E449 Lecture Notes

1. Vector Operations and Linear Algebra

Vectors in $\mathbb{R}^n$:

We can think of a vector $\vec{v}$ as a directed line segment. It has a magnitude (length), denoted by $|\vec{v}|$, and a direction. We associate any vector with the position vector having the same length and direction as $\vec{v}$ and whose tail sits at the origin and whose head sits at the point $(v_1, v_2, \ldots, v_n)$. Then from the distance formula (basically Pythagoras' Theorem), we find

$$|\vec{v}| = \sqrt{v_1^2 + v_2^2 + \ldots + v_n^2}.$$ 

• **Vector addition**: Just add coordinates. So if say $\vec{v} \in \mathbb{R}^3$ is given by $\vec{v} = (1, 2, 3)$ and $\vec{w} \in \mathbb{R}^3$ is given by $\vec{w} = (4, 5, 6)$ then $\vec{v} + \vec{w} = (5, 7, 9)$. Geometrically, $\vec{v} + \vec{w}$ has position vector given by placing the tail of $\vec{w}$ on top of the head of $\vec{v}$ and then joining the origin to the resulting location of the head of $\vec{w}$.

• **Multiplication of a vector and a scalar**: if $\vec{v}$ is a vector in $\mathbb{R}^n$ and $c \in \mathbb{R}$ is any number (scalar), then $c\vec{v}$ is the vector of magnitude $|c||\vec{v}|$ that has the same direction as $\vec{v}$ (if $c > 0$) and opposite direction as $\vec{v}$ if $c < 0$.

In coordinates, if $\vec{v} = (v_1, v_2, \ldots, v_n)$, then $c\vec{v} = (cv_1, cv_2, \ldots, cv_n)$.

Two notion of product for vectors

• **Dot product**: The dot produce (or “inner product”) of two vectors in $\vec{v} = (v_1, v_2, \ldots, v_n)$ and $\vec{w} = (w_1, w_2, \ldots, w_n)$ in $\mathbb{R}^n$ is denoted by $\vec{v} \cdot \vec{w}$ and is defined by the formula

$$\vec{v} \cdot \vec{w} = v_1w_1 + v_2w_2 + \ldots + v_nw_n.$$

Note that the dot product of two vectors is a number, i.e. a scalar. It is NOT another vector. A key property of the dot product is the formula

$$\vec{v} \cdot \vec{w} = |\vec{v}| |\vec{w}| \cos \theta \quad \text{where } \theta \text{ is the angle between } \vec{v} \text{ and } \vec{w}.$$
(This formula follows by drawing the triangle formed by $\vec{v}$ and $\vec{w}$ and applying the law of cosines.)

It follows that two vectors meet at a right angle, that is are orthogonal to each other, if and only if their dot product is zero.

Other properties of dot product (you can check these yourself from the definition)

\[
\begin{align*}
\ast & \quad \vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v} \\
\ast & \quad \vec{v} \cdot \vec{v} = |\vec{v}|^2
\end{align*}
\]

- **Cross Product**: The definition of the cross product is motivated by the attempt to find a vector, denoted by $\vec{v} \times \vec{w}$, that is orthogonal to any two other given vectors $\vec{v} = (v_1, v_2, v_3)$ and $\vec{w} = (w_1, w_2, w_3)$ in $\mathbb{R}^3$. After some algebra, one arrives at the formula

\[
\vec{v} \times \vec{w} = (v_2w_3 - v_3w_2, v_3w_1 - w_3v_1, v_1w_2 - v_2w_1).
\]

If you’re familiar with the notion of the determinant of a $3 \times 3$ matrix (which we will discuss soon in the course), then a device for remembering this formula is given by

\[
\vec{v} \times \vec{w} = \det \begin{pmatrix}
\vec{i} & \vec{j} & \vec{k} \\
v_1 & v_2 & v_3 \\
w_1 & w_2 & w_3
\end{pmatrix}
\]

where $\vec{i}$, $\vec{j}$ and $\vec{k}$ denote the three “standard basis vectors” $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ respectively.

**Key properties of cross product:**

\[
\begin{align*}
\ast & \quad \vec{v} \times \vec{w} = - (\vec{w} \times \vec{v}) \\
\ast & \quad |\vec{v} \times \vec{w}| = |\vec{v}| |\vec{w}| \sin \theta \\
\ast & \quad \vec{v} \times \vec{w} = \vec{0} \quad \text{if and only if } \vec{v} \text{ and } \vec{w} \text{ are parallel.}
\end{align*}
\]

- **Parametric equation of a line in $\mathbb{R}^n$:**

The parametric equation of a line in $\mathbb{R}^3$ passing through a point $P = (x_0, y_0, z_0)$ that moves
tangent to the direction given by the vector \( \vec{v} = (v_1, v_2, v_3) \) is given by:

\[
\vec{r}(t) = \vec{P} + tv = (x_0 + tv_1, y_0 + tv_2, z_0 + tv_3).
\]

The formula is similar for \( P \) and \( \vec{v} \) in \( \mathbb{R}^2 \) or in \( \mathbb{R}^n \) in general. If the parameter \( t \) is allowed to run through all real numbers then we get a line. If \( t \) is restricted to take values in an interval, say \( a \leq t \leq b \), then we get a line segment.

• **Equation of a plane in \( \mathbb{R}^3 \):**

A normal vector to a plane in \( \mathbb{R}^3 \) is a vector \( \vec{n} \) with the property that it is orthogonal to all vectors that are parallel to the plane. If \( \vec{n} \) is any such vector, then so is any multiple of \( \vec{n} \). However, there are always two unit normals to any plane—normal vectors of magnitude one, namely

\[
\pm \frac{\vec{n}}{|\vec{n}|} \text{ where } \vec{n} \text{ is any normal vector.}
\]

A plane is completely determined by the property that all vectors parallel to the plane are orthogonal (perpendicular) to a given normal \( \vec{n} = (n_1, n_2, n_3) \) along with the specification that the plane contains a given point, say \( P_1 = (x_1, y_1, z_1) \).

Here’s how it works: If \((x, y, z)\) is any point in this plane, then the vector with tail \( P_1 \) and head located at \((x, y, z)\) is parallel to the plane and is given by \((x - x_1, y - y_1, z - z_1)\). The condition that this vector is parallel to the plane means that it is orthogonal to \( \vec{n} \). This leads us to a condition determining all the points \((x, y, z)\) in the plane:

\[
(n_1, n_2, n_3) \cdot (x - x_1, y - y_1, z - z_1) = 0.
\]

Multiplying out the dot product we arrive at

\[
n_1x + n_2y + n_3z = d \text{ where } d = n_1x_1 + n_2y_1 + n_3z_1.
\]

Note that this is a linear relationship between the three variables \( x, y \) and \( z \). It follows that any equation of the form

\[
ax + by + cz = d
\]
determines a plane whose normal direction is given by \( \vec{n} = (a, b, c) \) (as long as \( a, b \) and \( c \) are not all zero).
Some key concepts regarding a set of vectors

Suppose $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is a set of $k$ vectors in $\mathbb{R}^n$ where $n \geq 2$.

Given any $k$ constants $c_1, c_2, \ldots, c_k$ the vector

$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_k\vec{v}_k$$

is called a \textit{linear combination} of the vectors in the set $S$.

The collection of \textit{all} linear combinations of the vectors in the set $S$ is called the \textit{span} of the set $S$, and is denoted $\text{span } S$.

\textbf{Example:} If $\vec{v}_1 = (1, 0)$ and $\vec{v}_2 = (0, 1)$ and $S = \{\vec{v}_1, \vec{v}_2\}$ then a linear combination of the two vectors in this set takes the form

$$c_1(1, 0) + c_2(0, 1) = (c_1, c_2).$$

Clearly, by considering \textit{all} linear combinations, that is, by letting $c_1$ and $c_2$ be any real numbers, we can build every vector in the plane. Thus, we find that $\text{span } S = \mathbb{R}^2$.

Now comes a crucial definition in linear algebra:

A set $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ of $k$ vectors in $\mathbb{R}^n$ is said to be \textit{linearly independent} if the only choice of constants $c_1, c_2, \ldots, c_k$ for which the equation

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_k\vec{v}_k = \vec{0}$$

holds is the “trivial choice”: $c_1 = c_2 = \ldots = c_k = 0$.

If the set of vectors $S$ is not linearly independent, then it is said to be \textit{linearly dependent}.

\textbf{Examples:} $\{(1, 0), (0, 1)\}$ is a linearly independent set since the only way to solve the equation

$$c_1(1, 0) + c_2(0, 1) = (0, 0)$$

is to choose $c_1 = c_2 = 0$.

On the other hand $\{1, 2, 3), (2, 4, 6)\}$ is a linearly dependent set since choosing $c_1 = 2$ and $c_2 = -1$ we find

$$2(1, 2, 3) - 1(2, 4, 6) = (0, 0, 0).$$
Suppose \( S = \{ \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \} \) is a set of \( k \) vectors in \( \mathbb{R}^n \) where \( n \geq 2 \). Let’s write these vectors as column vectors having \( n \) components as follows:

\[
\vec{v}_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{pmatrix}, \quad \ldots, \quad \vec{v}_k = \begin{pmatrix} a_{1k} \\ a_{2k} \\ \vdots \\ a_{nk} \end{pmatrix}.
\]

As we’ve seen, the question of whether \( S \) constitutes a linearly independent set of vectors is the question of whether or not there are \( k \) numbers \( c_1, c_2, \ldots, c_k \) not all zero such that

\[
c_1 \vec{v}_1 + c_2 \vec{v}_2 + \ldots + c_k \vec{v}_k = \vec{0}.
\]

But if we write this vector equation out, component by component, it is equivalent to a system of \( n \) linear equations in \( k \) unknowns:

\[
\begin{align*}
a_{11}c_1 + a_{12}c_2 + \ldots + a_{1k}c_k &= 0 \\
a_{21}c_1 + a_{22}c_2 + \ldots + a_{2k}c_k &= 0 \\
\vdots \\
a_{n1}c_1 + a_{n2}c_2 + \ldots + a_{nk}c_k &= 0
\end{align*}
\]

We call this system homogeneous since the right-hand side is zero.

Similarly, the question of whether some other vector, say \( \vec{w} \in \mathbb{R}^n \) is in the span of \( S \) is equivalent to the question of whether there exist \( k \) numbers \( c_1, c_2, \ldots, c_k \) such that

\[
\begin{align*}
a_{11}c_1 + a_{12}c_2 + \ldots + a_{1k}c_k &= b_1 \\
a_{21}c_1 + a_{22}c_2 + \ldots + a_{2k}c_k &= b_2 \\
\vdots \\
a_{n1}c_1 + a_{n2}c_2 + \ldots + a_{nk}c_k &= b_n
\end{align*}
\]
where

\[ \vec{w} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}, \]

This system is called inhomogeneous due to the non-zero right-hand side.

In part to answer questions about linear independence and span, we turn now to the topic of solving systems of linear equations.

- **Gaussian elimination**

  There are three operations you can perform on such systems that lead to a new system with exactly the same solution set:

  1. You can multiply any one equation by any non-zero constant.
  2. You can add any multiple of one equation to another equation.
  3. You can interchange the order of any two of the equations.

  These three operations are called the *elementary row operations*.

