find an appropriate interval $\Omega$ and take $\alpha$ to be a negative constant.*
Assignment 3. Due Tuesday, February 14

Out of Evans: Chapter 2: 10, 12, 14, 15, 16

A. Assume \( f : \mathbb{R} \to \mathbb{R} \) is a \( C_c^\infty(\mathbb{R}) \) function. Use Fourier transform (in \( x \)) to obtain the Poisson integral formula solution to Laplace’s equation in the upper half-plane that we derived (by other means) in class:

\[
  u_{xx} + u_{yy} = 0 \text{ for } -\infty < x < \infty, y > 0, \; u(x, 0) = f(x), \; |u| \text{ bounded.}
\]

(Note: you should end up with Formula (33) on page 37 of Evans.)

B. Suppose we wish to find a formula for the solution to Poisson’s equation \(-\Delta u = F(x)\) in the upper half-space \( \{x : x_n > 0\} \) subject to Neumann boundary conditions \( \nabla u \cdot \nu = g \) for \( x \in \{x : x_n = 0\} \) where \( \nu = (0, \ldots, 0, -1) \) and \( g = g(x_1, \ldots, x_{n-1}) \) is a given bounded, continuous function. Obtain this formula by first finding the Green’s function for the Laplacian in the upper half-space subject to Neumann boundary conditions. You may assume the given function \( F \) is smooth and compactly supported. You do not need to verify that your formula solves the problem, just find the Green’s function \( G(x, y) \) and then write down the appropriate integral formula in terms of \( G, g \) and \( F \).

C. Let \( u \) be the solution derived in class to the \( n \)-dimensional heat equation \( u_t = \Delta u \) for \( x \in \mathbb{R}^n, \; t > 0 \) subject to initial condition \( u(x, 0) = f(x) \), where \( f : \mathbb{R}^n \to \mathbb{R} \) is assumed to be continuous and of compact support. Prove that \( u \) converges uniformly (in \( x \)) to zero as \( t \) approaches infinity by showing:

\[
  |u(x, t)| \leq \frac{C}{t^{n/2}}
\]

for all \( x \in \mathbb{R}^n \) and all \( t > 0 \) where \( C \) is a constant.
Assignment 4. Due Tuesday March 7

Out of Evans: Chapter 2: 21a

A. Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set. Suppose $F : \mathbb{R} \to \mathbb{R}$ is a $C^1$ satisfying the condition $F'(u) < 0$ for all $u \in \mathbb{R}$. Suppose, for some $T > 0$, that $u_1$ and $u_2$ are $C^2(\Omega \times (0, T])$ functions that are continuous on $\overline{\Omega} \times [0, T]$ and that both satisfy the heat equation with a nonlinear source term:

$$u_t = \Delta u + F(u) \text{ for } x \in \Omega, \ 0 < t \leq T.$$ 

Assume that $u_1(x, t) \leq u_2(x, t)$ for all $(x, t) \in (\partial \Omega \times [0, T]) \cup (\Omega \times \{0\})$.

*(You’ll recognize this last set as the parabolic boundary.)*

Prove that $u_1(x, t) \leq u_2(x, t)$ for all $(x, t) \in \Omega \times (0, T]$.

*Hint:* Use the mean value theorem to handle the expression $F(u_1) - F(u_2)$.

B. An infinite string vibrates according to the homogeneous wave equation $u_{tt} - u_{xx} = 0$ with initial data given by

$$u(x, 0) = f(x) \text{ and } u_t(x, 0) = g(x) \text{ for } -\infty < x < \infty$$

where both $f$ and $g$ are smooth functions that are positive on the intervals $-4 < x < -3$ and $2 < x < 3$ and both zero everywhere else along the $x$-axis. A person stands at location $x = 0$. During what intervals of time will the person notice the string vibrating? Include a sketch in the $x-t$ plane with your explanation.

