Zero Sets for Spaces of Analytic Functions

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Abstract
We show that under mild conditions, a Gaussian analytic function $F$ that a.s. does not belong to a given weighted Bergman space or Bargmann–Fock space has the property that a.s. no non-zero function in that space vanishes where $F$ does. This establishes a conjecture of Shapiro (1979) on Bergman spaces and allows us to resolve a question of Zhu (1993) on Bargmann–Fock spaces. We also give a similar result on the union of two (or more) such zero sets, thereby establishing another conjecture of Shapiro (1979) on Bergman spaces and allowing us to strengthen a result of Zhu (1993) on Bargmann–Fock spaces.

1 Introduction
Zeros of Gaussian analytic functions were originally studied by Paley and Wiener (1934), Kac (1943a,b), and Rice (1944, 1945). Since then, many more mathematicians and physicists have been interested in such zero sets. For some of the history, see Sodin (2005) and Hough, Krishnapur, Peres, and Virág (2009). Those sources also give surveys of certain aspects of zero sets of Gaussian analytic functions as random objects. The topic of the present paper, however, is not mainly zero sets of Gaussian analytic functions as random objects, but as tools to understand zero sets in standard spaces of analytic functions. In particular, we consider the (weighted) Bergman spaces in the unit disk and the (weighted) Bargmann–Fock spaces in the entire plane, for which we give a unified treatment. In Subsection 1.1, we give a brief history of what is known for zero sets of functions in these spaces, focused on results relevant to ours. More can be found in Chapter 4 of Hedenmalm, Korenblum, and Zhu (2000) and Chapter 4 of Duren and Schuster (2004), which are devoted to zero sets of Bergman spaces, and Chapter 5 of Zhu (2012), which is devoted to zero sets of Bargmann–Fock spaces.

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Let $\mu$ be a finite measure on $(0, \infty)$, not identically 0. Write $r_\mu := \inf\{r : \mu(r, \infty) = 0\} \in (0, \infty]$, and assume that $\mu(\{r_\mu\}) = 0$. For $p \in (0, \infty)$, write $A^p(\mu)$ for the set of analytic functions $f$ defined for $|z| < r_\mu$ that satisfy

$$\int_0^{r_\mu} \int_0^1 |f(re^{2\pi i \theta})|^p \, d\theta \, d\mu(r) < \infty.$$ 

When $r_\mu = 1$, these spaces are referred to as **weighted Bergman spaces**, whereas when $r_\mu = \infty$, they are called **weighted Bargmann–Fock spaces**. Clearly all spaces $A^p(\mu)$ when $r_\mu$ is finite are isomorphic to weighted Bergman spaces. Denote the unit disk by $\mathbb{D} := \{z : |z| < 1\}$. The unweighted Bergman spaces $A^p(\mathbb{D})$ correspond to $d\mu(r) = 2r \mathbf{1}_{[0,1]}(r) \, dr$, whereas the unweighted Bargmann–Fock spaces $B^p(\mathbb{C})$ correspond to $d\mu(r) = 2r e^{-r^2} \, dr$. The most-studied weights are $d\mu(r) = 2(1-r^2)^{\alpha} \mathbf{1}_{[0,1]}(r) \, dr$ ($\alpha > -1$), in which case the corresponding Bergman spaces are denoted $A^p_\alpha(\mathbb{D})$, and $d\mu(r) = 2ar e^{-ar^2} \, dr$ ($\alpha > 0$), in which case the corresponding Bargmann–Fock spaces are denoted $B^p_\alpha(\mathbb{C})$.

A **standard complex Gaussian** random variable is one whose density with respect to Lebesgue measure on $\mathbb{C}$ is $z \mapsto e^{-|z|^2}/\pi$. We always consider the zero set $Z(f)$ of an analytic function $f$ as a multiset or a sequence, where each zero $w$ is listed with its multiplicity, which is $m$ if $z \mapsto f(z)/(z-w)^m$ is analytic and does not vanish at $w$.

Our main result is the following.

**Theorem 1.1.** Let $\mu$ be a finite measure on $(0, \infty)$ with $\mu(\{r_\mu\}) = 0$. Let $p \in (0, \infty)$. Suppose that $a_n \geq 0$ satisfy $\limsup_{n \to \infty} a_n^{1/n} < \infty$ and $r \mapsto \sum_{n=0}^\infty a_n^2 r^{2n} \notin L^{p/2}(\mu)$. Let $F(z) := \sum_{n=0}^\infty a_n \zeta_n z^n$ for $|z| < r_\mu$, where $\zeta_n$ are independent complex Gaussian random variables. Then a.s. the only analytic function $f \in A^p(\mu)$ with $Z(f) \supseteq Z(F)$ is $f \equiv 0$.

Note that if $r \mapsto \sum_{n=0}^\infty a_n^2 r^{2n} \notin L^{p/2}(\mu)$ for all $p > p_0$, then by considering a countable set of $p > p_0$, we may conclude that a.s. for all $p > p_0$, the only analytic function $f \in A^p(\mu)$ with $Z(f) \supseteq Z(F)$ is $f \equiv 0$.

The following corollary, in the special case where $\mu_1 = \mu_2$, was known for Bergman spaces (Horowitz, 1974).

**Corollary 1.2.** Let $R \in (0, \infty)$. Let $\mu_i$ ($i = 1, 2$) be finite measures with $r_{\mu_i} = R$. Let $p_i \in (0, \infty)$ ($i = 1, 2$). Suppose that there exist $a_n \geq 0$ that satisfy $\limsup_{n \to \infty} a_n^{1/n} < \infty$ and $r \mapsto \sum_{n=0}^\infty a_n^2 r^{2n} \notin L^{p_i/2}(\mu_i)$ for all $p_i > p_{0i}$, and define $g \in A^{p_1}(\mu_1) \setminus L^{p_2}(\mu_2)$. Then there is a function $f \in A^{p_1}(\mu_1)$ such that the only $g \in A^{p_2}(\mu_2)$ with $Z(g) \supseteq Z(f)$ is $g \equiv 0$.