  The idea of Gaussian elimination is to use these three strategies to arrive at an equivalent system for which it’s easy to read off the solution. Another simplifying notion is to record the coefficients from either of the two systems above in the form of a matrix:

\[
M = \begin{pmatrix}
  a_{11} & a_{12} & \ldots & a_{1k} \\
  a_{21} & a_{22} & \ldots & a_{2k} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{n1} & a_{n2} & \ldots & a_{nk}
\end{pmatrix}
\]
in the case of the homogeneous system above and

\[
M = \begin{pmatrix}
    a_{11} & a_{12} & \ldots & a_{1k} & | & b_1 \\
    a_{21} & a_{22} & \ldots & a_{2k} & | & b_1 \\
    \vdots & \vdots & \ddots & \vdots & | & \vdots \\
    a_{n1} & a_{n2} & \ldots & a_{nk} & | & b_k
\end{pmatrix}
\]

in the case of the inhomogeneous system. In this case, \( M \) is called an “augmented matrix” since we have augmented the coefficient matrix consisting of all the numbers \( a_{ij} \) with the column consisting of the components of \( \vec{b} \).

In Gaussian elimination, the goal is to use the three legal moves listed earlier to obtain an equivalent system (matrix) that has a triangular form. You can read about solving linear systems and the process of Gaussian elimination in the book by Penney *Linear Algebra Ideas and Applications* available for free online at IUCAT. Take a look at sections 1.2 and 1.3.

Once you’ve used Gaussian elimination to find an equivalent augmented matrix in Row Echelon Form, then it’s easy to read off the solution.

*Row Echelon Form*

A matrix is in row echelon form if

- Any row of zeros appears at the bottom of the matrix
- The leading entry of any row lies in a column that is to the right of any column in which a leading entry of a previous row sits.

(Note: By a “leading entry” of a row, we mean the leftmost non-zero entry of that row.)

It’s even easier to read of the answers if you go a bit further put your answer in

*Row Reduced Echelon Form.*

This means that additionally, you make sure:

- Every leading entry is a 1.
- In every column that holds a leading entry, all the other entries of that column are zero.
Matrix Operations

Given two $m \times n$ matrices $A$ and $B$ (i.e. $m$ rows and $n$ columns) we define the matrix $A + B$ as the $m \times n$ matrix whose $ij^{th}$ entry is the sum of the $ij^{th}$ entry of $A$ and the $ij^{th}$ entry of $B$:

$$(A + B)_{ij} = a_{ij} + b_{ij}.$$ 

Likewise, we define a scalar $c \in \mathbb{R}$ times a matrix $A$ as the matrix $cA$ obtained by multiplying every entry of $A$ by $c$.

- **Matrix multiplying a column vector**

Suppose $A$ is an $m \times n$ matrix whose $n$ columns consist of the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ where each of these column vectors has $m$ components. Now suppose $\vec{x}$ is a column vector in $\mathbb{R}^n$. Then we define the product of $A$ with $\vec{x}$ using the notion of dot product by the formula:

$$A\vec{x} = \begin{pmatrix}
    (\text{Row 1 of } A) \cdot \vec{x} \\
    (\text{Row 2 of } A) \cdot \vec{x} \\
    \vdots \\
    (\text{Row } m \text{ of } A) \cdot \vec{x}
\end{pmatrix}$$

Note that the product of an $m \times n$ matrix with a $n \times 1$ matrix (i.e. a column vector in $\mathbb{R}^n$) yields an $m \times 1$ matrix, that is a column vector in $\mathbb{R}^m$.

Alternatively one finds that this formula is equivalent to defining the product $A\vec{x}$ as the linear combination of the columns of $A$ with coefficients given by the components of $\vec{x}$, that is

$$A\vec{x} = x_1\vec{v}_1 + x_2\vec{v}_2 + \ldots + x_n\vec{v}_n.$$ 

With this definition in hand, we can make three connections between the various concepts we’ve been studying:

If $A$ is an $m \times n$ matrix whose $n$ columns consist of the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ in $\mathbb{R}^m$ and if $\vec{b}$ is a vector in $\mathbb{R}^m$ then
1. Finding all the ways in which $\vec{b}$ can be written as a linear combination of the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ is equivalent to finding all vectors $\vec{x}$ in $\mathbb{R}^n$ that solve the linear system $A\vec{x} = \vec{b}$. In particular, $\vec{b}$ is in the span of the columns of $A$ if and only if there exists a solution $\vec{x}$ to the equation $A\vec{x} = \vec{b}$.

2. The span of the set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is the set of all vectors $A\vec{x}$ where $\vec{x}$ is allowed to range over all vectors in $\mathbb{R}^n$.

3. The set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is linearly independent if and only if the only solution to the linear system $A\vec{x} = \vec{0}$ is the vector $\vec{x} = \vec{0}$.

**Multiplication by a matrix as an example of a linear transformation**

More and more in this class, we’ll be thinking of a matrix not just as a box of numbers but as something that “acts” on a vector, transforming it into another vector. That is, consider the function $\vec{x} \mapsto A\vec{x}$ where $A$ is an $m \times n$ matrix and $\vec{x}$ is a vector in $\mathbb{R}^n$. Then from our definition of $A\vec{x}$ this function (or “map” or “transformation”) eats a vector in $\mathbb{R}^n$ and the output is a vector in $\mathbb{R}^m$. This function is an example of a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$:

A function, say $T: \mathbb{R}^n \to \mathbb{R}^m$ is called a linear transformation if for any two vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^n$ and any two real numbers $c_1$ and $c_2$ one has the linearity property:

$$T(c_1\vec{v} + c_2\vec{w}) = c_1T(\vec{v}) + c_2T(\vec{w}).$$

If you go back to our definition of multiplication of a vector by a matrix, you see that for $T(\vec{x}) := A\vec{x}$, we have this property; that is:

$$A(c_1\vec{v} + c_2\vec{w}) = c_1A\vec{v} + c_2A\vec{w}.$$
A subspace of $\mathbb{R}^n$ is a subset $S$ of vectors in $\mathbb{R}^n$ that is closed under vector addition and multiplication by a (real) scalar. In other words, if $\vec{v}$ and $\vec{w}$ are any two vectors in $S$ and $c_1$ and $c_2$ are any two scalars, then $c_1\vec{v} + c_2\vec{w}$ is also in $S$.

Note that through the choice $c_1 = c_2 = 0$ we see that any subspace of $\mathbb{R}^n$ must contain the zero vector $\vec{0}$.

**Examples of Subspaces:**

Ex 1. Given any set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ in $\mathbb{R}^n$, the span of this set is automatically a subspace—this follows since the span is just the set of all linear combinations so of course it is closed under vector addition and scalar multiplication.

Ex 2. Given any $m \times n$ matrix $A$, the **column space** of $A$ is the set of all vectors in $\mathbb{R}^m$ of the form $A\vec{x}$ where $\vec{x}$ is allowed to be any vector in $\mathbb{R}^n$. In other words, it’s the set of all output, or really the range of the linear transformation $x \mapsto A\vec{x}$.

Going back to our characterization of $A\vec{x}$ as the linear combination of the columns of $A$, we see that the column space is really just the span of all the column vectors of $A$, hence the name.

The column space is another example of a subspace. Let’s see why: Suppose $\vec{b}_1$ and $\vec{b}_2$ are any two vectors in $\mathbb{R}^m$ that are in the column space of $A$. That means there must exist vectors $\vec{v}$ and $\vec{w}$ such that $A\vec{v} = \vec{b}_1$ and $A\vec{w} = \vec{b}_2$. But then for any two scalars $c_1$ and $c_2$ we have

$$A(c_1\vec{v} + c_2\vec{w}) = c_1A\vec{v} + c_2A\vec{w} = c_1\vec{b}_1 + c_2\vec{b}_2$$

so $c_1\vec{b}_1 + c_2\vec{b}_2$ is in the column space as well. Thus, the column space is closed under vector addition and scalar multiplication.

We sometimes write $\text{col}(A)$ to denote the column space of $A$.

Ex 3. Given any $m \times n$ matrix $A$, the **null space** of $A$ is the set of all vectors $\vec{x}$ in $\mathbb{R}^n$ such that $A\vec{x} = \vec{0}$. You can check for yourself that this is also a set of vectors that is closed under vector addition and scalar multiplication.

We sometimes write $\text{nul}(A)$ to denote the null space of $A$. 

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Connections between all the concepts in linear algebra we’ve studied

Here are some crucial relationships/facts:

1. A set of vectors \( \{ \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \} \) is linearly independent if and only if the matrix whose columns are these vectors has null space equal to just \( \{ \vec{0} \} \). This is because multiplying this matrix by the column vector \( \vec{c} \) consisting of entries \( c_1, c_2, \ldots, c_n \) is equivalent to taking a linear combination of the \( v_i \)'s.

2. Given an \( m \times n \) matrix \( A \), a vector \( \vec{b} \in \mathbb{R}^m \) is in the the column space of \( A \) if and only if the system of linear equations \( A\vec{x} = \vec{b} \) is solvable for some \( \vec{x} \in \mathbb{R}^n \). Furthermore, suppose it happens that \( A\vec{x} = \vec{b} \) is solvable for two different vectors \( \vec{v}_1 \) and \( \vec{v}_2 \) in \( \mathbb{R}^n \). Then note that

\[
A\vec{v}_1 - A\vec{v}_2 = \vec{b} - \vec{b} = \vec{0} \implies A(\vec{v}_1 - \vec{v}_2) = \vec{0} \implies \vec{v}_1 - \vec{v}_2 \text{ lies in the null space of } A.
\]

So, since the difference between any two solutions to this inhomogeneous system of equations lies in the null space of \( A \), we conclude that the set of all solutions to \( A\vec{x} = \vec{b} \) is either empty (that is, the system is inconsistent meaning the augmented matrix leads to an impossible to solve system) or else this set consists of any one solution (let’s call it \( \vec{v}_p \)) added to any linear combination of elements of the null space. Of course, it could be the case that \( \text{null}(A) = \{ \vec{0} \} \) in which case \( \vec{v}_p \) is the unique solution.

**Basis and Dimension**

Let’s recall that a subspace of \( \mathbb{R}^n \) is a set of vectors in \( \mathbb{R}^n \) that’s closed under vector addition and multiplication by a scalar. Our next concept is to come up with a notion of the size of a subspace as well as to describe an optimal set of vectors out of which one can build this subspace through linear combinations.

Let’s focus first on the case of \( \mathbb{R}^3 \). In this case, there are four kinds of subspaces:

1) \( S = \{ \vec{0} \} \). This is the trivial subspace having only one element.

2) \( S = \text{span} \{ \vec{u} \} \) where \( \vec{u} \) is any non-zero vector in \( \mathbb{R}^3 \). Then the subspace \( S \) is just the line
through the origin in the direction of $\vec{u}$.

3) $S = \text{span} \{ \vec{u}, \vec{v} \}$ where $\vec{u}$ and $\vec{v}$ are two linearly independent vectors in $\mathbb{R}^3$. Then the subspace $S$ is the plane passing through the origin and containing these two vectors. (It is also a plane having normal vector $\vec{u} \times \vec{v}$.)

4) $S = \text{span} \{ \vec{u}, \vec{v}, \vec{w} \}$ where $\vec{u}$, $\vec{v}$ and $\vec{w}$ are three linearly independent vectors in $\mathbb{R}^3$. Note that if these vectors are linearly independent then they don’t all lie in the same plane. In this case $S = \mathbb{R}^3$, since it turns out we can build any vector in $\mathbb{R}^3$ out of these three vectors.

