Comparing Return Probabilities

Russell Lyons
Continuous-Time Random Walk

Let \((G, w)\) be a finite graph with \(w : E(G) \to [0, \infty)\). Continuous-time random walk crosses an incident edge \(e\) at rate \(w(e)\). It thus leaves \(x \in V(G)\) at rate \(w(x) := \sum_{e \sim x} w(e)\). Its infinitesimal generator is the negative of the Laplacian \(\Delta_{(G,w)}\) with entries

\[
\Delta(x, y) := \begin{cases} 
-w(x, y) & \text{if } x \neq y \text{ and } x \sim y, \\
0 & \text{if } x \neq y \text{ and } x \not\sim y, \\
w(x) & \text{if } x = y.
\end{cases}
\]

The transition probability \(p_t(x, y)\) is the \((x, y)\)-entry of \(e^{-t\Delta}\), i.e., \(\langle e^{-t\Delta}1_y, 1_x \rangle\). The stationary distribution is uniform.
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The transition probability \(p_t(x, y)\) is the \((x, y)\)-entry of \(e^{-t\Delta}\), i.e., \(\langle e^{-t\Delta} 1_y, 1_x \rangle\). The stationary distribution is uniform (\(1\) is an eigenvector of \(\Delta\) with eigenvalue 0).
The Laplacian $\Delta_{(G,w)}$ is positive semidefinite (written $\Delta \geq 0$):

Proof 1: Each edge corresponds to a multiple of $\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \geq 0$, so $\Delta$ is a sum of p.s.d. matrices.

Proof 2: $\langle \Delta f, f \rangle = \sum_{e \in E(G)} w(e) (f(e^-) - f(e^+))^2 \geq 0$.

The return probabilities $p_t(x,x)$ are monotonic decreasing in $t$:

Proof 1: The eigenvalues $\lambda_i$ of $\Delta$ are nonnegative. If the unit eigenvectors are $f_i$, then $p_t(x,x) = \langle e^{-t\Delta}1_x, 1_x \rangle = |V(G)| \sum_{i=1}^n \lambda_i (f_i(x))^2$.

Proof 2: $d/dt \langle e^{-t\Delta}1_x, 1_x \rangle = \langle -\Delta e^{-t\Delta}1_x, 1_x \rangle = -\langle \Delta e^{-t\Delta/2}1_x, e^{-t\Delta/2}1_x \rangle \leq 0$. 


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  $$\frac{d}{dt} \langle e^{-t\Delta}1_x, 1_x \rangle = \langle -\Delta e^{-t\Delta}1_x, 1_x \rangle = -\langle \Delta e^{-t\Delta/2}1_x, e^{-t\Delta/2}1_x \rangle \leq 0.$$
Individual Return Probabilities
Individual Return Probabilities: Time 1

Are they monotonic in $w$?
Average Return Probability: Time 1

![Graph showing the average return probability over time with nodes labeled 1, w, and 10.]
Theorem, Benjamini and Schramm (pub. by Heicklen and Hoffman (2005))

On every finite graph, $G$, the average return probability at each time is monotonic decreasing in the weights, $w_G$. 

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Proof.

If $A \preceq B$, then the $i$th eigenvalue of $A$ is at most the $i$th eigenvalue of $B$ by the min-max principle.
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If \( A \leq B \), then the \( i \)th eigenvalue of \( A \) is at most the \( i \)th eigenvalue of \( B \) by the min-max principle. The average return probability equals

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|V(G)|^{-1} \sum_{i=1}^{\left|V(G)\right|} e^{-t\lambda_i}.
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The average return probability equals $|V(G)|^{-1} \text{tr} e^{-t\Delta_G} =: \text{Tr} e^{-t\Delta_G}$. If $G$ is transitive, then this equals $p_t(o, o)$.

Open Question, L.