We will actually prove a quantitative version of Theorem 1.1. Write $A(\mu)$ for the set of functions that are analytic in $\{z : |z| < r_\mu\}$. For $s \leq r_\mu$ and $f \in A(\mu)$, write $Z_s(f)$ for the multiset of $z$ with $f(z) = 0$ and $0 < |z| < s$. Denote

$$\|f\|_{A^p(\mu,s)} := \left(\int_0^s \int_0^1 |f(re^{2\pi i \theta})|^p \, d\theta \, d\mu(r)\right)^{1/p}.$$ 

We also abbreviate

$$\|f\|_{A^p(\mu)} := \|f\|_{A^p(\mu,r_\mu)}.$$
Given a sequence \( \{a_n; n \geq 0\} \), write \( a^{(r)} \) for the sequence \( \{a_n r^n; n \geq 0\} \) and \( \|a^{(r)}\|_2 \) for its \( \ell^2 \)-norm.

**Theorem 1.3.** Let \( a_n \geq 0 \) satisfy \( R^{-1} := \limsup_{n \to \infty} a_n^{1/n} < \infty \) and \( a_0 \neq 0 \). Let \( F(z) := \sum_{n=0}^{\infty} a_n \zeta_n z^n \) for \( |z| < R \), where \( \zeta_n \) are independent complex Gaussian random variables. Then for all finite measures \( \mu \) with \( r_\mu = R \) and \( \mu(\{ R \}) = 0 \), all \( p \in (0, \infty) \), and all \( s \in (0, R] \),

\[
E \left[ \max \left\{ \frac{|f(0)|}{\|f\|_{A^p(\mu, s)}} \; : \; 0 \neq f \in A(\mu), \; Z(f) \supset Z_s(F) \right\} \right] \leq \frac{\sqrt{\pi} a_0}{(\int_0^s \|a^{(r)}\|_2^p d\mu(r))^{1/p}}.
\]  

(1.1)

**Proof of Theorem 1.1 from Theorem 1.3.** Consider \( 0 \neq f \in A(\mu) \) with \( Z(F) \subset Z(f) \); we will show that \( \|f\|_{A^p(\mu)} = \infty \).

Without loss of generality, we may shift the indices of the \( a_n \) so that \( a_0 \neq 0 \), since this does not affect the condition \( \|a^{(r)}\|_2 \notin L^p(\mu) \), and it does not change \( Z_R(F) \). Thus, Theorem 1.3 applies.

If \( f(0) \neq 0 \), then the result follows directly from (1.1) by taking \( s = R \). Otherwise, we may reduce to this case: Let \( m \) denote the order of vanishing of \( f \) at 0, and let \( g(z) := f(z)/z^m \), so that \( g \in A(\mu) \) and \( g(0) \neq 0 \). We then have \( Z(g) \supset Z_R(F) \), from which we conclude that \( \|g\|_{A^p(\mu)} = \infty \). This clearly implies that also \( \|f\|_{A^p(\mu)} = \infty \), as desired. \( \Box \)

We also establish the following theorem, which relates to unions of zero sets in the special case \( b \equiv -1 \) upon observing that \( Z(F^N - 1) = \bigcup_{i=0}^{N-1} Z(F - e^{2\pi i k/N}) \).

**Theorem 1.4.** Let \( \mu \) be a finite measure on \((0, \infty)\) with \( \mu(\{ r_\mu \}) = 0 \). Let \( p \in (0, \infty) \).

Suppose that \( a_n \geq 0 \) satisfy \( \limsup_{n \to \infty} a_n^{1/n} < \infty \) and \( r \mapsto \sum_{n=0}^{\infty} a_n^2 r^{2n} \notin L^{p/2}(\mu) \). Let \( F(z) := \sum_{n=0}^{\infty} a_n \zeta_n z^n \) for \( |z| < r_\mu \), where \( \zeta_n \) are independent complex Gaussian random variables. Let \( b \in A(\mu) \) and \( N \) be a positive integer. Then a.s. the only analytic function \( f \in A^{p/N}(\mu) \) with \( Z(f) \supset Z(F^N + b) \) is \( f \equiv 0 \).

This establishes the full conjecture of Shapiro (1979) and enlarges the set of \( b \) to which it applies. Since \( Z(F \pm 1) \) are a.s. simple (see Peres and Virág (2005, Lemma 28)) and \( Z(F \pm 1) \) are disjoint, we obtain the following corollary.

**Corollary 1.5.** Let \( R \in (0, \infty] \). Let \( \mu_i \) \( (i = 1, 2) \) be finite measures with \( r_{\mu_i} = R \) and \( \mu_i(\{ R \}) = 0 \). Let \( p_i \in (0, \infty) \) \( (i = 1, 2) \). Suppose that there exist \( a_n \geq 0 \) that satisfy \( \limsup_{n \to \infty} a_n^{1/n} < \infty \) and \( r \mapsto \sum_{n=0}^{\infty} a_n^2 r^{2n} \in L^{p_i/2}(\mu_1) \setminus L^{p_2/2}(\mu_2) \). Then there are functions \( f_1, f_2 \in A^{p_i}(\mu_1) \) such that \( Z(f_1) \cap Z(f_2) = \emptyset \) and the only \( g \in A^{p_2/2}(\mu_2) \) with \( Z(g) \supseteq Z(f_1) \cup Z(f_2) \) is \( g \equiv 0 \).