Clearly, these four subspaces seem to have different ”sizes”. Now we want to make this precise. To define the size of a subspace we will seek the smallest set of vectors out of which we can build all of the other vectors. We arrive at the following notion:

A basis for a subspace $S$ in $\mathbb{R}^n$ is any linearly independent set of vectors, say $\{ \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \}$ that spans the subspace $S$.

For example, if $S$ is the $xy$-plane in $\mathbb{R}^3$ then a basis would be $\mathcal{B} = \{ (1, 0, 0), (0, 1, 0) \}$ since we can build any vector, say $(a, b, 0)$ in the $xy$-plane by taking $a(1, 0, 0) + b(0, 1, 0).$ Also one can quickly check that $\mathcal{B}$ is linearly independent.

Note, however, that another basis for the $xy$-plane would be, for example,

$$\tilde{\mathcal{B}} = \{ (1, 1, 0), (-1, 1, 0) \}$$

since $\tilde{\mathcal{B}}$ is linearly independent (recall that for just two vectors, linear independence is equivalent to the vectors not being multiples of each other) and by row-reducing the augmented matrix

$$
\begin{align*}
\begin{pmatrix}
1 & -1 & | & a \\
1 & 1 & | & b \\
0 & 0 & | & 0
\end{pmatrix}
& \rightarrow
\begin{pmatrix}
1 & 0 & | & \frac{a+b}{2} \\
0 & 1 & | & \frac{b-a}{2} \\
0 & 0 & | & 0
\end{pmatrix}
\end{align*}
$$

we see that $(a, b, 0)$ is in the span of $\tilde{\mathcal{B}}$ as well.

This illustrates the important fact that, except for the case where $S = \{ \vec{0} \}$, a basis is not in general unique.
Here are some other ways to think about a basis:

(1) A basis for a subspace is a maximal set (in terms of number of vectors) of linearly independent vectors that can be formed from vectors in the subspace.

(2) A basis is a minimal set of vectors whose span is the subspace.

Now it turns out—as we will discuss below—that while there are in general infinitely many bases, any two bases have the same number of elements. And that makes the following definition a reasonable one:

The **dimension** of a subspace is the number of elements in any basis for that subspace.

Thus, going back, for example to subspaces of \( \mathbb{R}^3 \), a line through the origin has dimension 1, while a plane through the origin has dimension 2 (and the subspace \( \{\vec{0}\} \) has dimension 0).

For the case where the subspace is all of \( \mathbb{R}^n \) then one convenient choice of basis is the ‘natural basis’ given by

\[
\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots \quad \vec{e}_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.
\]

Of course, the dimension of \( \mathbb{R}^n \) then ends up being \( n \).

Here are a couple of facts you may explore in homework problems:

1) If a subspace \( S \) of \( \mathbb{R}^n \) has a basis with \( k \) elements, then any collection of more than \( k \) vectors in \( S \) must be linearly dependent.

2) If a subspace \( S \) of \( \mathbb{R}^n \) has a basis with \( k \) elements then any collection of less than \( k \) vectors in \( S \) cannot span all of \( S \).

Do you see how it follows from these two facts that any two bases for the same subspace must always have the same number of elements? This makes our definition of dimension meaningful.

Next question: Suppose we’re presented with a subspace of \( \mathbb{R}^n \) as the span of a collection
of $k$ vectors, say

$$S = \text{Span} \{ \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \}$$

and we want to determine a basis for $S$. If this set of vectors is linearly independent then we’re done—it certainly spans so it forms a basis. But perhaps it’s not linearly independent. Below is a procedure for finding a basis for $S$. We will use the fact that under the three operations in the row-reduction process, namely (1) interchanging two rows (2) multiplying a row by a non-zero constant and (3) adding a multiple of one row to another row, the span of the collection of rows (what is called the ‘row space’) doesn’t change.

1) Form the $k \times n$ matrix, let’s call it $A$, having first row $\vec{v}_1$, second row $\vec{v}_2$, etc.

2) Perform the row reduction process to reach the row-reduced echelon form $\text{RREF}(A)$ that has leading ones and zeros above and below all leading ones.

3) The collection of rows of this row-reduced matrix has the same span as the original collection of rows, namely $S$ but because of its structure with 1’s and 0’s, it will be linearly independent and therefore it forms a basis for $S$.

Here’s an example:

Problem: Determine a basis for the span of the set of vectors

$$\{(1, 1, 8, 2), (0, 1, 5, 3) (1, 0, 3, 0) (1, 1, 8, 5)\}$$

We form the matrix having these vectors as rows:

$$A = \begin{pmatrix} 1 & 1 & 8 & 2 \\ 0 & 1 & 5 & 3 \\ 1 & 0 & 3 & 0 \\ 1 & 1 & 8 & 5 \end{pmatrix}$$

After row-reduction, we end up with

$$\text{RREF}(A) = \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
so a basis for \( S \) is
\[
\{(1, 0, 3, 0), (0, 1, 5, 0), (0, 0, 0, 1)\}.
\]
Note that the dimension of this subspace \( S \) is then necessarily 3.

**Linear Transformations Represented by Matrix Multiplication**

Let’s recall that a linear transformation on Euclidean space is a map \( T : \mathbb{R}^n \to \mathbb{R}^m \) such that two things hold:

1. \( T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w}) \)
2. \( T(c\vec{v}) = cT(\vec{v}) \)

for all vectors \( \vec{v} \) and \( \vec{w} \) in \( \mathbb{R}^n \) and all constants \( c \).

Now suppose we start with a basis \( B \) for \( \mathbb{R}^n \). To keep things (relatively) simple, let’s work with the ‘standard basis’ for \( \mathbb{R}^n \) so that \( B = \{ \vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n \} \) where for each \( j \), \( \vec{e}_j \) is the vector with a 1 in the \( j^{th} \) slot and zeros everywhere else. Let’s consider the situation where we know (or we know how to find out) the values obtained when a given linear transformation \( T \) “acts” on each of these basis vectors. That is, suppose we have determined the \( n \) vectors \( T(\vec{e}_j) \) for \( j = 1, 2, \ldots, n \). Let’s call \( \vec{w}_j \) these \( n \) vectors in \( \mathbb{R}^m \); that is \( T(\vec{e}_j) = \vec{w}_j \) for \( j = 1, 2, \ldots, n \). Then using the linearity of \( T \) we can determine the value of \( T(\vec{v}) \) for any vector \( \vec{v} \in \mathbb{R}^n \) as follows:

Write the arbitrary vector \( \vec{v} \) as a linear combination of the basis vectors, which in this case we’re taking to be \( \vec{e}_j \):
\[
\vec{v} = c_1\vec{e}_1 + \ldots + c_n\vec{e}_n.
\]
We call the \( c_j \)'s the ‘coordinates’ of the vector \( \vec{v} \) with respect to the basis \( B \). Then using the two properties (1) and (2) above satisfied by a linear transformation, we find
\[
T(\vec{v}) = T(c_1\vec{e}_1 + \ldots + c_n\vec{e}_n) = c_1T(\vec{e}_1) + \ldots + c_nT(\vec{e}_n) = c_1\vec{w}_1 + \ldots + c_n\vec{w}_n.
\]
Recalling the way we defined the product of a matrix times a column vector we can phrase this formula as: \( T(\vec{v}) = A\vec{v} \) where \( A \) is the \( m \times n \) matrix whose columns are given the \( n \)
vectors $\vec{w}_1, \ldots, \vec{w}_n$:

$$A = \begin{pmatrix}
\uparrow & \uparrow & \uparrow & \uparrow \\
\vec{w}_1 & \vec{w}_2 & \ldots & \vec{w}_n \\
\downarrow & \downarrow & \downarrow & \downarrow
\end{pmatrix}$$

This matrix $A$ is called the matrix representing the linear transformation $T$ with respect to the basis $B$ (which again, we’ve taken here to just be the standard basis for simplicity).

**Examples:**

1) Suppose $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is the linear transformation that rotates every vector in a counter-clockwise direction by $30^\circ$ and we want to find its matrix representation. Using the procedure outlined above, we just need to make two calculations: $T(1, 0)$ and $T(0, 1)$ and then we use these two vectors as column vectors. Using the values of $\cos 30^\circ$ and $\sin 30^\circ$ we find that $T$ maps the vector $(1, 0)$ to the vector $(\sqrt{3}/2, 1/2)$ and it rotates the vector $(0, 1)$ to the vector $(-1/2, \sqrt{3}/2)$. Thus this rotation $T$ is represented as follows:

$$T(\vec{v}) = B\vec{v} \quad \text{where} \quad B = \begin{pmatrix}
\sqrt{3}/2 & -1/2 \\
1/2 & \sqrt{3}/2
\end{pmatrix}.$$  

2) Now suppose $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is the linear transformation that stretches the magnitude of every vector by a factor of 2. Then $T(1, 0) = (2, 0)$ and $T(0, 1) = (0, 2)$ so this stretching transformation $T$ is represented as:

$$T(\vec{v}) = A\vec{v} \quad \text{where} \quad A = \begin{pmatrix}
2 & 0 \\
0 & 2
\end{pmatrix}.$$  

3) Now suppose $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is the linear transformation that rotates every vector in a counter-clockwise direction by $30^\circ$ and then stretches the magnitude of the rotated vector by a factor of 2. Then we find that $T(1, 0) = (\sqrt{3}, 1)$ while $T(0, 1) = (-1, \sqrt{3})$. That means that for any vector $\vec{v} \in \mathbb{R}^2$ we have

$$T(\vec{v}) = C\vec{v} \quad \text{where} \quad C = \begin{pmatrix}
\sqrt{3} & -1 \\
1 & \sqrt{3}
\end{pmatrix}.$$
Here we found the matrix $C$ by following the path of $(1,0)$ and then $(0,1)$ as they are first rotated and then stretched, but is there a relation between the matrices $A$, $B$ and $C$? This leads us to a notion of product that we have not previously encountered in these notes: the product of a matrix with another matrix.

### Matrix Multiplication

For the matrices $A$ and $B$ in the examples above, suppose we form a product $AB$ in the following way: We multiply $A$ times the first column of $B$ and call the result the first column of the product. Then we multiply the matrix $A$ times the second column of $B$ and call that the second column of the product.

Since $A$ times $\begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix} = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}$ and $A$ times $\begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix} = \begin{pmatrix} -1 \\ \sqrt{3} \end{pmatrix}$ we see that the matrix with these columns is precisely $C$, i.e. $AB = C$. Going back to the examples on linear transformations this shows that if one follows one linear transformation by another one then the product of the second matrix representation times the first yields the matrix for the combined transformation.

We can now more generally describe what it means to multiply other matrices using this example as a starting point. Given a matrix $A$ and another matrix $B$ we will define the product $AB$ as the matrix whose first column is the product of $A$ and the first column of $B$, whose second column is the product of $A$ and the second column of $B$, etc.

Here’s another example:

Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ and let $B = \begin{pmatrix} 2 & 0 & -1 & 1 \\ 1 & 2 & -1 & 3 \\ 2 & 0 & 1 & 0 \end{pmatrix}$.

Then

$A$ times first column of $B = \begin{pmatrix} 10 \\ 25 \end{pmatrix}$, $A$ times second column of $B = \begin{pmatrix} 4 \\ 10 \end{pmatrix}$.
A times third column of $B = \begin{pmatrix} 0 \\ -3 \end{pmatrix}$, $A$ times fourth column of $B = \begin{pmatrix} 7 \\ 19 \end{pmatrix}$.