If $G$ is a transitive infinite graph, is $p_t(o, o)$ monotonic decreasing in the weights, $w_G$, among transitive weight functions?
Theorem, K. Brown (pub. by Pittet and Saloff-Coste (2000))

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This is because for equivariant operators $A$, $\text{Tr} A := \langle A1_o, 1_o \rangle$ defines a normalized trace, meaning that $A \mapsto \text{Tr} A$ is linear, $\text{Tr} A \geq 0$ for $A \geq 0$, $\text{Tr} I = 1$, and $\text{Tr}(AB) = \text{Tr}(BA)$. 
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Open Question, Fontes and Mathieu

If \( G \) is an infinite Cayley graph, is \( \mathbb{E}[p_t(o, o)] \) monotonic decreasing in the weights, \( w_G \), among random weight functions with invariant law?
Theorem, Fontes and Mathieu (2006)

If $G$ is an amenable Cayley graph, then $\mathbb{E}[p_t(o, o)]$ monotonic decreasing in the weights, $w_G$, among random invariant weight functions.
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If $G$ is a Cayley graph, then $\mathbb{E}[p_t(o, o)]$ monotonic decreasing in the weights, $w_G$, among random invariant weight functions when there is an invariant monotone coupling of the weights.
Cayley Graphs: Return Probability

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This depends on the fact that for random equivariant operators $A$, we have a trace $A \mapsto E[\langle A1_o, 1_o \rangle]$. 
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This depends on the fact that for random equivariant operators $A$, we have a trace $A \mapsto \mathbb{E}[\langle A1_o, 1_o \rangle]$. More generally, if $\mu$ is a unimodular probability measure on rooted networks, then the class of equivariant operators $(G, o) \mapsto A_{(G, o)} = A_G$ has a trace

$$\text{Tr}_\mu : A \mapsto \int \langle A_G1_o, 1_o \rangle \, d\mu(G, o) = \mathbb{E}[\langle A1_o, 1_o \rangle].$$
A probability measure $\mu$ on (isomorphism classes of) rooted connected networks $(G, o)$ is called unimodular if it satisfies the following mass-transport principle: for every isomorphism-invariant nonnegative Borel function $f$,

$$
\int \sum_{x \in V(G)} f_G(o, x) \, d\mu(G, o) = \int \sum_{x \in V(G)} f_G(x, o) \, d\mu(G, o).
$$

Every finite network induces a unimodular probability measures via uniform rooting: this is just interchange of order of summation. Every Cayley graph (with arbitrary rooting) induces a unimodular measure: use $o :=$ the identity, $f(o, x) = f(x - 1 o, x - 1 x) = f(x - 1, o)$, and that inversion is a bijection.
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E[\langle AB 1_o, 1_o \rangle] = E[\langle B 1_o, A^* 1_o \rangle] = E\left[ \sum_{x \in V(G)} \langle B 1_o, 1_x \rangle \langle 1_x, A^* 1_o \rangle \right] = E\left[ \sum_{x \in V(G)} \langle B 1_o, 1_x \rangle \langle A 1_x, 1_o \rangle \right]
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This leads to the following extension.

If $\nu$ is a unimodular probability measure on rooted graphs with a pair of
weight functions, $w_1$ and $w_2$, with $w_1 \leq w_2$ a.s., then
$$\int p_t(o, o; w_1) \, d\nu \geq \int p_t(o, o; w_2) \, d\nu$$
for all $t > 0$. 


If $\mu_1$ and $\mu_2$ are unimodular probability measures on rooted networks
$(G, o, w_i)$ such that there is a coupling
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then is
$$\int p_t(o, o) \, d\mu_1 \geq \int p_t(o, o) \, d\mu_2$$
for all $t > 0$?

I.e., is
$$\text{Tr} \, \mu_1 \, e^{-t \Delta} \geq \text{Tr} \, \mu_2 \, e^{-t \Delta}?$$

When there is a unimodular monotone coupling $\nu$, we have
$$\text{Tr} \, \mu_i \, e^{-t \Delta} = \text{Tr} \, \nu \, e^{-t \Delta_i}.$$

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\int p_t(o, o; w_1) \, d\nu \geq \int p_t(o, o; w_2) \, d\nu \text{ for all } t > 0.
\]

Proof.

We have $\Delta_1 \leq \Delta_2$, so for all $t > 0$, we have $-t\Delta_1 \geq -t\Delta_2$. Therefore
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\int p_t(o, o; w_1) \, d\nu = \text{Tr}_\nu \, e^{-t\Delta_1} \geq \text{Tr}_\nu \, e^{-t\Delta_2} = \int p_t(o, o; w_2) \, d\nu.
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If $\mu_1$ and $\mu_2$ are unimodular probability measures on rooted networks $(G, o, w_i)$ such that there is a coupling $(G, o, w_1, w_2)$ with $w_1 \leq w_2$ a.s., then is $\int p_t(o, o) \, d\mu_1 \geq \int p_t(o, o) \, d\mu_2$ for all $t > 0$?

