Let us begin with some problems on real sequences. There are generically two important ways to conclude convergence/divergence of sequences. Monotone sequences that are bounded are convergent. Let us begin with an illustration of this simple principle using A14/2.

**Problem 1.** A14/2.

We are told \( a_n \) is bounded. A common method to show convergence of a bounded sequence is to show it is monotone. Suppose first there exists \( n \) so that \( a_n \leq a_{n-1} \). Then the given condition says that \( \frac{a_{n+1} - a_n}{2} \leq \frac{a_n - a_{n-1}}{2} \leq 0 \). By induction the sequence is monotone decreasing from that point. If there does not exist such an integer, the sequence is monotone increasing.

... Inequalities form the grammar in analysis. K. O. Friedrichs is known to have said that inequalities make the world go round. Never mind the world for the time being, let us study the following problem using some elementary but very useful inequalities.

**Problem 2.** Fix a positive number \( \alpha \), and choose \( x_1 > \alpha^{1/p} \), where \( p \) is a fixed positive integer. Does the sequence of numbers \( x_n \), defined by

\[
x_{n+1} = \frac{p-1}{p} x_n + \frac{\alpha}{p} x_n^{-p+1}
\]

converge? Determine the limit.

**Solution.** As a homework exercise, follow what you did in Ciprian’s lecture to apply the contraction mapping principle in studying this problem. Let us instead solve it directly by showing \( x_n \) is monotone and bounded. A useful tool in this is the so called arithmetic mean-geometric mean inequality. We have,

\[
x_{n+1} = \frac{1}{p} \left( x_n + \cdots + x_n + \alpha x_n^{-p+1} \right) \geq \alpha^{1/p}.
\]

This shows that the sequence is bounded below. On the other hand since by induction, \( x_n > 0 \) for all \( n \),

\[
x_{n+1} - x_n = -\frac{1}{p} x_n + \frac{\alpha}{p} x_n^{-p+1} = \frac{1}{px_n^p} \left( -x_n^p + \alpha \right) \leq 0.
\]
This shows that \( x_n \) is decreasing by induction. From this step onward, this approach is equivalent to the fixed point approach, in order to find the limit. The limit \( L \) satisfies

\[
L = \frac{p-1}{p} L + \frac{\alpha}{p} L^{-p+1}.
\]

Hence \( L = \alpha^{1/p} \). ■

Oscillations play an important role in analysis, as they capture averaging procedures via cancellations. A poor illustration of this is \( a_n = (-1)^n \) fails to converge, since \(-1 = \lim \inf a_n < \lim \sup a_n = 1\). However, quite often oscillations result in cancellations: if one can "control" oscillations, one can often conclude convergence of some sort. Here’s an illustration of this vague but important idea.

Problem 3. A13/2.

Solution. Yes, the sequence converges to 0. Use the facts that cosine is Lipschitz-1 and \( \cos(n\pi + \pi/2) = 0 \) to get \( |\cos(\pi\sqrt{n^2 + n})| = |\cos(\pi\sqrt{n^2 + n}) - \cos(n\pi + \pi/2)| \leq \pi\sqrt{n^2 + n} - n - 1/2 \). The term in absolute value goes to 0 since \( \sqrt{n^2 + n} - n - 1/2 = \sqrt{n^2 + n^{1/2}} \). ■

As you saw in this morning’s lecture, the contraction mapping principle is another tool to show convergence of sequences (not just of numbers). We will show an application of the contraction mapping principle on Thursday.

Sequences permit us to extend the notion of addition from finitely to infinitely many quantities. This is the notion of a series, we call an infinite series of real numbers convergent if the (numerical) sequence of its partial sums converge. A necessary condition for a series to converge, is that its general term tend to 0. However, this condition is not sufficient (alternate series). A very low-tech, but one of the most useful devices for showing convergence of a series, is comparison. If \( \sum a_n \) is a series, and \( |a_n| \leq c_n \) for every \( n \), and finally if \( \sum c_n \) converges, then \( \sum a_n \) converges (absolutely!). To see this, the partial sums form a Cauchy sequence: \( \sum_{p}^{q} a_n \leq \sum_{p}^{q} |a_n| \leq \sum_{p}^{q} c_n < \varepsilon \) if \( p, q \) are large.


Solution. Use induction to prove the entries of \( A^n \) are dominated by \( 2^{n-1}(\sup\{|a|, |b|, |c|, |d|\})^n \). Then, in each of the four entries of the matrix, we have an infinite sum, each of whose terms \( p_n \) is controlled in absolute value by \( |p_n| \leq \frac{2^n M^{2n+1}}{(2n+1)!} \). The latter series converges to something which is at most \( \frac{1}{2} e^{2M} \). ■

A surprisingly powerful tool for summing series whose terms are positive, and decreasing is the Cauchy Condensation test. This says that if \( a_n \) satisfies these conditions, then \( \sum a_n \) and \( \sum 2^n a_{2n} \) either both converge or both diverge. As an illustration,

Problem 5. J12/2.