So we find

$$AB = \begin{pmatrix} 10 & 4 & 0 & 7 \\ 25 & 10 & -3 & 19 \end{pmatrix}.$$  

Notice that the product of a $2 \times 3$ matrix and a $3 \times 4$ matrix is a $2 \times 4$ matrix. In general, by this procedure we can define the product of an $m \times n$ matrix $A$ and an $n \times k$ matrix $B$ to obtain an $m \times k$ matrix $AB$. However, this notion requires that the number of columns of the first matrix $A$ be the same as the number of rows of the second matrix $B$. If this is not the case, then the product $AB$ is not well-defined.

One particularly important matrix from the standpoint of matrix multiplication is the identity matrix. For any dimension $n$, the $n \times n$ identity matrix, usually denoted simply by $I$, is the matrix with columns given by $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$ so that it has all 1’s along its diagonal and all 0’s everywhere else. It plays the same role for matrix multiplication that the number 1 does for multiplication of numbers, namely:

$$IA = AI = A \quad \text{for any } n \times n \text{ matrix } A.$$  

You can find many more examples and more discussion of matrix products starting on page 164 of the Linear Algebra book by Penney available online.

### Inverse Matrices

There is no good notion of division by a matrix but the closest thing is multiplication by what is called the inverse of a square matrix. Given an $n \times n$ matrix $A$ we say an $n \times n$ matrix, denoted by $A^{-1}$, is the inverse of $A$ if it holds that

$$AA^{-1} = A^{-1}A = I.$$  

Such a matrix is especially helpful to have if one seeks to solve the basic linear system of $n$ equations in $n$ unknowns:

$$A\vec{x} = \vec{b} \quad \text{for some vector } \vec{b} \in \mathbb{R}^n.$$  

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Multiplying on both sides of this equation by $A^{-1}$ one finds

$$A^{-1}A\vec{x} = A^{-1}\vec{b} \quad \text{so that} \quad \vec{x} = I\vec{x} = A^{-1}\vec{b}.$$  

Of course we already saw that the system $A\vec{x} = \vec{b}$ can be solved via Gaussian elimination by augmenting the matrix $A$ with the column vector $\vec{b}$ but suppose, for instance, that one needed to solve the system $A\vec{x} = \vec{b}$ for lots of different vectors $\vec{b}$ but the same matrix $A$. Then, rather than using Gaussian elimination lots of times one could just multiply each $\vec{b}$ by $A^{-1}$ on the left to get the answers.

So does every matrix have an inverse? And how does one determine the inverse?

The answer to the first question is ‘no’ as can easily be seen, for example, by considering the case where $A$ is the zero matrix (the matrix with every entry 0). Then clearly the product of $A$ with any other matrix is again the zero matrix, and so it is certainly not equal to $I$.

There are in fact infinitely many other matrices that have no inverses and in a subsequent section we will reveal various ways of checking for this, but for now, let’s discuss a method for attempting to find an inverse, and in the process we’ll uncover one criterion for showing an inverse doesn’t exist.

Given a square matrix $A$, suppose we call the columns of the yet to be determined matrix $A^{-1}$, say $\vec{v}_1$, $\vec{v}_2$, ..., $\vec{v}_n$. Then recalling how matrix multiplication works, the desired condition $AA^{-1} = I$ means that we want

$$A\vec{v}_1 = \vec{e}_1, \quad A\vec{v}_2 = \vec{e}_2, \quad \ldots, \quad A\vec{v}_n = \vec{e}_n.$$  

So this suggests we could find each of these unknown column vectors by augmenting $A$ with $\vec{e}_1$, $\vec{e}_2$, etc. and row-reducing using Gaussian elimination. Of course, this requires solving $n$ separate systems. More efficient is simply solving them all at once by augmenting $A$ with the entire identity matrix $I$.

**Example 1**: Find the inverse of the matrix

$$A = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}.$$  

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We form the augmented $2 \times 4$ matrix
\[
\begin{pmatrix}
2 & 1 & 1 & 0 \\
3 & 2 & 0 & 1
\end{pmatrix}.
\]
Now we convert the matrix on the left to its row-reduced echelon form:
\[
\begin{pmatrix}
2 & 1 & 1 & 0 \\
3 & 2 & 0 & 1
\end{pmatrix} \rightarrow
\begin{pmatrix}
1 & 1/2 & 1/2 & 0 \\
3 & 2 & 0 & 1
\end{pmatrix} \rightarrow
\begin{pmatrix}
1 & 1/2 & 1/2 & 0 \\
0 & 1/2 & -3/2 & 1
\end{pmatrix} \rightarrow
\begin{pmatrix}
1 & 1/2 & 1/2 & 0 \\
0 & 1 & -3 & 2
\end{pmatrix} \rightarrow
\begin{pmatrix}
1 & 0 & 2 & -1 \\
0 & 1 & -3 & 2
\end{pmatrix}.
\]
So the first column vector of the inverse should be $(2, -3)^T$ and the second column vector should be $(-1, 2)^T$. (Here the notation $T$ stands for the transpose, which in the case of vectors simply converts a row vector into the corresponding column vector and vice versa.) One easily checks that indeed $AA^{-1} = A^{-1}A = I$ for the matrix
\[
A^{-1} = \begin{pmatrix}
2 & -1 \\
-3 & 2
\end{pmatrix}.
\]

**Example 2:** Find the inverse of the matrix
\[
B = \begin{pmatrix}
2 & -2 \\
1 & -1
\end{pmatrix}.
\]
We form the augmented $2 \times 4$ matrix
\[
\begin{pmatrix}
2 & -2 & 1 & 0 \\
1 & -1 & 0 & 1
\end{pmatrix}.
\]
However, now when we row-reduce we find
\[
\begin{pmatrix}
2 & -2 & 1 & 0 \\
1 & -1 & 0 & 1
\end{pmatrix} \rightarrow
\begin{pmatrix}
1 & -1 & 1/2 & 0 \\
1 & -1 & 0 & 1
\end{pmatrix} \rightarrow
\begin{pmatrix}
1 & -1 & 1/2 & 0 \\
0 & 0 & -1/2 & 1
\end{pmatrix}.
\]
The second row of zeros on the left in the last augmented matrix tells us the system is inconsistent. Apparently, there is no inverse, for if there were an inverse, its columns would
have to solve the associated systems. When a matrix has an inverse, we say it is *invertible* or *nonsingular*. When it does not have an inverse, we say it is not invertible or *singular*. In subsequent sections we will find a whole list of criteria that can be used to determine whether or not a matrix is invertible but from this section we have one criterion:

An \( n \times n \) matrix \( A \) is invertible if and only if after augmenting \( A \) with the identity matrix \( I \) and row-reducing, the row-reduced echelon form of the matrix on the left is itself the identity matrix \( I \).

Note that for a square matrix \( A \), there are only two possibilities: either \( RREF(A) = I \) or else the \( RREF(A) \) has at least one row of zeros and in the latter case, the augmented system will be inconsistent.

Relating all of this back to the original motivation of solving linear systems, we can also draw the following conclusion from the discussion in this section:

An \( n \times n \) matrix \( A \) is invertible if and only if the system \( A\vec{x} = \vec{b} \) is uniquely solvable for every vector \( \vec{b} \in \mathbb{R}^n \).

Note that if \( A \) is invertible, then multiplication by \( A^{-1} \) shows the solution must be \( \vec{x} = A^{-1}\vec{b} \), hence the claim of uniqueness.

**Determinants**

For a \( 2 \times 2 \) matrix \( A \) given by

\[
A = \begin{pmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{pmatrix},
\]

let’s recall that the determinant of \( A \) is the number given by the formula

\[
\det A = a_{11}a_{22} - a_{12}a_{21}.
\]

For a \( 3 \times 3 \) matrix

\[
A = \begin{pmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{pmatrix},
\]
the definition is a bit more complicated but it relies on the $2 \times 2$ definition as follows:

$$\det A = a_{11} \det \begin{pmatrix} a_{22} & a_{33} \\ a_{32} & a_{33} \end{pmatrix} - a_{12} \det \begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} + a_{13} \det \begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}.$$  

For an $n \times n$ matrix there is a similar definition in terms of the determinant of an $(n-1) \times (n-1)$ matrix.

Now this expression has some remarkable properties. To begin with, we list the effect on the determinant of performing any of our three elementary row operations.

Given an $n \times n$ matrix $A$ suppose we alter it to form a new matrix $B$ by:

1. interchanging any two rows of $A$. Then $\det B = -\det A$.
2. multiplying any row of $A$ by a non-zero constant $c$, Then $\det B = c \det A$.
3. adding any multiple of one row of $A$ to another row of $A$. Then $\det B = \det A$.

For a $2 \times 2$ or $3 \times 3$ matrix these are easy to checking directly. For instance, let’s check property (3) for a $2 \times 2$ where we add $c$ times the first row to the second row. Then the new matrix $B$ is given by

$$B = \begin{pmatrix} a_{11} & a_{12} \\ c a_{11} + a_{21} & c a_{12} + a_{22} \end{pmatrix},$$

and the determinant of $B$ is given by

$$\det B = a_{11}(c a_{12} + a_{22}) - a_{12}(c a_{11} + a_{21}) = ca_{11}a_{12} + a_{11}a_{22} - ca_{12}a_{11} - a_{12}a_{21} = a_{11}a_{22} - a_{12}a_{21} = \det A.$$  

The three facts listed above about determinants have a lot of consequences. For example, suppose an $n \times n$ matrix $A$ has a row of zeros and suppose we create a new matrix $B$ by multiplying this zero row by 2. Then according to rule (2) above, we would find $\det B = 2 \det A$. But clearly multiplying a zero row by 2 doesn’t change the matrix, so $A = B$ and necessarily $\det A = \det B$. The only way both of these equations can hold is if $\det A = 0$. Thus, we have the following fact:

*Any matrix with a row of zeros has zero determinant.*

Another use of these three facts is that it can simplify the arithmetic involving in computing a determinant. The idea is that given a matrix, if we perform row operation (3) in order
to create some zeros in the new matrix then the calculation will be much easier. Here’s an example:

**Example:**

Let

\[
A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & -1 & 3 \\ -1 & 0 & 4 \end{pmatrix}.
\]

Then let’s add twice the second row to the first row, followed by five times the third row added to the first row:

\[
\begin{pmatrix} 1 & 2 & -1 \\ 2 & -1 & 3 \\ -1 & 0 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 5 & 0 & 5 \\ 2 & -1 & 3 \\ -1 & 0 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 & 25 \\ 2 & -1 & 3 \\ -1 & 0 & 4 \end{pmatrix}.
\]

These three matrices all have the same determinant by property (3) but for the last one, due to the zeros in the first row, we find simply that

\[
det A = 25 \det \begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix} = -25.
\]

A few other very useful properties of the determinant which we will not go into further are the following:

(i) If \( A^T \) denotes the transpose of an \( n \times n \) matrix \( A \) (that is, the matrix whose \( ij^{th} \) entry is \( a_{ji} \)), then

\[
det A^T = det A.
\]

(ii) If \( A \) and \( B \) are any two \( n \times n \) matrices, then

\[
det (AB) = det A \det B.
\]

Note that from (ii), it follows that

\[
1 = \det I = \det (AA^{-1}) = \det A \det A^{-1} \quad \text{so} \quad \det A^{-1} = \frac{1}{\det A}.
\]
Determinants and Invertibility

Given an $n \times n$ matrix, we saw in the last section that the three elementary row operations either leave the determinant unchanged, multiply the determinant by $-1$ or multiply the determinant by some non-zero factor. In light of this, we can draw the following conclusion:

Either $\det A$ and $\det \text{RREF}(A)$ are both non-zero or both zero.

