Random Orderings and Unique Ergodicity of Automorphism Groups

Russell Lyons
Indiana University, Bloomington
Joint work with Omer Angel and Alexander Kechris, JEMS, 2014

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Talk will have two parts:

- a concrete part about finite graphs
- an abstract part about automorphism groups
Special vertices?
Preserved by isomorphism and induced subgraphs?
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Special vertices? Preserved by **isomorphism**
Special vertices? Preserved by **isomorphism** and **induced subgraphs**?
More precisely, we allow linear (total) orderings of \( V(G) \) at random:

\[ G = (V, E) \mapsto \mu_G \text{on} |V| \text{ orderings of} V \text{ that are consistent:} \]

\[ \phi: G \to G' \text{ isomorphism} \Rightarrow \phi^* \mu_G = \mu_G' \]

\[ \text{induced subgraph of} G \Rightarrow \mu_G \text{ induces} \mu_H \text{ by restriction} \]
More precisely, we allow linear (total) orderings of $V(G)$ at random:

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It must be uniform on empty graphs.
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**Question**

The uniform ordering is always consistent. Is there any other?
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Consider the universe only of paths.
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But there is a non-uniform consistent random ordering in this universe.
Consider the universe of trees.
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There is a non-uniform consistent random ordering in this universe.
Consider the universe of forests.
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Every consistent random ordering in this universe is uniform: Balister-Bollobás-Janson (2015+).
Other structures, such as hypergraphs or metric spaces?
The answer for the universe of all finite graphs:
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We prove that graphs have only the uniform ordering as a consistent ordering.

The application:

This implies that the automorphism group of “the random graph” (an infinite graph) is uniquely ergodic, i.e., every minimal action has a unique invariant probability measure.
A quantitative version:

**Theorem (Angel-Kechrís-L.)**

Suppose $G \mapsto \mu_G$ is a consistent ordering on graphs of size $\leq n$. Then for every $H$ of size $k \leq n$ and for every ordering $<_H$ on $V(H)$,

$$\left| \mu_H(<_H) - \frac{1}{k!} \right| \leq C(k) \sqrt{\frac{\log n}{n}}.$$
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To find \(G\), we choose it at random among all \(n\)-vertex graphs (the Erdős-Rényi model).
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To find \(G\), we choose it at random among all \(n\)-vertex graphs (the Erdős-Rényi model). The number of embeddings is very concentrated about its expectation. Given \(<_H\) and \(<_G\), the number of order-preserving embeddings is also very concentrated about its expectation. The ratio of these expectations is exactly \(1/k!\). Now take the union bound over all \(n!\) orderings \(<_G\).
Theorem (Angel-Kechris-L.)

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We don’t know how sharp our upper bound is.
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We don’t know how sharp our upper bound is. We have a lower bound that there is a consistent assignment $G \mapsto \mu_G$ on graphs of size $\leq n$ such that for all $k \in [3, n]$ there is some $H$ of size $k$ and some $<_H$ with

$$\left| \mu_H(<_H) - \frac{1}{k!} \right| \geq \frac{c(k)}{n}.$$
Theorem (Angel-Kechris-L.)

*The only consistent ordering on finite graphs is the uniform random ordering.*
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Let $\Gamma$ be a topological group and $X$ be a compact Hausdorff space. Suppose that $\Gamma$ acts continuously by homeomorphisms on $X$ ($X$ is a $\Gamma$-flow).