Solution. Since \( (a_n) \) are monotone decreasing and positive, we can use Cauchy’s condensation test to conclude that \( \sum a_n \) converges if and only if \( \sum 2^n a_{2^n} \) converges. Thus \( 2^n a_{2^n} \to 0 \). But then \( 2^{n+1} a_{2^{n+1}} \to 0 \), too, from which we deduce the convergence of the whole sequence. For if \( m \in [2^n, 2^{n+1}] \), \( m a_m \leq 2^{n+1} a_{2n} \). ■

As a further illustration, if \( p > 0 \), then \( \sum_{k=1}^{\infty} k^{-p} \) converges if and only if \( \sum_{k=1}^{\infty} 2^k k^{-p} \) converges, i.e. \( \sum_{k=1}^{\infty} 2^k \frac{1}{k^p} \) converges. The general term of this sequence does not go to 0 if \( p \leq 1 \). The series is geometric if \( p > 1 \). An elegant application of this is

Solution. Note $x_{n+1} = x_1 + \sum_{k=1}^{n} \sqrt{|x_k|}$. So if we could majorize each $x_k$ by a fixed constant times $k^q$ for some $q < 2$, the summability of $\sum_{k=1}^{\infty} k^p$ for $p < -1$ would imply that $\sum_{k=1}^{\infty} \sqrt{|x_k|}$ converges and we’d be done. Now define $\alpha = \sqrt{|x_1|} + 1$ and prove by induction that $|x_k| \leq k\alpha^2$ (nothing special about this $\alpha$). ■

Ratio and root tests are also absolute convergence. A good review of a number of summation tests (that are worth remembering for the tier exam) are reviewed via


Solution. First use ratio test: $\limsup_{k \to \infty} a_{k+1}/a_k = e$. Thus the series certainly converges for $e|x| < 1$ and diverges for $e|x| > 1$. It remains to investigate points $x = \pm 1/e$. Take $x = -1/e$.

Then use the Stirling’s formula given and the alternating series test to conclude convergence at this point. Finally take $x = 1/e$. Use the limit comparison test by rearranging the given Stirling formula, to conclude that the series diverges at this point, since $\sum_{k=1}^{\infty} k^{-1/2} = \infty$. ■

We used the alternate series test, where the series converged due to cancellations/oscillations.

Another low-tech but very powerful machinery to capture cancellations is summation/integration by parts. Let us illustrate this using a few simple problems.

Problem 8. J11/9

Solution. This just follows from the summation by parts formula (which you discussed in the lecture). Apply summation by parts to tails, i.e. consider sums $\sum_{k=m}^{n} a_k/k$. Summing by parts, we find,

$$A_n = a_1 + \cdots + a_n,$$

$$\sum_{k=m}^{n} \frac{a_k}{k} = \sum_{k=m}^{n} \frac{A_{k+1} - A_k}{k} = \frac{A_{m+1}}{m} - \frac{A_m}{m} + \frac{A_{m+2}}{m+1} - \frac{A_{m+1}}{m+1} \cdots + \frac{A_n}{n-1} - \frac{A_{n-1}}{n} + \frac{A_{n+1}}{n} - \frac{A_n}{n}$$

$$= \frac{-A_m}{m} + \frac{A_{n+1}}{n} + \sum_{k=m+1}^{n} A_k \left( \frac{1}{k-1} - \frac{1}{k} \right),$$

$$= \frac{1}{\sqrt{m}} + \frac{1}{\sqrt{n}} + \sum_{k=m+1}^{n} \frac{1}{\sqrt{k(k-1)}} < \varepsilon$$

provided $m, n$ are large. ■

Here’s a continuous version of this phenomenon.

Problem 9. J14/6

Solution. When $p = 1$, write $\int_{\pi}^{\infty} \sin x \, dx = \int_{0}^{\pi} \sin x \, dx + \int_{\pi}^{\infty} \sin x \, dx$. The first integral is fine. The second integral does not converge as an improper integral: take the sequence $x_n = n\pi$ and note that $\int_{\pi}^{\pi n} \sin x \, dx$ oscillates between 0 and $-2$. Next for $p < 0$, let us again break up the integral as before, $\sin xp$ is continuous on $[0, \pi)$ and is bounded, and hence this piece is bounded (continuous functions are Riemann integrable). Now when $p < 0$, notice that as $x \to \infty$, $\sin(xp) \approx xp$, and so $\int_{\pi}^{\infty} \sin(xp) \, dx$ converges if and only if $p < -1$. Finally, for $p > 1$, use Dirichlet’s test. Namely, $\int_{\pi}^{\infty} \sin(xp) \, dx = p^{-1} \int_{\pi}^{\infty} \sin u \frac{du}{u^{p-1}}$. The function $\sin u$ has integral that is uniformly bounded on bounded intervals due to cancellations. Hence using summation/integration by parts we are done. ■

Thanks to calculus, we’re often more comfortable with integrals than sums. when you have a series with positive decreasing terms, you can precisely conclude convergence (resp. divergence) by
majorizing (resp. minorizing) by an integral. Before working out a problem from a recent tier that uses this, let us give a quick (rather crude) proof of Stirling’s formula. Observe by integration by parts that

\[ n! = \int_0^\infty t^n e^{-t} \, dt. \]