C. Find an explicit formula for the solution $u = u(x, t)$ to the problem

$$u_{tt} - c^2 u_{xx} = 0 \text{ for } x > 0, \ t > 0,$$

$$u(0, t) = h(t) \text{ for } t > 0, \ u(x, 0) = f(x), \ u_t(x, 0) = g(x) \text{ for } x \geq 0$$

where $f, g$ and $h$ are all $C^2$ for non-negative arguments and satisfy the compatibility conditions

$$h(0) = f(0), \ h'(0) = g(0), \ h''(0) = c^2 f''(0).$$

Verify that the formula for $u$ you obtain has continuous second derivatives everywhere in the region $x > 0, \ t > 0$. 
D. Now consider the wave equation for a half-infinite string as in Problem C but change the boundary conditions to homogeneous Neumann:

\[ u_{tt} - c^2 u_{xx} = 0 \text{ for } x > 0, \ t > 0, \]
\[ u_x(0, t) = 0 \text{ for } t > 0, \ u(x, 0) = f(x), \ u_t(x, 0) = g(x) \text{ for } x \geq 0. \]

First write down compatibility conditions for \( f \) and \( g \) at \( x = 0 \). Then determine a formula for the solution. Finally, consider the special case where \( g \equiv 0 \) and the function \( f \) is zero except for a positive, smooth ”bump” in some interval \( 0 < a < x < b \). Graph a “snapshot” of the solution as a function of \( x \) at several times, focusing on how the wave interacts with the boundary \( x = 0 \) under these boundary conditions.

Assignment 5. Due Thursday March 30

Out of Evans: Chapter 2: 24

A.

(i) Suppose \( u : \mathbb{R}^3 \times (0, T] \to \mathbb{R} \) is a radial solution to the 3d wave equation in the sense that \( u(x, t) = \tilde{u}(r, t) \) where \( r = |x| \) for all \( x \in \mathbb{R}^3 \) for some function \( \tilde{u} : [0, \infty) \times (0, T] \to \mathbb{R} \). Define \( v(r, t) = ru(r, t) \) and making use of the formula for the radial Laplacian (see top of page 22, Evans) derive the PDE satisfied by \( v \).

(ii) Use part (i) to find the solution to the 3d wave equation with radial initial conditions \( u(x, 0) = f(|x|), \ u_t(x, 0) = g(|x|) \) for given smooth functions \( f : [0, \infty) \to \mathbb{R} \) and \( g : [0, \infty) \to \mathbb{R} \).
B. Let \( f : \mathbb{R}^3 \to \mathbb{R} \) and \( g : \mathbb{R}^3 \to \mathbb{R} \) be given \( C^\infty \) functions and suppose \( u = u(x,t) \) is the solution to the homogenous 3d wave equation subject to the initial conditions \( u(x,0) = f(x) \) and \( u_t(x,0) = g(x) \) for all \( x \in \mathbb{R}^3 \) given by

\[
u(x,t) = \frac{1}{4\pi c^2 t^2} \int_{\{y : |y-x|=ct\}} \left\{ f(y) + tg(y) + \nabla f(y) \cdot (y-x) \right\} dS_y
\]

for \( t > 0 \).

Carefully verify that

\[
\lim_{t \to 0^+} u(x,t) = f(x) \quad \text{and} \quad \lim_{t \to 0^+} u_t(x,t) = g(x) \quad \text{for all} \quad x \in \mathbb{R}^3.
\]

C.

(i) Suppose \( f : \mathbb{R}^3 \to \mathbb{R} \) and \( g : \mathbb{R}^3 \to \mathbb{R} \) are smooth and positive in a spherical shell \( B(0,\rho_2) \setminus B(0,\rho_1) \) for some constants \( 0 < \rho_1 < \rho_2 \) and zero elsewhere. Let \( u \) solve the 3d wave equation with initial data \( u(x,0) = f(x) \) and \( u_t(x,0) = g(x) \) for all positive values of \( t \) describe the set

\[
\{ x \in \mathbb{R}^n : u(x,t) \text{ must vanish} \}.
\]

(Your answer should involve \( \rho_1, \rho_2 \) and \( c \).)

(ii) Now answer the same question when \( u \) solves the 2d wave equation.

D. For given \( b > 0 \) and \( c \in \mathbb{R} \), find a formula (it will take the form of a double integral) for the solution \( w = w(x_1,t) \) to the problem

\[
w_{tt} - c^2 w_{x_1 x_1} + b^2 w = 0 \quad \text{for} \quad x_1 \in \mathbb{R}, \quad t > 0, \quad w(x_1,0) = 0, \quad w_t(x_1,0) = \psi(x_1),
\]

where \( \psi : \mathbb{R} \to \mathbb{R} \) is given. Here’s how to do it: Look for a solution \( u = u(x_1,x_2,t) \) to the 2d wave equation of the specific form \( u = w(x_1,t) \cos \left( \frac{b}{c} x_2 \right) \). (Use the explicit formula derived in class for that equation.) Argue, by ‘descending to 1-d’ that \( w \) solves the problem above (which is known as a version of the telegraph equation).
Assignment 6. Due Thursday April 13