On the other hand, we already learned that for a square matrix either $\text{RREF}(A) = I$ or else $\text{RREF}(A)$ has at least one row of zeros. In the first case, the determinant is 1 while in the second the determinant is zero. Putting this information together, we see that:

A matrix is invertible if and only if its determinant is non-zero.

Let’s look for some more equivalences along these lines. For example, the row space of $\text{RREF}(A)$ is the same as the row space of $A$, so if $\text{RREF}(A) = I$ then the span of the $n$ row vectors of $A$ is all of $\mathbb{R}^n$ (since the span of the rows of $I$, namely $\vec{e}_1, \ldots, \vec{e}_n$, is certainly $\mathbb{R}^n$). This means the $n$ rows of $A$ are linearly independent. On the other hand, if $\text{RREF}(A)$ has a row of zeros then these $n$ vectors do not span all of $\mathbb{R}^n$ and must be linearly dependent.

We’ve now established the following list of equivalences:

For any $n \times n$ matrix $A$, the following conditions are equivalent; that is, either they all hold or none of them holds:
1) $A$ is invertible
2) $\text{RREF}(A) = I$
3) For every vector $\vec{b} \in \mathbb{R}^n$ there is a unique solution to the equation $A\vec{x} = \vec{b}$.
4) The rows of $A$ are linearly independent.
5) The columns of $A$ are linearly independent.
6) $\det A \neq 0$.  

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Item 5 is on the list because we know \( \det A^T = \det A \) and \( A^T \) is simply the matrix with the rows and columns of \( A \) interchanged. So \( \det A \neq 0 \rightarrow \det A^T \neq 0 \rightarrow \) item 5.

There is one more item to be added to this list. Recall that the null space of an \( n \times n \) matrix \( A \) is set of all vectors \( \vec{x} \in \mathbb{R}^n \) such that \( A\vec{x} = \vec{0} \). To find all the vectors in the null space, we simply augment \( A \) with a column of zeros and row reduce. Remember that \( \vec{0} \) is always in the null space since \( A\vec{0} = \vec{0} \), but for any other vectors to solve this equation, one would have to have \( RREF(A) \neq I \).

Seeing it another way, by item 3 above, if the items in the list above all hold, then \( \{\vec{0}\} \) must be the unique element of the null space. On the other hand, if the null space includes any other, necessarily non-zero vector, say \( \vec{x} \), in addition to \( \{\vec{0}\} \) then from the definition of matrix times vector product, we would have

\[
\vec{0} = A\vec{x} = x_1 \vec{v}_1 + \ldots + x_n \vec{v}_n
\]

where \( \vec{v}_1, \ldots, \vec{v}_n \) are the columns of \( A \) and \( \vec{x} = (x_1, \ldots, x_n)^T \). This would mean the columns are linearly dependent, negating item 5 above.

In particular, this means we can determine whether or not the null space of a matrix \( A \) is trivial simply by checking whether or not the determinant of \( A \) is non-zero.

Example:

Let \( A := \begin{pmatrix} 2 & 6 \\ 1 & 3 \end{pmatrix} \). Then \( \det A = (2)(3) - (6)(1) = 0 \).

Thus, \( A \) must have a nontrivial null space. Let's find a basis for it by solving \( A\vec{x} = \vec{0} \) via row reduction:

\[
\begin{pmatrix} 2 & 6 & | & 0 \\ 1 & 3 & | & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 3 & | & 0 \\ 2 & 6 & | & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 3 & | & 0 \\ 0 & 0 & | & 0 \end{pmatrix}
\]

So \( \vec{x} = (x_1, x_2)^T \) satisfies \( x_1 + 3x_2 = 0 \) which means \( \vec{x} \) is any multiple of \( (-3, 1)^T \). In other words a basis for the null space of \( A \) is \( \{(-3, 1)\} \).
**Eigenvalues and Eigenvectors**

One of the most crucial notions attached to square matrices is that of eigenvalues and eigenvectors. Let’s start with their definition:

Given an $n \times n$ matrix $A$, we say that a (perhaps complex) number $\lambda$ is an **eigenvalue** of the matrix $A$ if there exists a non-zero vector $\vec{v} \in \mathbb{R}^n$ such that

$$A\vec{v} = \lambda \vec{v}.$$  

Such a vector $\vec{v}$ is called an **eigenvector** corresponding to the eigenvalue $\lambda$ and together, the number $\lambda$ and corresponding eigenvector $\vec{v}$ are called an **eigenpair** for the matrix $A$.

For example, if

$$A = \begin{pmatrix} 5 & 2 & 1 \\ -2 & 1 & -1 \\ 2 & 2 & 4 \end{pmatrix}$$

and $\vec{v} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$,

then we can check that $A\vec{v} = 4\vec{v}$ so 4 is an eigenvalue of $A$ with corresponding eigenvector $\vec{v}$.

Before we get into how we find eigenpairs, let’s look at one of the many uses of eigenvalues and eigenvectors.

**Solving Systems of Linear ODE’s using Eigenvalues and Eigenvectors**

Suppose $\lambda$ and $\vec{v}$ are an eigenpair for a matrix $A$ and suppose we are trying to solve the following system of linear ODE’s:

$$\begin{pmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_n \end{pmatrix} = A \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

where $A$ is a given $n \times n$ matrix and $x_1(t), \ldots, x_n(t)$ are $n$ unknown functions. Then it turns out that the vector-valued function $\vec{x}(t) = e^{\lambda t} \vec{v}$ is one solution to this system. This is because:

$$\vec{x}'(t) = (e^{\lambda t} \vec{v})' = \lambda e^{\lambda t} \vec{v} \quad \text{while} \quad A(e^{\lambda t} \vec{v}) = e^{\lambda t} A\vec{v} = e^{\lambda t} \lambda \vec{v}$$
so indeed $\vec{x}'(t) = A\vec{x}(t)$.

Taking this further, if we can find many eigenpairs, say even $n$ of them, $(\lambda_1, \vec{v}_1), (\lambda_2, \vec{v}_2), \ldots, (\lambda_n, \vec{v}_n)$ then since this ODE is linear and homogeneous, we can invoke the superposition principle to assert that any linear combination

$$\vec{x}(t) = c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2 + \ldots + c_n e^{\lambda_n t} \vec{v}_n$$

is also a solution and that way we can find what is called the general solution of the system.

How to find eigenvalues and eigenvectors

One starts with first determining the eigenvalues. Here’s how: We first rewrite the equation $A\vec{v} = \lambda \vec{v}$ in the equivalent form

$$(A - \lambda I)\vec{v} = \vec{0},$$

where we are using that the identity matrix $I$ has the property $I\vec{v} = \vec{v}$ for any vector $\vec{v} \in \mathbb{R}^n$. Viewing the problem like this, we see that we are looking for a value or values of $\lambda$ such that the matrix $A - \lambda I$ has a vector in its null space other than $\vec{0}$. Now in the previous section we saw that a square matrix has a nontrivial null vector if and only if its determinant is zero. This leads us to the requirement

$$\det (A - \lambda I) = 0.$$ 

The left-hand side of this equation is actually a polynomial of degree $n$ in the variable $\lambda$ and it is called the characteristic polynomial associated with $A$.

As we know, a polynomial of degree $n$ always has at least one distinct root and can have at most $n$ roots. Thus, it can happen that a matrix $A$ has only one distinct eigenvalue or as many as $n$ distinct eigenvalues.

Anyway, once we find a value of $\lambda$ that is a root of the characteristic polynomial we plug that value of $\lambda$ into the matrix $A - \lambda I$ and then we find all of the null vectors for that matrix by row-reducing. These will be the eigenvectors. It could be that there is only one linearly independent eigenvector but it could also be that the dimension of the null space of this matrix is more than one.
Since we saw earlier than the null space of a matrix is a subspace, it follows that the set of all eigenvectors corresponding to a given eigenvalue is also a subspace– that is, any linear combination of such eigenvectors is again an eigenvector.

**Example:** Find all eigenvalues and eigenvectors of the matrix

\[
A = \begin{pmatrix}
-4 & -3 \\
3 & 6
\end{pmatrix}.
\]

We start by setting the characteristic polynomial equation to zero:

\[
\det(A - \lambda I) = \det \begin{pmatrix}
-4 - \lambda & -3 \\
3 & 6 - \lambda
\end{pmatrix} = (-4 - \lambda)(6 - \lambda) + 9 = 0.
\]

This last equation simplifies to \(\lambda^2 - 2\lambda - 15 = 0\) which factors as

\[(\lambda - 5)(\lambda + 3) = 0\]

and so we find that \(\lambda_1 = 5\) and \(\lambda_2 = -3\) are the two eigenvalues of \(A\). To find all eigenvectors corresponding to \(\lambda_1\) we solve the system \((A - 5 I)\vec{v}_1 = 0\). Writing \(\vec{v}_1 = (x_1, x_2)^T\) we must solve

\[
\begin{pmatrix}
-9 & -3 \\
3 & 1
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix} = \begin{pmatrix}
0 \\
0
\end{pmatrix}.
\]

We find that \(3x_1 + x_2 = 0\) so any multiple of the vector \(\vec{v}_1 = (1, -3)^T\) is an eigenvector.

A similar approach, working with the matrix

\[
A - \lambda_2 I = A + 3 I = \begin{pmatrix}
-1 & -3 \\
3 & 9
\end{pmatrix}
\]

and solving \((A + 3 I)\vec{v}_2 = 0\) allows us to find that any multiple of \(\vec{v}_2 = (-3, 1)^T\) is an eigenvector corresponding to \(\lambda_2\).

Even if a matrix has only real entries, it doesn’t necessarily mean that the eigenvalues and eigenvectors will be real. This follows since we know that the roots of polynomials with real coefficients can certainly be complex. However, what we *can* say, is that if the entries
of the matrix are all real, then any complex eigenvalues come in conjugate pairs since that fact is true about the roots of real polynomials. The same is true about the corresponding eigenvectors. Here is an example:

**Example:** Find all eigenvalues and eigenvectors of

$$B = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}.$$  

As in the previous example, we start by setting the characteristic polynomial equation to zero:

$$\det \left( B - \lambda I \right) = \det \begin{pmatrix} 1 - \lambda & -1 \\ 1 & 2 - \lambda \end{pmatrix} = (1 - \lambda)(2 - \lambda) + 1 = 0.$$  

Simplifying we must solve $\lambda^2 - 3\lambda + 3 = 0$ and using the quadratic formula we obtain

$$\lambda = \frac{3}{2} \pm \frac{\sqrt{3}}{2} i.$$  

Working with the plus sign, if we call $\lambda_1 = \frac{3}{2} + \frac{\sqrt{3}}{2} i$, then we can find any corresponding eigenvectors $\vec{v}_1$ by solving

$$(B - \lambda_1 I)\vec{v}_1 = \begin{pmatrix} -1 - \frac{\sqrt{3}}{2} i & -1 \\ 1 & 1 + \frac{\sqrt{3}}{2} i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$  

Solving, we find that the corresponding eigenvector $\vec{v}_1$ is given by

$$\vec{v}_1 = \begin{pmatrix} 1 \\ \frac{1}{2} + \frac{\sqrt{3}}{2} i \end{pmatrix}.$$  

For the second eigenvalue $\lambda_2 = \frac{3}{2} - \frac{\sqrt{3}}{2} i$ which is the conjugate of $\lambda_1$ one can check that the corresponding eigenvector $\vec{v}_2$ is the conjugate of the vector $\vec{v}_1$, that is

$$\vec{v}_2 = \begin{pmatrix} 1 \\ \frac{1}{2} - \frac{\sqrt{3}}{2} i \end{pmatrix}.$$  

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2. Introduction to Multi-Variable Calculus

For a function, say $f$, that depends on two variables whose output is a scalar, we write $f : \mathbb{R}^2 \to \mathbb{R}$. For a function depending on $n$ variables, we write $f : \mathbb{R}^n \to \mathbb{R}$. The calculus of such functions follows a similar development to one variable calculus. Here we begin with the notion of derivative:

Partial Derivatives

For a function $f = f(x, y)$ depending on two variables, we can think about measuring the rate of change of the function with respect to a shift in either $x$ or $y$. This rate of change is called a partial derivative. It is defined by “freezing” one variable and looking at the limit of an increment in the other one.