Again, we prove a quantitative version of Theorem 1.4:

**Theorem 1.6.** Let \( a_n \geq 0 \) satisfy \( R^{-1} := \limsup_{n \to \infty} a_n^{1/n} < \infty \). Let \( F(z) := \sum_{n=0}^{\infty} a_n \zeta_n z^n \) for \( |z| < R \), where \( \zeta_n \) are independent complex Gaussian random variables. Then for all finite
measures $\mu$ with $r_\mu = R$ and $\mu(\{R\}) = 0$, all $p \in (0, \infty)$, all $b \in A(\mu)$, all positive integers $N$, and all $s \in (0, R]$,

$$
E\left[ \max\left\{ \frac{|f(0)|^{1/N}}{\|f\|_{A^p(\mu,s)}^{1/N}} : 0 \neq f \in A(\mu), Z(f) \supset Z_s(\mathcal{F}^N + b) \right\} \right] \leq \frac{c}{(\int_0^s \|a^{(r)}\|_2^p d\mu(r))^{1/p}},
$$

where

$$
c := \left( a_0^{4N}(2N)! + 4|b(0)|^2 a_0^{2N}N! + |b(0)|^4 \right)^{1/4N} \Gamma\left( \frac{2N - 1}{4N - 1} \right) \frac{4N-1}{4N}.
$$

Therefore, if $\|a^{(r)}\|_2 \notin L^p(\mu)$, then a.s. every $f \in A(\mu)$ with $Z(f) \supset Z(\mathcal{F}^N + b)$ satisfies $\|f\|_{A^p(\mu)} = \infty$.

Of course, what allows Gaussian series to have these properties is that such series have many zeros. A quantitative form of this property is what lies behind our results. Recall that by the arithmetic mean-geometric mean inequality (or Jensen’s inequality) and Jensen’s formula, every $f \in A(\mu)$ with $f(0) \neq 0$ satisfies

$$
\|f\|_{A^p(\mu)}^p = \int_0^{r_\mu} \int_0^1 |f(re^{2\pi i \theta})|^p d\theta d\mu(r) \geq \int_0^{r_\mu} \exp \int_0^1 \log |f(re^{2\pi i \theta})|^p d\theta d\mu(r)
= \int_0^{r_\mu} |f(0)|^p \prod_{z \in Z(f)} \max\left\{ \frac{r^p}{|z|^p}, 1 \right\} d\mu(r).
$$

In general, this inequality can be very far from an equality; for two simple examples, consider $f(z) := (1 - z)^{-1}$ and $p \geq 2$ or $f(z) := e^{(1 - z)^{-1}}$ and all $p$. What we will show, in contrast, is that for $f = \mathcal{F}$, a.s. finiteness of the right-hand side of (1.2) implies a.s. finiteness of the left-hand side and even finiteness of the expectation of the left-hand side. This is reminiscent of Fernique’s theorem (Appendix A), but the functional on the right-hand side does not satisfy the hypotheses of Fernique’s theorem. Moreover, Fernique’s theorem gives finiteness of a moment defined in terms of the original functional, whereas here, the $A^p(\mu)$-norm is, as we just illustrated, not in any way a function of the right-hand side.

**Theorem 1.7.** Let $a_n \geq 0$ satisfy $R^{-1} := \limsup_{n \to \infty} a_1^{1/n} < \infty$ and $a_0 \neq 0$. Let $\mathcal{F}(z) := \sum_{n=0}^\infty a_n \zeta_n z^n$ for $|z| < R$, where $\zeta_n$ are independent complex Gaussian random variables. Then for all finite measures $\mu$ with $r_\mu = R$ and $\mu(\{R\}) = 0$ and all $p \in (0, \infty)$, the following are equivalent:

(i) $\int_0^R \exp \int_0^1 \log |\mathcal{F}(re^{2\pi i \theta})|^p d\theta d\mu(r) < \infty$ a.s.;

(ii) $E[\|\mathcal{F}\|_{A^p(\mu)}^p] < \infty$;

(iii) $E[\int_0^R \exp \int_0^1 \log |\mathcal{F}(re^{2\pi i \theta})|^p d\theta d\mu(r)] < \infty$;

(iv) $\int_0^R \exp \int_0^1 \log |\mathcal{F}(re^{2\pi i \theta})|^p d\theta d\mu(r) < \infty$ with positive probability.
Moreover, for all $s \in (0, R]$,

$$
\mathbb{E} \left[ \left( \int_0^s \exp \int_0^1 \log |F(re^{2\pi i \theta})|^p d\theta d\mu(r) \right)^{1/p} \right] \leq \frac{\sqrt{\pi} \Gamma(1 + p/2)^{1/p} a_0}{\mathbb{E}[\|F\|_{A_p(\mu,s)}^p]^{1/p}}. \tag{1.3}
$$

The equivalence shown here may be surprising; indeed, in discussing his conjecture, Shapiro (1979) wrote that the arithmetic mean-geometric mean inequality “seems to give away too much.”