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\[ \int p_t(o, o) \, d\mu_1 \geq \int p_t(o, o) \, d\mu_2 \text{ for all } t > 0? \] I.e., is
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If $\mu_1$ and $\mu_2$ are unimodular probability measures on rooted networks $(G, o, w_i)$ such that there is a coupling $(G, o, w_1, w_2)$ with $w_1 \leq w_2$ a.s., then is $\int p_t(o, o) \, d\mu_1 \geq \int p_t(o, o) \, d\mu_2$ for all $t > 0$? I.e., is
\[ \text{Tr}_{\mu_1} e^{-t\Delta} \geq \text{Tr}_{\mu_2} e^{-t\Delta}? \]

When there is a unimodular monotone coupling $\nu$, we have
\[ \text{Tr}_{\mu_i} e^{-t\Delta} = \text{Tr}_\nu e^{-t\Delta_i}. \]
What is required is to compare two different unimodular measures, each with its own trace. We attempt to attack this problem via similar questions for finite graphs.
What is required is to compare two different unimodular measures, each with its own trace. We attempt to attack this problem via similar questions for finite graphs. Since finite graphs are unimodular, this appears impossible. But the essence is to compare two different traces, so we use two different graphs.
Say that $G$ dominates $H$, written $G \succcurlyeq H$, if there is a probability measure on pairs $(X, Y) \in V(G) \times V(H)$ such that almost surely there is a rooted isomorphism from $(H, Y)$ to a subgraph of $(G, X)$ and such that the marginal distributions of $X$ and $Y$ are each uniform. The way to think of domination is that $G$ looks bigger than $H$ from the point of view of a typical vertex.

This graph dominates an edge.
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If there are weights on the edges, we require that the rooted isomorphism from \( (H, Y) \) to a subgraph of \( (G, X) \) is weight increasing.
The graph on the left dominates a triangle.
The graph on the left does not dominate the graph on the right:

Fractional Tiling

An edge fractionally tiles the graph on the left and tiles the graph on the right. $H$ fractionally tiles $G$ if there is an integer number of copies of $H$ in $G$ such that each vertex of $G$ is covered the same number of times by these copies of $H$. If that latter number is 1, then $H$ tiles $G$. 
The graph on the left does not dominate the graph on the right:

An edge fractionally tiles the graph on the left and tiles the graph on the right.

*H fractionally tiles* *G* if there is an integer number of copies of *H* in *G* such that each vertex of *G* is covered the same number of times by these copies of *H*.

If that latter number is 1, then *H tiles* *G*. 
If $H$ fractionally tiles $G$, then $G \succ H$: 
If $G \succeq H$, then does continuous-time simple random walk satisfy

$$\frac{1}{|V(G)|} \sum_{x \in V(G)} p_t(x, x; G) \leq \frac{1}{|V(H)|} \sum_{x \in V(H)} p_t(x, x; H)$$

for all $t > 0$?
Open Question, L. (2016)

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$$

for all $t > 0$?

This inequality holds near 0 and near $\infty$. 
Open Question, L. (2016)

If $G \succeq H$, then does continuous-time simple random walk satisfy

$$\frac{1}{|V(G)|} \sum_{x \in V(G)} p_t(x, x; G) \leq \frac{1}{|V(H)|} \sum_{x \in V(H)} p_t(x, x; H)$$

for all $t > 0$?

This inequality holds near 0 and near $\infty$.