Since \( t^n e^{-t} \) achieves its maximum at \( t = n \), we are led to make a substitution \( t = n + s \). Hence,

\[ n! = n^n e^{-n} \int_{-n}^\infty \exp\left(n \log \left(1 + \frac{s}{n}\right) - s\right) \, ds \approx n^n e^{-n} \int_{-n}^\infty \exp\left(-\frac{s^2}{2n}\right) \, ds \approx n^{1/2} n^n e^{-n} \int_{-\infty}^{\infty} e^{-x^2/2} \, dx = \sqrt{2\pi} n^{n+1/2} e^{-n}. \]

These estimates can be made precise, see Terry Tao’s blog post on this subject. Nevertheless, let’s consider the following problem which uses Stirling.

**Problem 10.** A14/8.

**Solution.** Let us first consider the case \( p > 1 \). The function \( g(t) = t^p \) is convex when \( t \geq 0 \). Consequently by Jensen’s inequality (i.e., the definition of convexity), one has that

\[ \log(n!) = n \log n - n + O(\log n). \]

Stirling’s formula then says that

\[ \log(n!) = n \log n - n \log n + O(\log n). \]

Consequently, \( \frac{\log n!}{n} \to n \log n - n + O(\log n) \to n \log n \) as \( n \to \infty \). To see that \( \sum_{n=1}^\infty \frac{\log n!}{n(n+1)} \) converges, majorize it by \( \int_2^\infty \frac{dt}{t(n \log t)^p} = \frac{u^{p-1}}{u^{p-1}} \left. \frac{du}{u^{p-1}} \right|_{u=\log n}^{u=\infty} = \frac{1}{p-1} (\log 2)^{1-p}. \) The function \( g(t) = t^p \) is concave when \( p < 1 \) and so the same argument easily gives divergence of the sum. When \( p = 1 \), use Stirling directly to obtain that the sum diverges. 

Let us now move on to continuity: continuous functions are precisely those that are well behaved under convergence of sequences.

**Problem 11.** A13/6.

**Solution.** Let \( y > x \), and let \( n \) be an integer that is to be chosen. Set \( t = \frac{y-x}{n} \). Then,

\[ f(x + \frac{t}{n}) \geq f(x) - \frac{t^2}{n^2}, \]

\[ f(x + \frac{2t}{n}) \geq f(x + \frac{t}{n}) - \frac{t^2}{n^2}, \]

\[ \cdots, \]

\[ f(y) \geq f(x + \frac{n-1}{n} t) - \frac{t^2}{n^2}. \]
Adding

\[ f(y) \geq f(x) - \frac{t^2}{n} \]

Let \( n \to \infty \). ■

Continuous maps have several important properties. Let’s review a few of them through problems.

**Problem 12.** A13/1

**Solution.** Denote the infimum of that set by \( \mu \). Let \( (r_m, a_m)_{m=1}^{\infty} \) be a minimizing sequence, where for ease of notation we denote \( r = (r_1, \cdots, r_N) \in [0, \infty)^N \) and \( a = (a_1, \cdots, a_N) \in (\mathbb{R}^n)^N \). By minimizing sequence we of course mean that \( \|r_m\|^2 = (r_m^1)^2 + \cdots + (r_m^N)^2 \to \mu \). Furthermore, these pairs are admissible in the sense that \( A \subset \bigcup_{k=1}^{N} B(a_k, r_k) \) for every \( m \). Then, wlog, we may assume that \( a_i^m \in 10^n A \) for each \( i = 1, \cdots, N, \) and for each \( m \). By compactness extract a convergent subsequence. As for the radii, we may assume for \( m \) big enough that \( \|r_m\|^2 \leq \mu + 1 \). Extract a further subsequence so that \( r_m \to r \in \mathbb{R}^N \). Now verify that \( A \subset \bigcup_{k=1}^{N} B(a_k, r_k) \) and that \( \sum_{k=1}^{N} r_k^2 = \mu \). ■

Uniform continuity is a very important idea.

**Problem 13.** A14/3

**Solution.** Thinking about a counter-example in the \( \varepsilon = 0 \) case tells you the answer in the general case. Take \( f(x) = \sqrt{x} : [0, 1] \to [0, 1] \). Check that there does not exist \( M > 0 \) such that \( |f(x) - f(y)| \leq M|x - y| \). Now let \( \varepsilon > 0 \). Since \( f : K \to \mathbb{R} \) is continuous and \( K \) is compact, \( f \) is in fact uniformly continuous. So obtain \( \delta \) such that if \( |x - y| \leq \delta \), \( |f(x) - f(y)| \leq \varepsilon \) (hence the inequality is satisfied if \( |x - y| \leq \delta \)). If \( |x - y| \geq \delta \), take \( M = 2 \sup |f|/\delta \). ■

Using the same idea, try the following problem. Let \( f : \mathbb{R} \to \mathbb{R} \) be uniformly continuous. Prove that there are positive numbers \( A, B \) such that

\[ |f(x)| \leq A|x| + B. \]