### 1.1 History of Zero Sets

Given a collection $A$ of analytic functions, say that $Z$ is an $A$-zero set if there is some function in $A$ whose zero set equals $Z$. There is no geometric characterization known for a set of points in $D$ to be an $A^p(D)$-zero set, but there are necessary conditions known that are not far from known sufficient conditions. It is also known that no condition depending solely on the moduli of the points can be both necessary and sufficient. For further discussion, let $z$ be a countable multiset in $D$ and write

$$
\varphi_z(r) := \sum_{z \in z, |z| \leq r} (1 - |z|).
$$

The situation for zeros of Bergman functions contrasts strongly with that for the Hardy spaces,

$$
H^p(D) := \{ f \in A^0(D); \sup_{r < 1} \int_0^1 |f(re^{2\pi i \theta})|^p d\theta < \infty \},
$$

where for all $p \in (0, \infty]$, the **Blaschke condition**

$$
\varphi_z(1) < \infty
$$

is necessary and sufficient to be an $H^p(D)$-zero set. For every $p \in (0, \infty)$, the Blaschke condition is sufficient to be an $A^p(D)$-zero set (since $H^p(D) \subset A^p(D)$), while the condition

$$
\sum_{z \in z \setminus \{0\}} \frac{(1 - |z|)}{\log^{1+\epsilon}(1 - |z|^{-1})} < \infty
$$

is known to be necessary for every $\epsilon > 0$ but not for $\epsilon = 0$ (Horowitz, 1974). On the other hand, if a subset of $z$ lies on a line (or in a Stolz angle), then the Blaschke condition for that subset is also necessary for $z$ to be an $A^p(D)$-zero set (Shapiro and Shields, 1962). Combining the preceding results, we deduce that the moduli alone do not determine whether a point set is an $A^p(D)$-zero set.
Horowitz (1974) showed that for $0 < p < q < \infty$, there exists $f \in A^p(\mathbb{D})$ whose zeros are not the zeros of any function in $A^q(\mathbb{D})$. In fact, he showed that if $f \in A^q(\mathbb{D})$ with zero set $\{z_k; k \geq 1\}$ ordered so that $|z_k|$ is increasing and $f(0) \neq 0$, then

$$\sup_n n^{-1/q} \prod_{k=1}^{n} \frac{1}{|z_k|} < \infty,$$  \hspace{1cm} (1.4)

whereas for every $p < q$, there is some $f \in A^p(\mathbb{D})$ with zero set $\{z_k; k \geq 1\}$ ordered so that $|z_k|$ is increasing and $f(0) \neq 0$ satisfying

$$\sup_n n^{-1/q} \prod_{k=1}^{n} \frac{1}{|z_k|} = \infty.$$

(Since (1.4) depends only on the moduli, it is not sufficient to be a zero set.) This distinction among the zero sets for different $p$ was refined by Shapiro (1979): for $0 < p < \infty$, there exists $f \in A^p(\mathbb{D})$ whose zeros are not the zeros of any function in $A^{p+}(\mathbb{D})$, where $A^{p+}(\mathbb{D}) := \bigcup_{q > p} A^q(\mathbb{D})$.\footnote{Shapiro (1979) did this by using random (Gaussian) series, as we detail soon.}

Later works by LeBlanc (1990), Bomash (1992), and Nowak and Waniurski (2003) considered random angles for fixed moduli, culminating in the following result.

**Theorem 1.8.** Let $0 < p < \infty$ and $z = \{z_n; n \geq 1\} \subset \mathbb{D}$. Let $\theta_n$ be independent uniform $[0,1]$ random variables. If there exists $\epsilon > 0$ such that

$$\int_{0}^{1} e^{p\varphi_\epsilon(r)} \log^{(1+\epsilon)}(1-r)^{-1} dr < \infty,$$ \hspace{1cm} (1.5)

then a.s. $\{z_n e^{2\pi i \theta_n}; n \geq 1\}$ is an $A^p(\mathbb{D})$-zero set. If $q > p$, then the condition (1.5) is not sufficient for $\{z_n e^{2\pi i \theta_n}; n \geq 1\}$ to be a.s. an $A^q(\mathbb{D})$-zero set.

The Blaschke condition shows that the union of two $H^p(\mathbb{D})$-zero sets is again an $H^p(\mathbb{D})$-zero set. Horowitz (1974) also showed that although the union of two $A^p(\mathbb{D})$-zero sets is an $A^{p/2}(\mathbb{D})$-zero set (trivially: just multiply the functions), it need not be an $A^q(\mathbb{D})$-zero set if $q > p/2$. This was again strengthened by Shapiro (1979) to show that it need not be an $A^{(p/2)+}(\mathbb{D})$-zero set.\footnote{Actually, there was a gap in his proof: in the middle of page 168 where the quantity $I(r)$ is being bounded below, going from the integral over $\mathbb{T}$ to $E^{\omega}(u)$ throws away a part that may be negative, so the inequality does not follow. Thus, it seems that our proof of Corollary 1.2 is the first valid proof of (Shapiro, 1979, Theorem 1 (i) implies (iii)).}

Many of the above results were extended to weighted Bergman spaces. For example, for $(p, \alpha) \in (0, \infty) \times (-1, \infty)$, Horowitz (1974) studied the zero sets of the spaces $A^p_\alpha(\mathbb{D})$, showing that they were distinct classes of sets for pairs with distinct values of $(\alpha + 1)/p$, provided

\footnote{The same gap as noted in the previous footnote applies to this result, but is filled by our Corollary 1.5.}
that $\alpha \geq 0$. He asked whether it sufficed that the pairs $(p, \alpha)$ be distinct. The proviso that $\alpha \geq 0$ was removed by Sedletskiǐ (1987). The full question was answered affirmatively by Sevast’yanov and Dolgoborodov (2013). Our Corollary 1.2 easily establishes the result of Sedletskiǐ: if $((a_1 + 1)/p_1 < (a_2 + 1)/p_2 =: \beta$, then the function $f(r) := (1 - r^2)^{-\beta}$ has nonnegative coefficients and satisfies $\int_0^1 f(r) r^n (1 - r)^{\alpha_1} \, dr < \infty$ and $\int_0^1 f(r) r^{p_2} (1 - r)^{\alpha_2} \, dr = \infty$, whence we may use the Gaussian analytic function corresponding to $a_n := |(z_n^\beta)|^{1/2}$ to obtain a (random) $A_{p_1}^p(\mathbb{D})$-zero set that is not an $A_{p_2}^p(\mathbb{D})$-zero set. We do not know whether the full result of Sevast’yanov and Dolgoborodov (2013) can be deduced from our result, Corollary 1.2.