Theorem, L. (2016)

This inequality holds if $H$ fractionally tiles $G$. 
Let $G$ be a graph with positive weights $w$ on its edges. Suppose that $H_i$ is a subgraph of $G$ with positive weights $w_i$ on its edges for $i = 1, \ldots, k$ with the following two properties:

1. There is a constant $m$ such that for every $x \in V(G)$,
   \[ |\{i; x \in V(H_i)\}| = m, \]
2. For every $e \in E(G)$,
   \[ w(e) \geq \frac{1}{m} \sum_{i=1}^{k} w_i(e). \]

Then for all $t > 0$, we have

\[
\frac{1}{|V(G)|} \sum_{x \in V(G)} p_t(x; G) \leq \frac{1}{\sum_{j=1}^{k} |V(H_j)|} \sum_{i=1}^{k} \sum_{x \in V(H_i)} p_t(x; H_i).\]
Theorem, L. (2016)

Let $G$ be a graph with positive weights $w$ on its edges. Suppose that $H_i$ is a subgraph of $G$ with positive weights $w_i$ on its edges for $i = 1, \ldots, k$ with the following two properties:

- there is a constant $m$ such that for every $x \in V(G)$,
  $$\left| \{ i \mid x \in V(H_i) \} \right| = m,$$

Then for all $t > 0$, we have

$$\frac{1}{|V(G)|} \sum_{x \in V(G)} p_t(x; G) \leq \frac{1}{\sum_{j=1}^{k} |V(H_j)|} \sum_{i=1}^{k} \sum_{x \in V(H_i)} p_t(x; H_i).$$
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The square lattice $\mathbb{Z}^2$ and the subgraph $H$ formed by deleting every vertex both of whose coordinates are odd.
Continuous-time simple random walk on each graph, where edges are crossed at rate 1.
Average Return Probabilities

\[ p_t((0, 0); \mathbb{Z}^2) \leq \frac{1}{3} p_{3t/2}((0, 0); H) + \frac{2}{3} p_{3t/2}((0, 1); H). \]
Theorem, L. (2016)

Let $G$ be a unimodular transitive graph and $H$ be a random subgraph of $G$ with edge weights $w_H$ such that the law of $(H, w_H)$ is $\text{Aut}(G)$-invariant. If

$$\forall e \sim o \quad \mathbb{E}[w_H(e) | e \in H] \cdot \mathbb{P}[e \in H | o \in H] \leq 1,$$

then continuous-time simple random walk on $G$ and the continuous-time network random walk on $(H, w_H)$ satisfy

$$p_t(o; G) \leq \mathbb{E}[p_t(o; H, w_H) | o \in H].$$
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$$p_t(o; G) \leq \mathbb{E}[p_t(o; H, w_H) \mid o \in H].$$
For example, suppose that $G$ is the usual nearest-neighbor graph on $\mathbb{Z}^d$ ($d \geq 2$) and $H$ is the infinite cluster of a supercritical Bernoulli percolation on $G$. Let $\delta := \mathbb{E}[\deg_H(o) \mid o \in H] / (2d)$. Then

$$\forall t \geq 0 \quad p_t(o; \mathbb{Z}^d) \leq \mathbb{E}[p_{t/\delta}(o; H) \mid o \in H].$$

This is obtained by using $w_H \equiv 1/\delta$. The above inequality is false for any larger value of $\delta$. 
Theorem, L. (2016)

Let $\mathbf{P}$ be a unimodular probability measure on rooted networks $(G, o)$ with positive weights $w_G$ on its edges and with a percolation subgraph $H$ of $G$ with positive weights $w_H$ on its edges. Let $\mathbf{P}_{(G,o)}$ denote the conditional law of $H$ given $(G, o)$. Assume that $\alpha := \mathbf{P}_{(G,o)}[o \in V(H)] > 0$ is a constant $\mathbf{P}$-a.s. If $\mathbf{P}$-a.s. whenever $e \in E(G)$ is adjacent to $o$,

$$E_{(G,o)}[w_H(e) \mid e \in E(H)] \mathbf{P}_{(G,o)}[e \in E(H) \mid o \in V(H)] \leq w_G(e),$$

then $E[p_t(o; G)] \leq E[p_t(o; H) \mid o \in V(H)].$
For example, let $(G, o)$ be any unimodular random rooted graph and consider Bernoulli($\alpha$) site percolation on $G$. Let $H$ be the induced subgraph. Then

$$\forall t \geq 0 \quad \mathbb{E}[p_t(o; G)] \leq \mathbb{E}[p_{t/\alpha}(o; H) \mid o \in H].$$

This is obtained by using $w_H \equiv 1/\alpha$. This is sharp: for all $\beta < \alpha$, there is some $t$ such that $\mathbb{E}[p_t(o; G)] > \mathbb{E}[p_{t/\beta}(o; H) \mid o \in H].$
Let $\tau(G)$ denote the number of spanning trees of a finite connected graph, $G$. 

