Notes: Day 3: Connectedness and the Riemann integral.

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August 6, 2015

Let us begin today with some exercises on the Riemann integral. Recall what a partition is. Associated with a partition $\pi$ of a bounded interval, one defines for each subinterval $I_k \in \pi$, two numbers, $m_k := \inf_{I_k} f$, and $M_k := \sup_{I_k} f$. Further one defines $U(f, \pi) := \sum_k M_k |I_k|, L(f, \pi) := \sum_k m_k |I_k|$, with $|\cdot|$ denoting length. Observe that $f$ being bounded is part of the definition of Riemann integrability. We say that $f$ is Riemann integrable if $U(f) := \sup_{\pi} U(f, \pi) = \inf_{\pi} L(f, \pi) = L(f)$, where the supremum and infimum are taken over partitions of the interval of integration.

This definition of Riemann integrability is not as convenient to verify, consequently one proves in courses that it is equivalent to the following Cauchy-type criterion for Riemann integrability. We say $f : I \rightarrow \mathbb{R}$ is Riemann integrable (some books call it Darboux integrable), if given $\epsilon > 0$ there exists a partition $\pi$ such that if for any partition $\pi' \supseteq \pi$ one has that $U(f, \pi') - I(f, \pi) < \epsilon$. Recall here that the mesh of a partition (some books call it the norm of the partition) $\pi$ is the size of the biggest subinterval defined by $\pi$.

An essential idea that these definitions of Riemann integrability consider, is the oscillation of $f$. Given an interval $I$, we define $\text{osc}_I f := \sup_I f - \inf_I f$. A function is Riemann integrable if we can make $U(f, \pi) - L(f, \pi)$ as small as possible. In terms of oscillations, this means that we can find a partition $\pi$ of the interval $I$ such that the oscillation of $f$ on most intervals is arbitrarily small, and the size of the intervals on which $f$ oscillates a lot can be made arbitrarily small. As an example, define $f_1(x) : [0, 1] \rightarrow \mathbb{R}$ by $f_1(0) = 2015, f_1(x) = 0$ when $0 < x \leq 1$. Then this argument shows that $f_1$ is Riemann integrable and $\int f_1 = 0$.

Using these ideas, let us solve problem

**Problem 1.** J11/1

J11/1. Use triangle inequality for finitely many real numbers applied to Riemann sums and pass to the limit. In other words, for each integer $n$, one has $|\frac{1}{n} \sum_{k=1}^{n} f(\frac{k}{n})| \leq \frac{1}{n} \sum_{k=1}^{n} |f(\frac{k}{n})|$. Since $f$ is continuous, it is Riemann integrable. $|f|$ is continuous also. Pass to the limit $n \rightarrow \infty$.

Here’s an alternate solution, using Cauchy-Schwarz. $|\int_{0}^{1} f(t)dt|^2 = (\int_{0}^{1} f(t)dt, \int_{0}^{1} f(t)dt) = \int_{0}^{1} |f(t)||\int_{0}^{1} f(t)dt|dt$. ■

Can you solve this problem if instead of $f$ being continuous, we only had $f$ Riemann integrable? Absolutely! This is because by the reverse triangle inequality, we can control the oscillation of $|f|$ by the oscillation of $f$. Hence if $f$ is Riemann integrable, so is $|f|$. This idea is important and is a recurring theme in analysis. Namely, if $f, g$ are bounded functions on some bounded interval $I$, and
if $g$ is Riemann integrable and the oscillation of $g$ controls oscillation of $f$, i.e. for eg.

$$|f(x) - f(y)| \leq C|g(x) - g(y)|, \quad x, y \in I,$$

then $f$ is Riemann integrable.

Let us now work out a problem based on approximating a function by its Riemann sum. This problem uses the idea of improper Riemann integrals, since it will be defined on an unbounded interval.

**Problem 2. A12/6**

**Solution.** We compute $\lim_{n \to \infty} \sum_{k=1}^{n} \frac{n^{-1}}{n+k^r} = \lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{n+\left(k/n\right)^r} = \int_{0}^{\infty} \frac{dx}{1+x^r}$.