Very little is known about the zero sets of functions in the Bargmann–Fock spaces, even for $p = 2$. Zhu (1993) showed that if $f \in B_0^p(\mathbb{C})$ with $f(0) \neq 0$ and we write $Z(f) = z$ as a sequence in increasing order of modulus, then $\inf_n |z_n| / \sqrt{n} > 0$. On the other hand, classical results show that if $z$ satisfies $\sum_n |z_n|^{-2} < \infty$, then there is some $f \in B_0^p(\mathbb{C})$ with $Z(f) = z$ (see Zhu (2012, Theorem 5.3)).

Chistyakov, Lyubarskii, and Pastur (2001) considered particular stationary random point processes and showed that for $p = 2$, the critical density for being a $B^p(\mathbb{C})$-zero set is 1; no information was provided about processes with density 1.

Our results give new proofs of results of Zhu (1993) and answer his question, showing that the zero sets of $B_0^p(\mathbb{C})$ depend on $p$ for fixed $\alpha$; he had shown that they differ for differing $\alpha$, whether or not $p$ is fixed. We also strengthen his result that there is a union of two disjoint $B_0^p(\mathbb{C})$-zero sets that is not a $B_0^p(\mathbb{C})$-zero set; indeed, we can find such sets $Z_1, Z_2$ such that for all $q > p$ (simultaneously), $Z_1 \cup Z_2$ is not contained in a $B_0^{q/2}(\mathbb{C})$-zero set.

### 1.2 Shapiro’s Approach

Consider $F(z) := \sum_{n=0}^\infty a_n \zeta_n z^n$, where $\zeta_n$ are IID standard complex Gaussian random variables and $a_n > 0$ satisfy $\limsup_n a_n^{1/n} \leq 1$. Because $\mathbb{E} \left[ \log^+ |\zeta_0| \right] < \infty$, we also have $\limsup_n |\zeta_n|^{1/n} = 1$ a.s. by the Borel-Cantelli lemma, whence a.s. $F(z)$ converges for all $z \in \mathbb{D}$ to an analytic function.

Let $\mu$ be a finite measure with $r_\mu = 1$. Write $L^{p+} := \bigcup_{q>p} L^q(\mu)$. Shapiro (1979) showed that the following are equivalent:

1. $r \mapsto \|a^{(r)}\|_2 \in L^p(\mu) \setminus L^{p+}(\mu)$;
2. a.s. $F \in A^p(\mu) \setminus A^{p+}(\mu)$;
3. a.s. $F \in A^p(\mu)$ and the only function in $A^{p+}(\mu)$ that vanishes everywhere that $F$ does is the 0 function.

In addition, he showed that when (1) holds,

a.s. $F \pm 1 \in A^p(\mu)$ and the only function in $A^{(p/2)+}(\mu)$ that vanishes on $Z(F^2 - 1)$ is the 0 function.
He conjectured that the following strengthening holds:
\[
    r \mapsto \|a^{(r)}\|_2 \notin L^p(\mu) \implies \\
    \text{a.s. the only function in } A^p(\mu) \text{ that vanishes on } Z(\mathbf{F}) \text{ is the 0 function and } \\
    \text{the only function in } A^{p/2}(\mu) \text{ that vanishes on } Z(\mathbf{F}^2 - 1) \text{ is the 0 function.}
\]

More generally, he conjectured Theorem 1.4 when \( r_\mu = 1 \) and \( b \) satisfies a certain restriction. The equivalence of (1) and (2) follows from the following equivalence:
\[
r \mapsto \|a^{(r)}\|_2 \in L^p(\mu) \iff \text{a.s. } \mathbf{F} \in A^p(\mu).
\]
(1.6)

To see this, note that for each \( z \in \mathbb{D} \), the random variable \( \mathbf{F}(z) \) has the same distribution as \( \|a(|z|)\|_2 \zeta_0 \). Thus, Tonelli’s theorem yields
\[
    \mathbb{E}[\|\mathbf{F}\|_{A^p(\mu)}^p] = \mathbb{E}[\zeta_0^p] \cdot \int_0^1 \|a^{(r)}\|_2^p \, d\mu(r).
\]
(1.7)

The forward implication of (1.6) is now immediate. The reverse implication is a consequence of (1.7) and Fernique’s theorem, which tells us that if \( \mathbf{F} \) a.s. belongs to \( A^p(\mu) \), then there exist some \( c_0, c_1 > 0 \) such that \( \mathbb{E}[\exp(c_0 \|\mathbf{F}\|_{A^p(\mu)}^c)] < \infty \). (See Appendix A for a statement and proof of a general form of Fernique’s theorem.)

The usefulness of Shapiro’s approach comes partly from his implicit observation that given \( \mu \) and \( p \), there exists \( r \mapsto \sum_{n=0}^\infty a_n^2 r^{2n} \in L^{p/2}(\mu) \setminus \bigcup_{q>p} L^{q/2}(\mu) \). This follows from the lemma in Section 3 of Shapiro (1977), where he considers analytic functions, not just real power series. For completeness, we give a short proof here. (It will not matter whether \( r_\mu \) is finite.) For each \( M > 0 \), let \( q := p + 1/M \) and let \( \delta = \delta(M) \) be small enough so that \( \mu(r_\mu - \delta, r_\mu) \leq M^{-p/q(q-p)} \). We can find \( N = N(M) \) large enough so that
\[
    \int_{r_\mu - \delta}^{r_\mu} r^{Np} \, d\mu(r) \geq \frac{1}{2} \int_0^{r_\mu} r^{Np} \, d\mu(r).
\]
We also have by the power-mean inequality that
\[
    \left( \int_{r_\mu - \delta}^{r_\mu} r^{Nq} \, d\mu(r) \right)^\frac{1}{q} \geq \mu(r_\mu - \delta, r_\mu)^\frac{1}{q} \left( \int_{r_\mu - \delta}^{r_\mu} r^{Np} \, d\mu(r) \right)^\frac{1}{p} \geq M^2 \left( \int_{r_\mu - \delta}^{r_\mu} r^{Np} \, d\mu(r) \right)^\frac{1}{p}.
\]