**Definition.** The partial derivative of a function $f : \mathbb{R}^2 \to \mathbb{R}$ with respect to $x$, evaluated at a point $(x, y)$ is defined by

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

provided this limit exists. For notation we write either $\frac{\partial f}{\partial x}$ or simply $f_x$.

In a similar way, we define the partial derivative with respect to $y$ via:

$$\lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

and write either $\frac{\partial f}{\partial y}$ or simply $f_y$.

**Computing partial derivatives:** To compute $\frac{\partial f}{\partial x}$ one just holds $y$ fixed—that is, treat $y$ as you would a constant—and use the usual rules (sum rule, product rule, quotient rule, chain rule) to take the derivative with respect to $x$ the same way you did in one variable calculus. Likewise, to compute $\frac{\partial f}{\partial y}$ you treat $x$ as a constant.

**Example:**

For $f(x, y) = x^2y^3 + e^{3x} + \ln (1 + y)$ we find

$$\frac{\partial f}{\partial x} = 2xy^3 + 3e^{3x} \quad \text{and} \quad \frac{\partial f}{\partial y} = 3x^2y^2 + \frac{1}{1 + y}.$$
In one variable calculus, we learn that geometrically, the value of the derivative yields the slope of the tangent line to the graph of a function. Of course, if a function is not differentiable, such as \( f(x) = |x| \) at \( x = 0 \) then there is no tangent line that just “grazes” the graph.

What is the corresponding geometric interpretation for a function \( f(x,y) \)? The graph of such a function is a \textit{surface} that one can draw in three dimensions using an \( x \), \( y \) and \( z \) axis, where the \( z \)-axis corresponds to the value of \( f(x,y) \). Then we would like to replace the notion of a tangent \textit{line} by the notion of a tangent \textit{plane}.

The equation of this plane at a point, say \((a,b)\) is also called the linear approximation to the function \( f \). Recall that the equation of the tangent line for \( f : \mathbb{R} \to \mathbb{R} \) at a point \( x = a \) is given by the formula \( P_1(x) := f(a) + f'(a)(x-a) \). It has the property that it agrees with \( f \) at \( a \), i.e. \( f(a) = P_1(a) \) and its first derivative does so as well, i.e. \( f'(a) = P'_1(a) \). We will ask the same thing of the linear approximation to a function of two variables, \( f(x,y) \), at a point \((a,b) \in \mathbb{R}^2 \). That is, if we call it \( P_1(x,y) \) then we ask that \( P_1 \) be a linear function of \( x \) and \( y \) such that

\[
P_1(a,b) = f(a,b) \quad \text{and also that} \quad \frac{\partial P_1}{\partial x}(a,b) = \frac{\partial f}{\partial x}(a,b) \quad \text{and} \quad \frac{\partial P_1}{\partial y}(a,b) = \frac{\partial f}{\partial y}(a,b).
\]

It’s easy to check that this forces \( P_1 \) to be given by the formula

\[
P_1(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b). \quad (TP)
\]

Rewriting this as \( z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \) or, even better as

\[
z - f_x(a,b)x - f_y(a,b)y = d \quad \text{where} \quad d := f(a,b) + af_x(a,b) + bf_y(a,b)
\]

we recognize from our earlier work on planes this semester that the linear approximation to \( f \) does indeed have a graph that is a plane. Recall that we can also readily identify a normal vector to this plane through this formula as \( \vec{n} = (-f_x(a,b), -f_y(a,b), 1) \).

**Example:** Find the equation of the tangent plane (i.e. the linear approximation to) the graph of \( f(x,y) = \frac{y-1}{x+1} \) at the origin.

We first use the quotient rule to compute that

\[
f_x(x,y) = -\frac{(y-1)}{(x+1)^2} \quad \text{and} \quad f_y(x,y) = \frac{1}{x+1}
\]
so \( f_x(0, 0) = 1 \) and \( f_y(0, 0) = 1 \). Since \( f(0, 0) = -1 \) we find using the formula \((TP)\) above that

\[
P_1(x, y) = -1 + x + y \quad \text{or} \quad z - x - y = -1.
\]

Recalling the example mentioned earlier about \(|x|\), we can ask here whether or not in general a tangent plane really does just “graze” the graph of \( f \), thus giving a good approximation to \( f \) if one stays near enough to the point \((a, b)\)? It turns out that existence of the partial derivatives \( f_x(a, b) \) and \( f_y(a, b) \) alone is not enough to guarantee this. However, the linear approximation \textit{does} give a good approximation if, for instance, one knows additionally that these two partial derivatives are \textit{continuous} at \((a, b)\). Continuity, say of \( f_x \), at \((a, b)\) means that

\[
\lim_{(x,y) \to (a,b)} f_x(x,y) = f_x(a,b).
\]

where the symbol \((x,y) \to (a,b)\) means that we must check this condition for points \((x,y)\) that approach \((a,b)\) along any possible path.

Anyway, as long as we are careful not to avoid “trouble” such as division by zero, taking a log of zero, etc. then most of the standard functions we will work with have this property.

When \( f \) is well approximated by it tangent plane near \((a, b)\), which we can state precisely as the condition that

\[
f(x, y) - P_1(x, y) = \varepsilon_1(x, y)(x - a) + \varepsilon_2(x, y)(y - b)
\]

where \(\varepsilon_1(x, y) \to 0\) and \(\varepsilon_2(x, y) \to 0\) as \((x, y) \to (a, b)\), then we say that \( f \) is \textit{differentiable} at \((a, b)\). Note, in particular, that being differentiable is a stronger condition that just having partials \( f_x \) and \( f_y \) that exist. On the other hand, being differentiable is implied by the condition that \( f_x \) and \( f_y \) exist and are continuous.

\textbf{The Gradient Vector and Directional Derivatives}

Let’s assume now that we’re dealing with a function \( f : \mathbb{R}^2 \to \mathbb{R} \) that is differentiable. So, in particular, the partial derivatives \( f_x(x, y) \) and \( f_y(x, y) \) exist. We now define the \textbf{gradient}
vector as the vector of partial derivatives of $f$. The notation for the gradient vector is $\nabla f(x, y)$ so we have

$$\nabla f(x, y) = (f_x(x, y), f_y(x, y)).$$

**Example:** Compute $\nabla f$ for $f(x, y) = e^{2x+4y^2}$.

We find

$$\nabla f(x, y) = (2e^{2x+4y^2}, 8ye^{2x+4y^2}) = e^{2x+4y^2}(2, 8y).$$

Before going on, we mention that everything discussed here for a function of two variables has an analog for a function of more than two variables. For example, the gradient of a function $g(x, y, z)$ is the vector $\nabla g = (g_x, g_y, g_z)$ where the components of this vector are now the three partial derivatives of $g$.

Next we consider the question of how we might compute the rate of change of a function $f(x, y)$ in a direction other than the coordinate directions $x$ and $y$. That is, suppose we pick a direction $\vec{v}$ other than $(1, 0)$ or $(0, 1)$ and ask for the rate of change of $f$ if we make a small shift in the direction $\vec{v}$. This is called the directional derivative of $f$ in the direction of $\vec{v}$ at a point $(a, b)$. It is defined in a way that is similar to partial derivatives:

$$D_{\vec{v}} f(a, b) := \lim_{h \to 0} \frac{f((a, b) + h\vec{v}) - f(a, b)}{h}.$$ 

Sometimes this is also denoted as $\frac{\partial f}{\partial \vec{v}}$.

As with partial derivatives, there is a simple way to compute directional derivatives. It is based on the linear approximation $P_1$ given by formula $(TP)$ a few pages back. If we assume $f$ is differentiable at a point $(x, y)$ (not necessary but let’s do it anyway), then in the limit above, writing $\vec{v}$ as $\vec{v} = (v_1, v_2)$, we find

$$f((a, b) + h\vec{v}) - f(a, b) \approx P_1(a + hv_1, b + hv_2) - f(a, b)$$

$$= f(a, b) + f_x(a, b)(a + v_1h - a) + f_y(a, b)(b + hv_2 - b) - f(a, b)$$

$$= f_x(a,b)hv_1 + f_y(a,b)hv_2 = h\nabla f(a, b) \cdot (a, b)$$

where the last expression is a dot product of two vectors. Thus, we find that

$$D_{\vec{v}} f(a, b) = \lim_{h \to 0} \frac{h\nabla f(a, b) \cdot (a, b)}{h} = \nabla f(a, b) \cdot \vec{v}.$$
In other words, to compute the directional derivative of a function \( f \) in a direction \( \vec{v} \), we just take the dot product of the gradient vector with the direction \( \vec{v} \).

**Example.** Compute the directional derivative of the function \( f(x, y) = \cos(\pi x + y) + 3y \) at the point \((2, 0)\) in the direction \( \vec{v} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) \).

We first compute

\[
\nabla f(x, y) = (-\pi \sin(\pi x + y), -\sin(\pi x + y) + 3),
\]

so \( \nabla f(2, 0) = (-\pi \sin(2\pi), -\sin(2\pi) + 3) = (0, 3) \). That means that

\[
D_{\vec{v}}f(2, 0) = (0, 3) \cdot (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = 0 + \frac{3}{\sqrt{2}} = \frac{3}{\sqrt{2}}.
\]

Next let’s pursue the following question: Given a function \( f : \mathbb{R}^2 \to \mathbb{R} \) and a point \((x, y)\), in which direction \( \vec{v} \) is the directional derivative of \( f \) at \((x, y)\) largest? In which direction is it smallest? Another way to phrase this is to ask: What is the direction of steepest ascent if one were to “climb” the graph of \( f \) starting at \((x, y)\) and similarly, what is the direction of steepest descent?