A. Consider the P.D.E.

\[ u_x + u_y = 1 \]

subject to the initial condition \( u(x, y) = h(x, y) \) for \((x, y) \in \Gamma \) where \( \Gamma \) is a given smooth curve and \( h : \Gamma \to \mathbb{R}^1 \) is a given smooth function.

a. Find a smooth initial curve \( \Gamma \) passing through the origin and a smooth function \( h : \Gamma \to \mathbb{R}^1 \) such that a solution to the problem exists in a neighborhood of every point of \( \Gamma \) except the origin. Verify this non-existence at the origin for your example.

b. Now find an initial curve \( \Gamma \) and function \( h : \Gamma \to \mathbb{R}^1 \) such that the problem has infinitely many solutions. Describe explicitly this infinite family of solutions.

B. Let \( u \) be a \( C^1 \) solution of

\[ a(x, y)u_x + b(x, y)u_y = -u \text{ for } (x, y) \in \overline{B(0, 1)}, \]

where \( B(0, 1) \) denotes the unit ball in \( \mathbb{R}^2 \). Assume that \( a(x, y)x + b(x, y)y > 0 \) for all \((x, y) \in \partial B(0, 1)\). Prove that \( u \) must vanish identically in \( \overline{B(0, 1)} \).

C. Find the solution to the PDE

\[ u_x + xu_t = u^2 \]

subject to the initial condition \( u(x, t) = \cos x \) along the line \( t = 2x + 1 \). Also sketch the characteristic projections in the \( xt \)-plane.

D. Solve the PDE

\[ u_x + u^2 u_y = 1 \]

subject to the initial condition \( u(x, 0) = 1 \).

E. Find an implicit solution via the method of characteristics for the problem

\[ u_y = xuu_x, \quad u(x, 0) = x \]

(Correct answer: \( x = ue^{-yu} \).)
F. Suppose $u_j(x, y), j = 1, 2$ are two smooth solutions to the same PDE

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u)$$

subject to two different initial conditions, where $a, b$ and $c$ are smooth. Suppose the graphs of these two solutions intersect transversally along a smooth curve $\gamma \subset \mathbb{R}^3$. (By *transversally* one means, for example, that at each point of $\gamma$, the normals to the two graphs are independent of each other.) Prove that $\gamma$ must be a characteristic for this equation. That is, show that at each point of $\gamma$, the tangent vector points in the direction $(a, b, c)$.

**Assignment 7. Due Tuesday April 25**

It turns out that shocks are only considered “physical,” (i.e. ‘good shocks’) if the characteristics run into the shock as time increases, rather than away from the shock. This is expressed as the pair of inequalities (17) on page 142 of Evans, where in (17), $\sigma = s'(t) = \text{the shock speed}$. In each of the problems below, only use a shock if it is physical in this sense.

A. Suppose $F : \mathbb{R} \to \mathbb{R}$ is a smooth, strictly convex function. Consider the conservation law

$$u_t + (F(u))_x = 0 \quad \text{for } -\infty < x < \infty, \ t > 0, \ u(x, 0) = u_0(x) \ \text{for } -\infty < x < \infty$$

where for real numbers $u_L$ and $u_R$, the initial condition $u_0$ takes the form

$$u_0(x) = \begin{cases} 
    u_L & \text{for } x < 0, \\
    u_R & \text{for } x > 0 
\end{cases}$$

Find a formula for the weak solution to this problem in case $u_L < u_R$. Then do the same thing in the case $u_R < u_L$. 
B. Find the solution to Burger’s equation

\[ u_t + uu_x = 0 \text{ for } -\infty < x < \infty, \ t > 0, \ u(x, 0) = u_0(x) \text{ for } -\infty < x < \infty \]

where the initial condition \( u_0 \) takes the form

\[ u_0(x) = \begin{cases} 
0 & \text{for } x < 1, \\
1 & \text{for } x > 1 
\end{cases} \]

C. Now solve Burger’s equation with initial condition

\[ u_0(x) = \begin{cases} 
1 & \text{for } x < 0, \\
0 & \text{for } 0 < x < 1 \\
1 & \text{for } x > 1 
\end{cases} \]

*NOTE: Be sure your answer continues to be valid beyond the time \( t = 2 \).*