Now, let \( n_k := N(2^k) \), and choose \( b_k > 0 \) so that \( \left( \int_0^{r_\mu} b_k^p r^{np} \, d\mu(r) \right)^{2/p} = 1/2^k \). Then \( \sum_{k=1}^\infty b_k^2 r^{2n_k} \) has the desired property: Write \( \|f\|_p := \left( \int_0^{r_\mu} |f(r)|^p \, d\mu(r) \right)^{1/p} \). If \( p \geq 2 \), then
\[
    \left\| \sum_{k=1}^\infty b_k^2 r^{2n_k} \right\|_{p/2} \leq \sum_{k=1}^\infty \left\| b_k^2 r^{2n_k} \right\|_{p/2} = 1,
\]
while if \( p < 2 \), then
\[
    \left\| \sum_{k=1}^\infty b_k^2 r^{2n_k} \right\|_{p/2} \leq \sum_{k=1}^\infty \left\| b_k^2 r^{2n_k} \right\|_{p/2} < \infty.
\]
At the same time, for each \( q > p \), consider any \( j \) with \( q > p + 1/2 \). Then
\[
\left\| \sum_{k=1}^{\infty} b_k^2 r^{2nk} \right\|_{q/2} \geq \left\| b_j^2 r^{2nj} \right\|_{q/2} \geq b_j^2 2^{2j} \left( \int_{r_{j-1}/2}^{r_j} r^{nj} d\mu(r) \right)^{2/p} \\
\geq b_j^2 2^{2j} \left( \frac{1}{2} \int_{0}^{r_j} r^{nj} d\mu(r) \right)^{2/p} = 2^{j-2/p}.
\]
Since this holds for all such \( j \), it follows that \( \left\| \sum_{k=1}^{\infty} b_k^2 r^{2nk} \right\|_{q/2} = \infty \).

2 Proofs

In this section, we prove Theorem 1.3 and then indicate the additional steps needed for the more general Theorem 1.6. At the end, we prove Theorem 1.7.

**Proof of Theorem 1.3.** Note that the density of \(|\zeta_0|\) with respect to Lebesgue measure on \( \mathbb{R}^+ \) is \( r \mapsto r e^{-r^2/2} \). It suffices to prove the theorem for \( s < r\mu \), since the case \( s = r\mu \) follows by taking limits.

Suppose that \( 0 \notin Z(f) \supseteq Z(F) \). Note that \( 0 \notin Z(F) \) a.s. Thus, for \( 0 < s < r\mu \), we have a.s. by the arithmetic mean-geometric mean inequality and Jensen’s formula that
\[
\|f\|_{A^p(\mu,s)}^p = \int_{0}^{s} \int_{0}^{1} |f(re^{2\pi i \theta})|^p d\theta d\mu(r) \geq \int_{0}^{s} \exp \int_{0}^{1} \log |f(re^{2\pi i \theta})|^p d\theta d\mu(r) \\
= \int_{0}^{s} |f(0)|^p \prod_{z \in Z_s(f)} \max\left\{ \frac{r^p}{|z|^p}, 1 \right\} d\mu(r) \geq \int_{0}^{s} |f(0)|^p \prod_{z \in Z_s(F)} \max\left\{ \frac{r^p}{|z|^p}, 1 \right\} d\mu(r) \\
= |f(0)/F(0)|^p \int_{0}^{s} \exp \int_{0}^{1} \log |F(re^{2\pi i \theta})|^p d\theta d\mu(r).
\]
Therefore,
\[
|f(0)| \leq a_0 |\zeta_0| \left( \int_{0}^{s} \exp \int_{0}^{1} \log |F(re^{2\pi i \theta})|^p d\theta d\mu(r) \right)^{-1/p}.
\]
Recall that for each \( r \) and \( \theta \), \( F(re^{2\pi i \theta}) \) is a Gaussian random variable with the same distribution as \( \|a(r)\|_2 \zeta_0 \), where \( a_n(r) := a_n r^n \). Write
\[
G_r(\theta) := F(re^{2\pi i \theta})/\|a(r)\|_2,
\]
so that \( G_r(\theta) \) is a standard complex Gaussian random variable for each \( r \) and \( \theta \). Hölder’s
inequality and the arithmetic mean-geometric mean inequality yield
\[
\left( \int_0^s \exp \int_0^1 \log |F(re^{2\pi i \theta})|^p \, d\theta \, d\mu(r) \right)^{-1/p} = \left( \int_0^s \|a(r)\|_p^p \exp \int_0^1 \log |G_r(\theta)|^p \, d\theta \, d\mu(r) \right)^{-1/p}
\leq \left( \int_0^s \|a(r)\|_p^p \exp \int_0^1 \log |G_r(\theta)|^{-1} \, d\theta \, d\mu(r) \right) \left( \int_0^s \|a(r)\|_2^2 \, d\mu(r) \right)^{1+1/p}
\leq \left( \int_0^s \|a(r)\|_p^p \right)^{1+1/p} \left( \frac{\int_0^s \|a(r)\|_2^2 \, d\mu(r)}{\int_0^s \|a(r)\|_2^2 \, d\mu(r)} \right).
\] (2.2)