Then $I_1$ is always finite. When $r > 1$, $I_2 < \int_{1}^{\infty} \frac{dx}{1+x^r} < \infty$. When $r \leq 1$, that integral diverges: $x^r \leq x$ since $x \geq 1$, and so $\int_{1}^{\infty} \frac{dx}{1+x^r} \geq \int_{1}^{\infty} \frac{dx}{1+x} = +\infty$. ■

The preceding problem consisted of integrals of monotone functions, and for such functions, it is a very useful tool in analysis that lower Riemann sums, upper Riemann sums and integrals are related. You might have seen a version of this as the integral test. We have already seen a proof of convergence / divergence of $\sum_{k=1}^{\infty} k^{-p}$ using the Cauchy condensation test. We can give a second proof using this integral test. Let us revisit some problems from Monday we didn’t have time for.

**Problem 3. A14/8.**

**Solution.** Let us first consider the case $p > 1$. The function $g(t) = t^p$ is convex when $t \geq 0$. Consequently by Jensen’s inequality (i.e., the definition of convexity), one has that $\frac{\log 2 + \cdots + \log n}{n-1} \geq \left(\frac{\log 2 + \cdots + \log n}{n-1}\right)^p$. In other words, $H_n(p) \geq (n-1)^{1-p}(\log 2 + \cdots + \log n)^p$. Stirling’s formula then says that $\log n! = n \log n - n + O(\log n)$. We see $a_n(p) \leq \frac{(n-1)^{p-1}}{(\log n)^p} = \frac{(n-1)^{p-1}}{n^{p-1}\log n} = c_n \frac{\log n}{n}$ where $c_n \to 1$ as $n \to \infty$, and hence is bounded. To see that $\sum_{n=2}^{\infty} \frac{1}{n^{(p-1)/r}}$ converges, majorize it by $\int_{1}^{\infty} \frac{dt}{t^{(p-1)/r}} = \int_{1}^{\infty} \frac{du}{u^{p-1}} = \int_{1}^{\infty} \left(\log 2\right)^{1-p}$. The function $g(t) = t^p$ is concave when $p < 1$ and so the same argument easily gives divergence of the sum. When $p = 1$, use Stirling directly to obtain that the sum diverges. ■

Since integration is defined as a limit, it is natural to ask how it behaves with other limiting processes. For instance, if $f_n$ are Riemann integrable, and $f_n \to f$, then is it true that $\int f_n \to \int f$ as $n \to \infty$? NO. It is not even true when $f_n$ are continuous. For example, take $f_n(x) = -n(x-n^{-1})$ on $[0, 1/n]$ and $0$ from $(1/n, 1]$. Then pointwise, $f_n \to 0$ on $(0, 1)$ but the integral and the limits don’t commute. The right condition to look for when $f_n$ are continuous is uniform convergence. Namely, if $f_n \to f$ uniformly, and $f_n$ are continuous, then $\int f_n \to \int f$. Here is a problem which uses this idea.

**Problem 4. J10/8**

**Solution.** Set $g_n(x) = \sum_{k=1}^{n} a_k x^k$ defined on $[0, 1]$. Then since $|a_k x^k| \leq k^2 2^{-k}$, and also $\sum_{k=1}^{\infty} k^2 2^{-k} < \infty$ (root test), by the Weierstrass M-test, $g_n(x)$ converge uniformly to a function $g(x)$. Consequently, we can pass the limit inside the integral (using uniform continuity of $f$ on a bounded set and the fact that $g_n$’s, $g$ have bounded range) and the required limit exists and is equal to $\int_{0}^{1} f(x, g(x)) \, dx$. ■
The relationship between uniform convergence and derivatives is slightly more complicated, and is indirect: it depends on the corresponding theorem for integrals via the FTC. Here is a theorem from Rudin about this. Again, its proof depends on the fundamental theorem of Calculus (whose proof you should know, since it was asked on a tier exam).

**Theorem 5.** Let \( f_n : (a, b) \to \mathbb{R} \) be a sequence of functions whose derivatives \( f'_n \) are Riemann integrable. Suppose \( f_n \to f \) pointwise and \( f'_n \to g \) uniformly where \( g \) is continuous, then

1. \( f_n \to f \) uniformly,
2. \( g = f' \), and \( f \) is continuously differentiable.

I was unable to find problems on this theorem in the last 5 years, but here are a couple from August 2009 (so you won’t find these on my solutions packet that I handed out on Day 1).

