Erratum to

Identities and Inequalities for Tree Entropy


by Russell Lyons

The proof of Theorem 3.1 made an incorrect appeal to the monotone convergence theorem and should be replaced by the following:

*Proof.* The hypothesis is equivalent to \( D \in \text{DetAlg} \). Write \( Q := (I + P)/2 \in \text{Alg} \), which is the transition operator for the lazy random walk associated to \( P \). Since \( I - cQ \in \text{Alg} \subseteq \text{DetAlg} \) for \( c \in \mathbb{R} \), it follows that \( 2D(I - cQ) \in \text{DetAlg} \); in particular for \( c = 1 \), we obtain \( \Delta \in \text{DetAlg} \). Since \( DQ \geq 0 \), we have \( 2D \geq 2D - 2cDQ \geq 2D - 2DQ = \Delta \) for \( 0 \leq c \leq 1 \). It is easy to see that \( 2D - 2cDQ \to \Delta \) in the measure topology as \( c \uparrow 1 \) (for its definition, see Fack and Kosaki (1986), §1.5), whence also \( \log(2D - 2cDQ) \to \log \Delta \) in the measure topology as \( c \uparrow 1 \). Because of (3.1), we may apply the dominated convergence theorem to \( \log^+(D - cDQ) \) (see Fack and Kosaki (1986), Theorem 3.6) and the monotone convergence theorem to \( \log^-(D - cDQ) \) (see Fack and Kosaki (1986), Theorem 3.5(ii)), obtaining

\[
\text{Det} \Delta = \lim_{c \uparrow 1} \text{Det}(2D - 2cDQ) . \quad (3.3)
\]

Since \( 2D - 2cDQ = 2D(I - cQ) \), the fundamental theorem of Fuglede and Kadison (1952) yields \( \text{Det}(2D - 2cDQ) = 2 \text{Det} D \cdot \text{Det}(I - cQ) \). On the other hand, for \( 0 < c < 1 \),

\[
\log \text{Det}(I - cQ) = \Re \text{Tr} \log(I - cQ)
\]

by Theorem 1 \((2^a)\) of Fuglede and Kadison (1952) (or Theorem I.6.10 of Dixmier (1981)) and

\[
\log(I - cQ) = - \sum_{k \geq 1} c^k Q^k / k
\]

(in the norm topology). Therefore,

\[
\log \text{Det}(I - cQ) = - \sum_{k \geq 1} \Re \text{Tr}_\rho c^k Q^k / k = - \sum_{k \geq 1} \text{Tr}_\rho c^k Q^k / k ,
\]

whose limit as \( c \uparrow 1 \) is

\[
- \sum_{k \geq 1} \text{Tr}_\rho Q^k / k = \int - \sum_{k \geq 1} \frac{1}{k} q_k(o; G) d\rho(G, o)
\]
by the monotone convergence theorem, where $q_k(o; G)$ is the $k$-step return probability for $Q$. By Lemma 3.5 of Lyons (2005),

$$\sum_{k \geq 1} \frac{1}{k} q_k(o; G) = \log 2 + \sum_{k \geq 1} \frac{1}{k} p_k(o; G).$$ (3.4)

Comparing (2.2) with equations (3.3), (2.6), and (3.4), we deduce the equality in (3.2). ■

The exhaustion of $G$ by finite subnetworks $G_n$ preceding Theorem 3.3 should specify that $G_n$ are induced subnetworks.

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