Multiplying both sides by \(a_0|\zeta_0|\) and using (2.1), we have
\[
\frac{|f(0)|}{\|f\|_{Ap(\mu,s)}} \leq \frac{a_0|\zeta_0| \cdot \int_0^s \|a(r)\|_p^p \int_0^1 |G_r(\theta)|^{-1} \, d\theta \, d\mu(r)}{\left( \int_0^s \|a(r)\|_2^2 \, d\mu(r) \right)^{1+1/p}}.
\] (2.3)

Recall that for each \(r\) and \(\theta\), \(G_r(\theta)\) and \(\zeta_0\) are both standard complex Gaussians, and \((G_r(\theta), \zeta_0)\) is jointly Gaussian. By a version of Slepian’s lemma due to Kahane (1986), we have
\[
E[|\zeta_0| : |G_r(\theta)|^{-1}] \leq E[|\zeta_0|] E[|G_r(\theta)|^{-1}] = 1 \cdot \sqrt{\pi}.
\]

Taking expectations in (2.3) and applying the above inequality finishes the proof, except for showing that the maximum on the left-hand side of (1.1) is achieved and is measurable.

To show these properties, note first that the maximum is achieved because of a standard normal-families argument (compare Duren and Schuster (2004, p. 120)). Next, for a finite multiset \(W\), let \(p_W(z) := \prod_{w \in W}(z - w)\) be the monic polynomial whose zeros are \(W\) (with multiplicity). For any analytic function \(f\) whose zeros include \(W\), the function \(f/p_W\) is analytic. Therefore,
\[
\max \left\{ \frac{|f(0)|}{\|f\|_{Ap(\mu,s)}} : 0 \neq f \in A(\mu), Z(f) \supset Z_s(F) \right\} = \max \left\{ \frac{|f(0)p_{Z_s(F)}(0)|}{\|fp_{Z_s(F)}\|_{Ap(\mu,s)}} : 0 \neq f \in A(\mu) \right\}.
\]

Restricting to polynomials \(f\) with rational coefficients, we see that this maximum is measurable provided \(p_{Z_s(F)}\) is measurable. Now there is a measurable set (of probability 0) where \(\limsup |\zeta_0|^{1/n} \geq 1\); off of this set, \(Z_s(F)\) is finite and can be determined by looking at the values of \(F\) on a fixed, countable, dense set of points, thereby proving the desired measurability.

\[\square\]

**Remark 2.1.** In fact, Theorem 1.1 may be deduced directly from (2.2) without using Slepian’s lemma: Simply take expectations of both sides and use the facts that \(E[|G_r(\theta)|^{-1}] = \sqrt{\pi}\) and \(|\zeta_0| < \infty\) to obtain
\[
E\left[ \left( \int_0^s \exp \int_0^1 \log |F(re^{2\pi i \theta})|^p \, d\theta \, d\mu(r) \right)^{-1/p} \right] \leq \frac{\sqrt{\pi}}{\left( \int_0^s \|a(r)\|_p^p \, d\mu(r) \right)^{1/p}}.
\]

As \(s \uparrow r_\mu\), the right-hand side tends to 0, which already gives Theorem 1.1 via (2.1).
Proof of Theorem 1.6. We may assume that \( a_0 \neq 0 \). Suppose that \( 0 \notin Z(f) \supseteq Z(F) \). As before, we have for \( 0 < s < r_\mu \)
\[
\left( \frac{|f(0)|}{\|f\|_{AP^{(N)}(\mu,s)}} \right)^{1/N} \leq |a_0^N \zeta_0^N + b(0)|^{1/N} \left( \int_0^s \exp \int_0^1 \log |F(re^{2\pi i \theta})^N + b(re^{2\pi i \theta})|^p d\theta d\mu(r) \right)^{-1/p}.
\]
Write
\[
H_r(\theta) := |F(re^{2\pi i \theta})^N + b(re^{2\pi i \theta})|/\|a(r)\|^N.
\]
In the same way as before, we obtain
\[
\left( \int_0^s \exp \int_0^1 \log |F(re^{2\pi i \theta})^N + b(re^{2\pi i \theta})|^p d\theta d\mu(r) \right)^{-1/p} = \left( \int_0^s \|a(r)\|_{2}^p \exp \int_0^1 \log H_r(\theta)^{p/N} d\theta d\mu(r) \right)^{-1/p}.
\]
\[
\leq \frac{\int_0^s \|a(r)\|_{2}^p \exp \int_0^1 \log H_r(\theta)^{-1/N} d\theta d\mu(r)}{\left( \int_0^s \|a(r)\|_{2}^p d\mu(r) \right)^{1+1/p}}.
\]
\[
\leq \frac{\int_0^s \|a(r)\|_{2}^p \exp \int_0^1 \log H_r(\theta)^{-1/N} d\theta d\mu(r)}{\left( \int_0^s \|a(r)\|_{2}^p d\mu(r) \right)^{1+1/p}}.
\]
(2.4)

We have for any \( \beta \) that
\[
\mathbb{E}[H_r(\theta)^{-\beta}] = \mathbb{E}[|\zeta_0^N + b(re^{2\pi i \theta})|/\|a(r)\|^N]^{-\beta}.
\]
Now \( \zeta_0^N \) has density \( \rho: z \mapsto c|z|^{-2(N-1)/N}e^{-|z|^2} \) (with respect to area measure \( \lambda_2 \), for some constant \( c \)) that is decreasing in \( |z| \). Therefore, given any \( \alpha \in \mathbb{C} \), the rearrangement inequality of Hardy and Littlewood yields
\[
P[|\zeta_0^N| < r] = \int_{|z|<r} \rho(z) d\lambda_2(z) \geq \int_{|z-\alpha|<r} \rho(z) d\lambda_2(z) = P[|\zeta_0^N - \alpha| < r],
\]
which is to say that \( |\zeta_0^N| \) is stochastically dominated by \( |\zeta_0^N - \alpha| \). Thus, for all \( 0 < \beta < 2/N \),
\[
\mathbb{E}[|\zeta_0^N - \alpha|^{-\beta}] \leq \mathbb{E}[|\zeta_0^N|^{-\beta}] = \int_0^\infty \frac{e^{-t}}{t^{3N/2}} dt = \Gamma(1 - \beta N/2).
\]
Therefore,
\[
\mathbb{E}[H_r(\theta)^{-\beta}] \leq \Gamma(1 - \beta N/2).
\]
(2.5)