**Problem 6.** A09/7

**Solution.** We are given that \( f_n(x) := f^{(n)}(x) \) exists for each \( n \), so \( f_n \) is \( C^\infty \). Moreover, notice that \( f_{n+1}(x) = f_n'(x) \). Fix a compact set \( K \subset \mathbb{R} \). We are given that \( f_n \) converges uniformly to \( g \).

The derivatives of \( f_n \), which as a sequence is the same, converges also uniformly to \( g \). Consequently, the preceding theorem applies, and we find that \( f_n \to g \), and \( f'_n = f_{n+1} \to g \), and \( g' = g \). It follows that \( g = e^x \).

Yesterday we talked a bit about bump functions. Here’s a problem on why they are important.

**Problem 7.** A10/8

**Solution.** We observe for any \(|h| < 2, h \neq 0\), \(|\frac{1}{h} \int_R (f(x-y) - f(x)) \phi(\frac{y}{h}) dy| = |\frac{1}{h} \int_{x-h}^{x} (f(z) - f(x)) \phi(\frac{z-x}{h}) dz| \leq \frac{1}{h} \int_{-5h}^{5h} |f(z) - f(x)| \phi(\frac{z-x}{h}) dz| = (*) \). Now \( f \) is uniformly continuous on the compact set \([x-10, x+10]\). Given \( \varepsilon > 0 \) select \( \delta > 0 \) such that \(|f(x) - f(y)| < \varepsilon/10 \), when \(|x-y| < \delta \). Thus for \( h \) such that \( 5h < \delta \), one has that if \(|x-\varepsilon| < 5h < \delta \), then \((*) \) \leq \frac{1}{h} \int_{-5h}^{5h} |f(z) - f(x)| \phi(\frac{z-x}{h}) dz \leq \frac{\varepsilon}{10} \|f\|_{\infty} 10 h \cdot \frac{1}{h} \to 0 \), as \( \varepsilon \to 0 \).

Let us conclude with an example of a bump function. It is a good construction that incorporates several ideas we have talked about on uniform convergence. I learned it from part 3 of the Rudin trilogy (Functional analysis, by W. Rudin). We will first construct a function \( g \in C^\infty(a, b) \), where \( a < b \) are finite numbers. Start with a sequence of positive numbers \( \delta_0, \delta_1, \cdots, \delta_n, \cdots \), such that the series \( \sum_{n=0}^{\infty} \delta_n = b - a \). Set

\[
m_n = \frac{2^n}{\delta_1 \cdots \delta_n}.
\]

Let \( f_0 \) be a continuous monotone function such that \( f_0 = 0 \) for \( x \leq a \) and \( f_0 = 1 \) for \( x \geq a + \delta_0 \). Define

\[
f_{n+1}(x) = \frac{1}{\delta_{n+1}} \int_{x-\delta_{n+1}}^{x} f_n(t) \, dt.
\]

Then it follows that \( f_n \) is \( n \) times continuously differentiable, (note that averaging each time increases the regularity by one), and that \(|D^n f_n| \leq m_n \). If \( r < n \), we see

\[
D^r f_n(x) = \frac{1}{\delta_n} \int_{0}^{\delta_n} (D^r f_{n-1}(x-t)) \, dt.
\]
and so by induction on \( n \), we see that if \( n \geq r \),

\[
|D^r f_n(x)| \leq m_r.
\]

For \( r \) fixed, and for any \( n \geq r + 2 \), we find that

\[
|D^r f_n(x) - D^r f_{n-1}(x)| \leq \frac{1}{\delta_n} \int_0^{\delta_n} |D^r f_{n-1}(x - t) - D^r f_{n-1}(x)| dt \leq m_{r+1}\delta_n.
\]

Since \( r \) is fixed and \( \sum \delta_n < \infty \), we find \( \delta_n \to 0 \) as \( n \to \infty \). It follows that \( D^r f_n \) converge uniformly as \( n \to \infty \) for each fixed \( r \). Hence theorem 5 applies, and says that \( f_n \) converges uniformly to a function \( g \), which is infinitely differentiable, and \( g(x) = 0 \) for \( x < a \), and \( g(x) = 1 \) for \( x > b \).