Using our formula for evaluating the directional derivative from the previous page, we can rephrase this question as: For which unit vector \( \vec{v} \) is \( \nabla f(x, y) \cdot \vec{v} \) largest or smallest? To answer this question, let’s use our formula from earlier this semester for the angle between two vectors in terms of the dot product:

\[
D_{\vec{v}}f(x, y) = \nabla f(x, y) \cdot \vec{v} = |\nabla f(x, y)| |\vec{v}| \cos \theta = |\nabla f(x, y)| \cos \theta,
\]

where \( \theta \) is the angle between the gradient vector and any particular direction \( \vec{v} \). From this formula, it’s easy to see that picking \( \vec{v} \) such that \( \theta = 0 \) gives the biggest value since then \( \cos \theta = 1 \) and picking \( \theta = \pi \) yields the smallest value since then \( \cos \theta = -1 \). But \( \theta = 0 \) means the direction coincides with the direction of the gradient and \( \theta = \pi \) means the direction is opposite that of the gradient. The conclusion is:

*The gradient always points in the direction of maximal ascent of a function and points opposite to the direction of maximal descent.*
Higher order partial derivatives. For a function \( f = f(x,y) \) there are four second order partials, \( f_{xx}, f_{yy}, f_{xy} \) and \( f_{yx} \). Alternatively, these four derivatives can be written as

\[
\frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial y^2}, \quad \frac{\partial^2 f}{\partial y \partial x} \quad \text{and} \quad \frac{\partial^2 f}{\partial x \partial y}.
\]

These second order derivatives are defined through limits in a manner analogous to the definitions of first order partials. For instance,

\[ f_{xy} = (f_x)_y := \lim_{\Delta y \to 0} \frac{f_x(x, y + \Delta y) - f_x(x, y)}{\Delta y} \]

and

\[ f_{yx} = (f_y)_x := \lim_{\Delta x \to 0} \frac{f_y(x + \Delta x, y) - f_y(x, y)}{\Delta x}. \]

To actually compute these derivatives, however, we just use the usual derivative rules as we did for first order partials, remembering, for example, that when computing \( f_{xy} \) we first calculate \( f_x \), treating \( y \) as a constant and then compute \((f_x)_y\), treating \( x \) as a constant.

**Example.** For the function \( f(x, y) = x^3 e^{2y} \), let’s compute all derivatives up to second order.

\[
f_x = 3x^2 e^{2y}, \quad f_y = 2x^3 e^{2y}, \quad f_{xx} = 6xe^{2y}, \quad f_{yy} = 4x^3 e^{2y}
\]

and

\[
f_{xy} = (3x^2 e^{2y})_y = 6x^2 e^{2y} \quad \text{and} \quad f_{yx} = (2x^3 e^{2y})_x = 6x^2 e^{2y}.
\]

Notice that in this example \( f_{xy} = f_{yx} \). This is not a coincidence. In fact, provided \( f \) and its partials up to second order are continuous, this will always happen. We say that for such a function with continuous derivatives up to second order, the mixed second partials commute.

We give below a rough idea as to why this happens, based on the limit definition of first and second partials:

We start with \( f_{xy} \). For \( \Delta y \) very small, we know

\[
f_{xy}(x, y) \approx \frac{f_x(x, y + \Delta y) - f_x(x, y)}{\Delta y}
\]
and then going back to the limit definition of $f_x$ we further argue that for $\Delta x$ also very small we have

$$f_{xy} \approx \frac{1}{\Delta y} \left\{ \frac{f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y)}{\Delta x} - \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} \right\}$$

$$= \frac{f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y) - f(x + \Delta x, y) + f(x, y)}{\Delta y \Delta x}.$$

On the other hand, for $\Delta x$ very small we know

$$f_{yx}(x, y) \approx \frac{f_y(x + \Delta x, y) - f_y(x, y)}{\Delta x}$$

and then going back to the limit definition of $f_y$ we further argue that for $\Delta y$ also very small we have

$$f_{yx} \approx \frac{1}{\Delta x} \left\{ \frac{f(x + \Delta x, y + \Delta y) - f(x + \Delta x, y)}{\Delta y} - \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} \right\}$$

$$= \frac{f(x + \Delta x, y + \Delta y) - f(x + \Delta x, y) - f(x, y + \Delta y) + f(x, y)}{\Delta x \Delta y}.$$

Comparing the approximate expressions for $f_{xy}$ and $f_{yx}$ respectively, we see that the two are equivalent.

Partial Differential Equations

A partial differential equation is an equation satisfied by a function of more than one variable and any of its partial derivatives. Below we give a few important examples:

1. **Laplace’s Equation.** Laplace’s equation for a function of two variables, say $u(x, y)$, is given by

$$u_{xx} + u_{yy} = 0.$$

More generally, for a function of $n$ variables, it is given by

$$u_{x_1 x_1} + u_{x_2 x_2} + \ldots + u_{x_n x_n} = 0.$$

This equation arises in many physical settings and the left-hand side of this equation involving the sum of all the pure second partials, is called the Laplacian and is denoted by either $\Delta u$
or $\nabla^2 u$. The equation describes, for example, the equilibrium temperature distribution of a uniform body, such as a chunk of metal or a body of water. It also describes the electrostatic potential (whose gradient is the electric field) in the absence of any charges. And there are \textit{many} more settings where it arises.

**Example.** Verify that the function $u(x,y) = \sin (2x) \sinh (2y)$ satisfies Laplace’s equation.

We first recall from earlier in the semester that for any $t$, $(\sinh t)' = \cosh t$ and $(\cosh t)' = \sinh t$. Now we calculate

\[
u_x = 2 \cos (2x) \sinh (2y) \quad \text{so} \quad u_{xx} = -4 \sin (2x) \sinh (2y)
\]

and

\[
u_y = 2 \sin (2x) \cosh (2y) \quad \text{so} \quad u_{yy} = 4 \sin (2x) \sinh (2y).
\]

Thus, $u_{xx} + u_{yy} = 0$. When a function satisfies Laplace’s equation, we say that that function is \textit{harmonic}. The function $u$ in this example is one of infinitely many harmonic functions.

2. **Heat Equation.** The heat equation, also known as the diffusion equation, for a function $u = u(x,t)$ is given by

\[
u_t = \kappa u_{xx} \quad \text{for some positive constant } \kappa,
\]

where we think of $x$ as being a spatial variable signifying location and $t$ denotes time. For a function depending on more than one spatial variable and time, the heat equation takes the form $u_t = \kappa \Delta u$ where the right-hand side is the Laplacian from the previous example. This equation describes the evolution in time of what is called a diffusion process. A diffusion process is one in which some quantity (for example, heat) is spreading out as time goes on. It might also describe, for example, the spreading out of some chemical or pollutant that has been dumped into a lake.

**Example.** Verify that the function

\[
u(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}
\]

satisfies the heat equation.
satisfies the heat equation when the diffusion coefficient $\kappa$ takes the value 1. That is, verify that $u_t = u_{xx}$.

Computing the partial derivative $u_t$ requires the product rule:

$$u_t = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} \left(-\frac{x^2}{4t}\right) + \left(\frac{1}{\sqrt{4\pi t}}\right)' e^{-\frac{x^2}{4t}}$$

$$= \left(\frac{1}{\sqrt{4\pi t}} \frac{x^2}{4t^2} - \frac{1}{2} \frac{4\pi}{(4\pi t)^{3/2}}\right) e^{-\frac{x^2}{4t}}$$

Then for the $x$-partials, we find

$$u_x = \left(-\frac{x}{2t}\right) \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$$

so, again using the product rule we get

$$u_{xx} = -\frac{1}{2t} \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} + \left(-\frac{x}{2t}\right)^2 \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$$

$$= \left(-\frac{1}{2t} + \frac{x^2}{4t^2}\right) \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$$

and with a little algebra we can check that indeed $u_t = u_{xx}$.

One Version of the Chain Rule

Let’s first recall the chain rule for differentiable functions $f : \mathbb{R} \to \mathbb{R}$, say $f = f(u)$, and $u : \mathbb{R} \to \mathbb{R}$, say $u = u(x)$. Then if we form the composition $f(u(x))$ the chain rule tells us that the derivative is given by

$$\frac{d}{dx} (f(u(x))) = \frac{df}{du} \frac{du}{dx}.$$

What is the corresponding rule when, for example, we have a function $u : \mathbb{R}^2 \to \mathbb{R}$, say $u = u(x, y)$ and then we plug into $u$ two scalar-valued functions, say $x = x(t)$ and $y = y(t)$ to form the composition $u(x(t), y(t))$?

Computing the derivative $\frac{d}{dt} u(x(t), y(t))$ requires a new form of the chain rule. Here is what it turns out to be:

$$\frac{d}{dt} u(x(t), y(t)) = u_x(x(t), y(t)) \frac{dx}{dt} + u_y(x(t), y(t)) \frac{dy}{dt}.$$
Notice the mixture of partial derivatives of \( u \) and ordinary derivatives of \( x \) and \( y \). We won’t prove this new rule, but will simply mention two general principles that differentiation rules involving functions of more than one variable always follow:

(1) There will be as many terms in the answer as there are appearances of the variable with respect to which you are differentiating.

(2) Always work ‘from the outside inward’ when taking a derivative.

What this second point means is that, for example in the above calculation, we start each term by differentiation of \( u \) and then work our way inside to differentiating \( x(t) \) and \( y(t) \).

**Example.** Take \( u(x, y) = xy^2 \), \( x(t) = \cos t \) and \( y(t) = \sin t \). Then following the chain rule we find

\[
\frac{d}{dt}u(x(t), y(t)) = y(t)^2(-\sin t) + 2x(t)y(t)(\cos t)
\]

\[
= \sin^2 t(-\sin t) + 2(\cos t)(\sin t)(\cos t) = -\sin^3 t + 2\cos^2 t \sin t.
\]

Of course in this example it would be a lot easier just to first form the composition \( u(x(t), y(t)) = \cos t \sin^2 t \) and then just use the product rule, but it’s good to see how the chain rule works before confronting tougher examples.

By the way, notice that \( x(t)^2 + y(t)^2 = 1 \) in this example so computing the derivative of this composition is really determining the rate of change of the values of the function \( u(x, y) \) when \( x \) and \( y \) are required to vary along the unit circle. We’ll come back to calculations like that later.

**Quadratic Approximation: the 2nd Degree Taylor Polynomial**

Earlier this semester we reviewed Taylor’s Theorem and Taylor polynomials for a function \( f : \mathbb{R} \to \mathbb{R} \) of one variable. Recall that the \( n^{th} \) degree Taylor polynomial approximation to \( f \), which we might call \( P_n(x) \), taken at a point say \( x = a \) is given by

\[
P_n(x) = f(a) + f'(a)(x - a) + \frac{1}{2!}f''(a)(x - a)^2 + \ldots + \frac{1}{n!}f^{(n)}(a)(x - a)^n,
\]

where \( f^{(n)}(a) \) denotes the \( n^{th} \) derivative of \( f \) evaluated at \( x = a \).
A few sections back, we looked at what happens for the case $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ if we seek a linear approximation at a point $(x, y) = (a, b)$. Recall that we found

$$f(x, y) \approx P_1(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

where the approximation is more dependable the closer $(x, y)$ is to $(a, b)$.

The graph of this function $P_1$ is a plane that we called the tangent plane approximation to $f$ at $(a, b)$ and it had the property that the value of $P_1$ along with the two first partials $\frac{\partial P_1}{\partial x}$ and $\frac{\partial P_1}{\partial y}$ at $(a, b)$ agreed with those of $f$ itself.

Now we ask: What is the form of the quadratic polynomial approximation to $f$ at $(a, b)$ that agrees with $f$ up to and including all of the second order partials when evaluated at $(a, b)$? We will use this approximation, that we call $P_2(x, y)$, when studying max/min problems in the next sections.