Recall that $\tau(G) = \det \Delta_o$, where $\Delta_o$ indicates striking the row and column of $\Delta$ corresponding to $o$.

Conjecture, L. (2016) If $G \succeq H$, then $\tau(G) \geq \frac{1}{|V(G)|} \tau(H)$.

Theorem, L. (2016) This holds if either $G$ or $H$ is transitive, or [J. Kahn] if $H$ fractionally tiles $G$.

Note that $\log \tau(G) = |V(G)| - 1 + \text{Tr log } \Delta_o$.

An infinitary version of the theorem holds. Define the tree entropy of $\mu$ as $h(\mu) := \text{Tr } \mu \log \Delta$. 
Let $\tau(G)$ denote the number of spanning trees of a finite connected graph, $G$. Recall that $\tau(G) = \det \Delta_o$, where $\Delta_o$ indicates striking the row and column of $\Delta$ corresponding to $o$.

Conjecture, L. (2016)

If $G \succeq H$, then

$$\tau(G)^{1/|V(G)|} \geq \tau(H)^{1/|V(H)|}.$$

Theorem, L. (2016)

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**Theorem, L. (2016)**

This holds if either $G$ or $H$ is transitive, or [J. Kahn] if $H$ fractionally tiles $G$.

Note that $\log \tau(G)^{1/|V(G)|} = V(G)^{-1} \text{tr} \log \Delta_o$.

An infinitary version of the theorem holds. Define the tree entropy of $\mu$ as

$$h(\mu) := \text{Tr}_\mu \log \Delta.$$
If $\mu_1 \neq \mu_2$ are unimodular probability measures on rooted weighted connected infinite graphs that both satisfy
\[
\int \log w_G(o) \, d\mu_i(G, o) \in [-\infty, \infty)
\]
and $\mu_1$ stochastically dominates $\mu_2$, then $h(\mu_1) > h(\mu_2)$. 

This depends on another representation for tree entropy:

**Theorem, L. (2010)**

If $\mu$ is a unimodular probability measure on rooted weighted infinite graphs that satisfies

\[
\int \log w_G(o) \, d\mu(G, o) \in [-\infty, \infty),
\]

then

\[
\mathbf{h}(\mu) = \int_0^\infty \left( \frac{s}{1 + s^2} - \int R(G, o, s) \, d\mu(G, o) \right) \, ds.
\]

Here, given a network $G$, one of its vertices $x$, and a positive number $s$, let $R(G, x, s)$ be the effective resistance between $x$ and infinity in the network $G^s$ formed from $G$ by adding an edge of conductance $s$ between every vertex and infinity, where $\infty$ is also a vertex of $G^s$.

This allows us to use Rayleigh’s monotonicity principle pointwise.
Supplementary Material
If $H$ is transitive, then $G \supseteq H$ iff every vertex of $G$ belongs to a copy of $H$. If $G$ is transitive, then $G \supseteq H$ iff $G$ contains a copy of $H$. In both cases, the independent coupling of roots works.
If $H$ is transitive, then $G \asymp H$ iff every vertex of $G$ belongs to a copy of $H$. If $G$ is transitive, then $G \asymp H$ iff $G$ contains a copy of $H$. In both cases, the independent coupling of roots works.

If $H$ fractionally tiles $G$, then $G \asymp H$. Conversely, if $G$ is transitive and dominates $H$, then $H$ fractionally tiles $G$. 
A similar proof shows that if $f$ is any decreasing convex function and $H$ fractionally tiles $G$, then

$$
\text{Tr } f(\Delta_G) \leq \text{Tr } f(\Delta_H).
$$
A similar proof shows that if $f$ is any decreasing convex function and $H$ fractionally tiles $G$, then

$$\text{Tr } f(\Delta_G) \leq \text{Tr } f(\Delta_H).$$

However, it is not true that this inequality holds whenever $G \succ H$; a counter-example is provided by taking $f(s) := (4 - s)^+$ for these graphs:

The graph $G$ on the left dominates the graph $H$ on the right.