Homework # 2: A Special Case of $L(1, \chi) \neq 0$.

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In this problem set you will work through Dirichlet’s proof that $L(1, \chi) \neq 0$ for $\chi = \left( \frac{\cdot}{p} \right)$ where $p$ is a prime number.

1. Use our formula for $\Gamma$ to prove, $\Gamma(s) = n^s x^{n-1} \int_0^1 (\log(\frac{1}{x}))^{s-1} dx$. Conclude that,
   \[ L \left( s, \left( \frac{1}{p} \right) \right) \Gamma(s) = -\int_0^1 \frac{f(x)}{x^p-1} \left( \log \left( \frac{1}{x} \right) \right)^{s-1} dx, \]
   for some polynomial $f$. Find $f$.

2. Let $\zeta_p = e^{\frac{2\pi i}{p}}$ be a primitive $p$th root of unity. Show that,
   \[ \frac{f(x)}{x^p-1} = \sum_{a=0}^{p-1} c_p^a f(\zeta_p^a) \frac{1}{x - \zeta_p^a}. \]
   Therefore,
   \[ L \left( 1, \left( \frac{1}{p} \right) \right) = -\frac{1}{p} \sum_{a=0}^{p-1} c_p^a f(\zeta_p^a) \int_0^1 \frac{dx}{x - \zeta_p^a}. \]

3. Compute $\int_0^1 \frac{dx}{x - \zeta_p^a}$. Plug this into your equation for $L \left( 1, \left( \frac{1}{p} \right) \right)$.

4. Notice that $\zeta_p^a f(\zeta_p^a) = \sum_{m=1}^{p-1} \left( \frac{m}{p} \right) c_p^a m$, which is called the Gauss sum, $g_a$. Show that $g_a \neq 0$. (In fact one can show, $g_a = \left( \frac{a}{p} \right) g_1$; and $g_1 = \left( \frac{-1}{p} \right) p$. To prove the latter statement, consider the sum $\sum_a g_a g_{-a}^* in two different ways.)

5. Prove
   \[ L \left( 1, \left( \frac{1}{p} \right) \right) = -g_1 \sum_{a=1}^{p-1} \left( \frac{a}{p} \right) \left( \log \left( \frac{2 \sin \frac{a\pi}{p}}{p} \right) \right). \]

6. Suppose that $p \equiv 3 \pmod 4$. Show that $L \left( 1, \left( \frac{1}{p} \right) \right) \neq 0$.

7. Suppose $p \equiv 1 \pmod 4$. Show that
   \[ L \left( 1, \left( \frac{1}{p} \right) \right) = -g_1 \sum_{a=1}^{p-1} \left( \frac{a}{p} \right) \log \frac{\prod_{a=\square} \sin \frac{az}{p}}{\prod_{a \neq \square} \sin \frac{az}{p}}. \]
   Show
   \[ \prod_{a=\square} \sin \frac{az}{p} = \prod_{a=\square} (1 - \zeta_p^a) = A, \]
   \[ \prod_{a \neq \square} \sin \frac{az}{p} = \prod_{a \neq \square} (1 - \zeta_p^a) = B. \]
   Notice that $A$ and $B$ live in the field $\mathbb{Q}(\zeta_p)$. Using Galois theory, show that we must have $A = x + y\sqrt{p}$ and $B = x - y\sqrt{p}$ with $x, y$ rational. Notice that since $AB = p$, $x^2 - py^2 = p$. Clearly this implies that $y \neq 0$. Conclude that $L \left( 1, \left( \frac{1}{p} \right) \right) \neq 0$. (Also notice that we can show the negative Pell’s equation $a^2 - pb^2 = -1$ has an integer solution when $p$ is a prime 1 modulo 4.)