To find a formula for $P_2$ we guess its form as

$$P_2(x, y) = P_1(x, y) + A(x - a)^2 + B(y - b)^2 + C(x - a)(y - b)$$

$$= f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) + A(x - a)^2 + B(y - b)^2 + C(x - a)(y - b).$$

for some constants $A$, $B$ and $C$. Then calculating the three second partials of $P_2$, namely $(P_2)_{xx}$, $(P_2)_{yy}$ and $(P_2)_{xy}$, plugging in $(x, y) = (a, b)$ and equating what we get to $f_{xx}(a, b)$, $f_{yy}(a, b)$ and $f_{xy}(a, b)$ respectively, we will find that $A = \frac{1}{2}f_{xx}(a, b)$, $B = \frac{1}{2}f_{yy}(a, b)$ and $C = f_{xy}(a, b)$. Thus, we arrive at the formula

$$P_2(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

$$+ \frac{1}{2}f_{xx}(a, b)(x - a)^2 + \frac{1}{2}f_{yy}(a, b)(y - b)^2 + f_{xy}(a, b)(x - a)(y - b).$$

In general, this formula, though admittedly more complicated, gives a better approximation to $f$ near $(a, b)$ than does the linear approximation $P_1(x, y)$.

**Example.** Determine the second degree Taylor polynomial approximation $P_2$ to the function

$$f(x, y) = 2x + 3xe^{2y} + y \cos x + 1$$
taken about the origin.
We will need 6 numbers: $f(0,0)$, $f_x(0,0)$, $f_y(0,0)$, $f_{xx}(0,0)$, $f_{yy}(0,0)$ and $f_{xy}(0,0)$. We find

$$f_x(x, y) = 2 + 3e^{2y} - y\sin x \quad \text{and} \quad f_y(x, y) = 6xe^{2y} + \cos x$$

so that

$$f_{xx}(x, y) = -y\cos x, \quad f_{yy}(x, y) = 12xe^{2y} \quad \text{and} \quad f_{xy}(x, y) = 6e^{2y} - \sin x.$$ 

Evaluating these 6 derivatives at $(0,0)$ we find $f(0,0) = 1$, $f_x(0,0) = 5$, $f_y(0,0) = 1$, $f_{xx}(0,0) = 0$, $f_{yy}(0,0) = 0$ and $f_{xy}(0,0) = 6$. Plugging these values into our general formula for $P_2$ with $(a,b) = (0,0)$ yields

$$P_2(x, y) = 1 + 5x + y + 6xy.$$ 

**Critical Points, Local and Absolute Maxima and Minima**

A central question in mathematics is that of determining when a given function is biggest or smallest. For say a function of two variables, $f(x,y)$, the place to look for the location of a point where $f$ is biggest or smallest is a point where the tangent plane is horizontal—anywhere else, the function can be made bigger or smaller by moving in the direction of the gradient or opposite the gradient. This leads us to make the following definition:

**Definition.** A **critical point** of a differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$ is any point $(a,b)$ where $\nabla f(a,b) = (0,0)$.

This is just the generalization of critical point for $f : \mathbb{R} \to \mathbb{R}$ where critical point just means $f'(a) = 0$.

We also need a few other definitions that are just generalizations of calculus I terms. We need to distinguish between a point where the value of $f$ is just bigger or smaller than its value at other nearby points and a point where the value of $f$ is bigger or smaller than its value at all other points.

**Definition.** A function $f : \mathbb{R}^2 \to \mathbb{R}$ has a **local maximum** at $(a,b)$ if $f(a,b) \geq f(x,y)$ for all other nearby points $(x,y)$ in some disc centered at $(a,b)$. If $f(a,b) \leq f(x,y)$ for all such nearby points then we say $f$ has a **local minimum** at $(a,b)$. 

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**Definition.** A function $f : \mathbb{R}^2 \to \mathbb{R}$ has an absolute (or global) maximum at $(a,b)$ if $f(a,b) \geq f(x,y)$ for all other points $(x,y)$ in $\mathbb{R}^2$. If $f(a,b) \leq f(x,y)$ for all other points $(x,y)$ in $\mathbb{R}^2$, then we say $f$ has an absolute (or global) minimum at $(a,b)$.

Then there’s one more possibility: It could be that $(a,b)$ is a critical point of a function $f$ that is the location of neither a local maximum nor a local minimum. This means that for some points $(x,y)$ arbitrarily close to $(a,b)$, the values of $f(x,y)$ are bigger than $f(a,b)$ and for other points $(x,y)$ arbitrarily close to $(a,b)$, the values of $f(x,y)$ are smaller than the number $f(a,b)$. In this case, we call the critical point $(a,b)$ a saddle point. It gets this name because near $(a,b)$, the graph of $f$ looks like a horse’s saddle—it goes up in one direction and down in another.

So if one seeks to locate local minima and local maxima and to distinguish them from saddle points, the first step is to find all critical points by simultaneously setting $f_x = 0$ and $f_y = 0$. But then, how does one tell a local maxima from a local minima from a saddle point? We will do this by looking at the quadratic approximation $P_2$ to the function $f$ taken about each critical point.

If one returns to the formula for $P_2$ from a few pages back and evaluates it at a critical point, one finds that the condition $\nabla f(a,b) = (0,0)$ leads to the simpler formula

$$P_2(x,y) = f(a,b) + \frac{1}{2}f_{xx}(a,b)(x-a)^2 + \frac{1}{2}f_{yy}(a,b)(y-b)^2 + f_{xy}(a,b)(x-a)(y-b).$$

This quadratic polynomial gives a good approximation to the function $f$ near a critical point $(a,b)$ and often by examining it, we can classify a critical point.

Here are some examples:

**Example:** Find all critical points of the function

$$f(x,y) = x^2 + y^2 - 2x - 6y + (x-1)^3 + 14$$

and classify them as either local maxima, local minima or neither.

**Step 1. Find the critical points.** We compute that

$$f_x = 2x - 2 + 3(x-1)^2 = (x-1)(3x-1) \quad \text{and} \quad f_y = 2y - 6.$$
Thus, \( \nabla f(x, y) = (0, 0) \) only if \( x = 1 \) or \( \frac{1}{3} \) and \( y = 3 \), so there are two critical points, \((1, 3)\) and \((\frac{1}{3}, 3)\).

Step 2. Find \( P_2 \) for each the critical point in order to classify it. Let’s start with \((1, 3)\). Since

\[
f_{xx}(x, y) = 2 + 6(x - 1), \quad f_{xy}(x, y) = 0 \quad \text{and} \quad f_{yy}(x, y) = 2
\]

we find, plugging in \( x = 1 \) and \( y = 3 \) into these three formulas, that for \((a, b) = (1, 3)\) we have

\[
f(x, y) \approx P_2(x, y) = f(1, 3) + (x - 1)^2 + (y - 3)^2 \quad \text{for} \ (x, y) \ \text{near} \ (1, 3).
\]

This formula shows that near \((1, 3)\), \( f(x, y) \geq f(1, 3) \) since the last two terms are non-negative. We conclude, therefore, that \((1, 3)\) is the location of a local minimum.

How about the other critical point \((\frac{1}{3}, 3)\)? Doing the same procedure for this critical point, we find that \( P_2 \) taken about this second critical point is given by

\[
P_2(x, y) = f(\frac{1}{3}, 3) - (x - \frac{1}{3})^2 + (y - 3)^2.
\]

What does this tell us? Notice that if we consider points near \((\frac{1}{3}, 3)\) of the form \((x, 3)\) then we get

\[
f(x, 3) \approx P_2(x, 3) = f(\frac{1}{3}, 3) - (x - \frac{1}{3})^2,
\]

and from this we see that there are points nearby \((\frac{1}{3}, 3)\) for which \( f(x, 3) < f(\frac{1}{3}, 3) \). On the other hand, by considering points of the form \((\frac{1}{3}, y)\) for \( y \) close to \( 3 \) we find

\[
f(\frac{1}{3}, y) \approx P_2(\frac{1}{3}, y) = f(\frac{1}{3}, 3) + (y - 3)^2,
\]

and so for such \( y \)-values, \( f(\frac{1}{3}, y) > f(\frac{1}{3}, 3) \). Thus, \((\frac{1}{3}, 3)\) is neither the location of a local maximum nor a local minimum. It is a saddle point.

Before going on to the next example, let’s consider whether or not the local minimum \((1, 3)\) might be an absolute minimum? In other words, could it be the case that for all \((x, y) \in \mathbb{R}^2\), one has \( f(x, y) \geq f(1, 3) \)? The answer is no. To see this, we go back to the formula for \( f \) itself and note that due to the presence of the term \((x - 1)^3\), if we plug in a huge negative number then this term will be dominant over all of the other terms and will push the value of
f down towards $-\infty$. This shows that f has no absolute minimum and a similar argument shows that f also has no absolute maximum.

**Example.** Let’s again look for all critical points and classify them, now for the function

$$g(x, y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4.$$ 

We find

$$g_x = 3x^2 + 3y^2 - 6x \quad \text{and} \quad g_y = 6xy - 6y.$$ 

Setting $g_y = 0$ we find that either $x = 1$ or $y = 0$. If $x = 1$, then for $g_x$ we find $g_x(1, y) = 3y^2 - 3$ which is zero when either $y = 1$ or $y = -1$. Thus, two critical points are $(1, 1)$ and $(1, -1)$. The other possibility is that $y = 0$ and since $g_x(x, 0) = 3x^2 - 6x$ we find that either $x = 0$ or $x = 2$. So there are two more critical points $(0, 0)$ and $(2, 0)$.

For each of these four critical points, we can calculate the value of $g_{xx}$, $g_{xy}$ and $g_{yy}$, plug these values into the formula to determine the quadratic approximation $P_2$ and in that way classify each of the critical points. If you try this, you should get:

- For $(0, 0)$ we find : $P_2(x, y) = 4 - 3x^2 - 3y^2$,
- For $(1, 1)$ we find : $P_2(x, y) = 2 + 6(x - 1)(y - 1)$,
- For $(2, 0)$ we find : $P_2(x, y) = 3(x - 2)^2 + 3y^2$,
- For $(1, -1)$ we find : $P_2(x, y) = 2 - 6(x - 1)(y + 1)$.

From these formulas we see that $(0, 0)$ is the location of a local maximum, $(2, 0)$ is the location of a local minimum, and for $(1, 1)$ and $(1, -1)$, we have saddle points since we can find points arbitrarily close to either such that $P_2$ (and therefore $g$ itself) has values both bigger and less than the value of $g$ the critical point itself.

It turns out that by studying carefully all possibilities for a general quadratic approximation to a function $f$ (also known as the second degree Taylor polynomial), one can establish the following:
Second Derivative Test for $f : \mathbb{R}^2 \rightarrow \mathbb{R}$

Suppose $f$ is twice differentiable and has a critical point at $(a, b)$.

Let $\Lambda := \det \begin{pmatrix} f_{xx}(a, b) & f_{xy}(a, b) \\ f_{xy}(a, b) & f_{yy}(a, b) \end{pmatrix}$.

(i) If $f_{xx}(a, b) > 0$ and $\Lambda > 0$ then $f$ has a local minimum at $(a, b)$.
(ii) If $f_{xx}(a, b) < 0$ and $\Lambda > 0$ then $f$ has a local maximum at $(a, b)$.
(iii) If $\Lambda < 0$ then $f$ has a saddle point at $(a, b)$.
(iv) If $\Lambda = 0$ then this test gives no information.

Though we don’t present the details here, it turns out that condition (i) says both eigenvalues of this matrix are positive, while condition (ii) says both are negative and condition (iii) says there are two eigenvalues of opposite sign.