Multiply both sides of the inequality (2.4) by \( |a_0^N \zeta_0^N + b(0)|^{1/N} \) and use Hölder’s inequality to bound the resulting expectation:
\[
\mathbb{E}[|a_0^N \zeta_0^N + b(0)|^{1/N} \cdot H_r(\theta)^{-\beta}] \leq \mathbb{E}[|a_0^N \zeta_0^N + b(0)|^4]^{1/4N} \mathbb{E}[H_r(\theta)^{-4/(4N-1)}]^{4N-1/4N}
\]
\[
\leq \left( a_0^{4N}(2N)! + 4|b(0)|^2 a_0^{2N} N! + |b(0)|^4 \right)^{1/4N} \Gamma \left( \frac{2N - 1}{4N - 1} \right)^{4N-1/4N},
\]
where in the last inequality, we used (2.5) with \( \beta := 4/(4N - 1) \).
Proof of Theorem 1.7. We established (1.3) during the proof of Theorem 1.3, where we rely on (1.7) and the fact that \( \mathbb{E}[|c_0|^p] = \Gamma(1 + p/2) \) for an equivalent expression on the right-hand side. That (ii) implies (iii) follows from the arithmetic mean-geometric mean inequality. That (iii) implies (i) and (i) implies (iv) are obvious. That (iv) implies (ii) follows from (1.3) with \( s = R \).

\[ \square \]

### A Fernique’s Theorem

We present here a general version of Fernique’s theorem, not only for use in deriving the background in Subsection 1.2, but also for comparison with our Theorem 1.7.

**Theorem A.1.** Let \( V \) be a separable topological vector space. Let \( \phi : V \to [0, \infty] \) be Borel measurable, \( c \in [1, \infty) \), and \( c_1, c_2 \in (1, \infty) \) satisfy for all \( x, y \in V \) that \( \phi(-x) = \phi(x) \), \( c_2 \phi(x) \leq \phi(\sqrt{2}x) \leq c_1 \phi(x) \), and \( \phi(x + y) \leq c(\phi(x) + \phi(y)) \). Let \( X \) be a random variable with values in \( V \) such that if \( Y \) has the same distribution as \( X \) and is independent of \( X \), then \( \langle \phi(X), \phi(Y) \rangle \) has the same joint distribution as \( \langle \phi(\sqrt{X-Y}), \phi(\sqrt{X+Y}) \rangle \). If \( \mathbb{P} [\phi(X) < \infty] = 1 \), then there are some \( \alpha, \beta > 0 \) so that \( \mathbb{E} e^{\alpha \phi(X)^\beta} < \infty \).

**Proof.** Suppose that \( \phi(\sqrt{2}) \leq \tau \) and \( \phi(\sqrt{2}) > t \). Then \( \phi(x - y) \leq c_1 \tau \) and \( \phi(x + y) > c_2 t \). Also, \( \phi(2y) = \phi(\sqrt{2}y) \leq c_1^2 \phi(y) \), whence \( \phi(x + y) \leq c\phi(x - y) + c c_2^2 \phi(y) \). Therefore \( \phi (y) > (c_2 t/c_1 \tau)/(c c_1^2) \). Symmetry gives the same lower bound on \( \phi (y) \). It follows that

\[
\mathbb{P} [\phi(X) \leq \tau] \mathbb{P} [\phi(Y) > t] = \mathbb{P} [\phi(\sqrt{X-Y}) \leq \tau, \phi(\sqrt{X+Y}) > t] \leq \mathbb{P} [\phi(X) > (c_2 t - c c_1 \tau)/(c c_1^2)]^2. \tag{A.1}
\]

Choose \( \tau < \infty \) so that \( \mathbb{P} [\phi(X) \leq \tau] \geq e/(1 + e) \). Define recursively \( t_0 := (c c_1/c_2) \tau \geq \tau \) and \( t_{n+1} := (c c_1^2/c_2) t_n + t_0 \) for some constant \( c_3 < \infty \). The display (A.1) yields

\[
\mathbb{P} [\phi(X) > t_{n+1}] = \mathbb{P} [\phi(X) > t_{n+1}] \leq \frac{1 + e}{e} \mathbb{P} [\phi(X) > t_{n}]^2,
\]

whence if we write \( y_n := \frac{1 + e}{e} \mathbb{P} [\phi(X) > t_{n}] \), then \( y_{n+1} \leq y_n^2 \), and so \( y_n \leq y_0^2 \). Therefore,

\[
\mathbb{P} [\phi(X) > c_3(c c_1^2/c_2)^n] \leq e^{-2^n} = e^{-(c c_1^2/c_2)^\beta n},
\]

where \( \beta := \log 2/\log(c c_1^2/c_2) > 0 \). This means that

\[
\mathbb{P} [\phi(X) > t] \leq e^{-c_4 t^\beta}
\]

for some \( c_4 > 0 \) and all \( t \geq t_0 \). With \( \alpha := c_4/2 \), the conclusion may be verified via integration by parts. \( \square \)
References