What can we say about $\Gamma$-invariant Borel probability measures on $X$? A measure $\mu$ on $X$ is $\Gamma$-invariant if $\mu(\gamma A) = \mu(A)$ for all Borel $A \subseteq X$ and all $\gamma \in \Gamma$. 
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Application

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Suppose from now on that $\Gamma$ is amenable. When is an invariant measure also unique? If $X$ decomposes into invariant compact pieces, then it will not be unique. What if $X$ is minimal, i.e., every $\Gamma$-orbit is dense?
Examples

Assumptions:

- $X$ is a $\Gamma$-flow;
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Examples:

- Suppose $\Gamma$ is compact. Then the invariant measure is unique.
- Suppose that $\Gamma = \mathbb{Z}$ acting by irrational rotation on the circle, $(n, x) \mapsto x + n\alpha \pmod{1}$. Then the invariant measure is unique. But there are minimal $\mathbb{Z}$-flows that have more than one invariant measure.
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- In fact, every countable infinite $\Gamma$ has a minimal flow that has more than one invariant measure, i.e., is not uniquely ergodic.
Application

- $X$ is a $\Gamma$-flow;
- $\Gamma$ is amenable;
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We say $X$ is **uniquely ergodic** if it has a unique $\Gamma$-invariant measure.
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When is every minimal $\Gamma$-flow uniquely ergodic (as when $\Gamma$ is compact, but not when $\Gamma$ is countably infinite)?
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We need consider only one $\Gamma$-flow, the **universal minimal $\Gamma$-flow**, $M(\Gamma)$. Every minimal $\Gamma$-flow is a $\Gamma$-factor of $M(\Gamma)$ (i.e., there is a continuous surjection $\phi: M(\Gamma) \rightarrow X$ that commutes with the $\Gamma$-actions).
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If $\Gamma$ is locally compact but not compact, $M(\Gamma)$ is uniquely ergodic iff every minimal $\Gamma$-flow is uniquely ergodic, in which case we call $\Gamma$ itself uniquely ergodic.
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[These universal flows are not metrizable when $\Gamma$ is locally compact but not compact.]
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**Examples:**

Let $S_\infty$ be the group of *all* permutations of $\mathbb{N}$, with the pointwise convergence topology. Glasner-Weiss (2002) showed that

$$M(S_\infty) = \{\text{all linear orders of } \mathbb{N}\},$$

This was the first example of a uniquely ergodic group other than groups that are compact or extremely amenable (i.e., every $\Gamma$-flow has a fixed point).

(The first natural example of an extremely amenable group is due to Gromov-Milman (1983): $U(\ell^2(\mathbb{N}))$.)

Kechris-Pestov-Todorcevic (2005) gave many more examples of universal minimal flows for closed subgroups of $S_\infty$, showing how this is related to model theory and Ramsey theory.
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Kechris-Pestov-Todorcevic (2005) gave many more examples of universal minimal flows for closed subgroups of $S_\infty$, showing how this is related to model theory and Ramsey theory.
For example, let $R$ be the random graph, i.e., the graph on $\mathbb{N}$ that is equal a.s. to the Erdős-Rényi random graph on $\mathbb{N}$. 
For example, let $\mathcal{R}$ be the random graph, i.e., the graph on $\mathbb{N}$ that is equal a.s. to the Erdős-Rényi random graph on $\mathbb{N}$. Then Kechris-Pestov-Todorcevic showed that

$$M(\text{Aut}(\mathcal{R})) = M(S_\infty) = \{\text{all linear orders of } \mathbb{N}\}.$$ 

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This example and others we give are the next examples of non-compact non-extremely-amenable uniquely ergodic groups.
For metric spaces, it is also true that only the uniform ordering is consistent.
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For graphs, the Ramsey property is the following theorem of Nešetřil and Rödl (1977): Consider only ordered graphs and $q \geq 1$. Suppose that $K$ is an induced subgraph of $H$. Then there is a graph $G$ containing an induced subgraph isomorphic to $H$ such that for any coloring $c : \binom{G}{K} \to \{1, \ldots, q\}$, there is $H' \in \binom{G}{H}$ such that $c|_{\binom{H'}{K}}$ is constant.

(When all graphs are empty, this is the classical theorem of Ramsey: [Equation])
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(When all graphs are empty, this is the classical theorem of Ramsey: If $k < h$, then there is $g$ sufficiently large such that for any coloring $c: \binom{[g]}{k} \to \{1, \ldots, q\}$, there is $H' \in \binom{[g]}{h}$ such that $c\upharpoonright \binom{H'}{k}$ is constant.)