Noah Snyder: Teaching Statement

1. Undergraduate teaching

My formative educational experience was an 8-week summer program in number theory, taught in a non-standard inquiry-based format where students proved all the main results themselves rather than seeing the proofs presented to them in lecture. I still think of this kind of inquiry-based approach as the gold standard, and I certainly learned this material better than any other subjects I’ve learned. On the other hand, such an approach is not always appropriate: it is extremely time-intensive for the students, the rate of learning material is slower, and it requires a tiny student-teacher ratio that is impossible to maintain in many courses. So instead, I hope to include some of the advantages of this approach in my teaching, while adapting it in ways appropriate to the situation. In addition, I know about myself that I’m much better at something the second time I do it, than I am the first time. I also think that one should be somewhat risk averse when teaching, because the downside of having a class that’s a disaster is significant. As a result my method is largely to begin with a relatively standard approach to a class the first time I teach it, and then to iterate and modify in the direction of incorporating more high-impact and inquiry-based techniques. I will illustrate my teaching philosophy by discussing Math M391 “Introduction to Mathematical Reasoning,” which was the first class I taught at Indiana, and the only class that I’ve taught twice at Indiana.

Let me begin by discussing what kind of a class Math M391 is, who the typical students are, and how it fits into the larger curriculum. In the United States, there is a significant qualitative difference between lower-level math classes (calculus, multivariable calculus, and linear algebra) and upper-level math classes (algebra, analysis, and topology), where the homework changes from calculating examples to proving theorems. Historically many students had difficulty navigating this large change. Most schools have thus moved to a system where there is a specific class aimed at teaching proof skills which serves as a bridge to higher mathematics, and at Indiana M391 serves this role. Thus the main goals of this class are to teach about proofs, to give students skills in making rigorous arguments and writing proofs, and to put them in position to succeed in their upper-level classes. This means that it is quite unusual among math classes in that there is no expectation that the class cover specific mathematical content, so the teacher has a great deal of flexibility in selecting topics. The population of Math M391 is also somewhat unusual. First, our honors curriculum introduces proofs much earlier, and so students who take honors classes typically do not take M391. Secondly, math education majors all take Math M391. As a result the class is a mix of about half non-honors math majors and math minors and about half math education majors. The presence of math education majors makes M391 a critically important class, because it is a key opportunity to affect teaching of mathematics in the next generation.

The first challenge when teaching M391 is to decide which mathematical topics to consider. Most important proof techniques can be learned using examples from any subject in mathematics, but there are still advantages and disadvantages to each. I decided that
what I wanted to do was teach elementary number theory, which means studying the properties of whole numbers, and related topics like prime numbers and clock arithmetic (where counting forward eventually ends you back at zero, like how 12 = 0 on a clock). This choice had several advantages. First, it is material I know extremely well and in complete detail, since it was the material that I was learning about when I learned about proofs. Second, I have a lot of experience with teaching students about proofs via this material from my time at the Ross summer program, and so have a good sense of what the harder and easier parts are. Third, I liked that it was clear exactly what axioms I wanted to start with, since I often find it frustrating in introduction to proof textbooks that it’s not clear to students exactly what they’re allowed to assume in their proofs. I then set about finding a textbook that was compatible with this approach, and found a very nice one called *The Art of Proof: Basic Training for Deeper Mathematics* by Beck and Geoghegan. In addition to covering the material I wanted, I also really liked about this book that it developed the subject systematically so that the theorems that the students prove are stated in the text in logical order, which makes it clear that you can use earlier theorems to prove later ones without any fear of circularity. On the other hand, it was somewhat different from my original plan in that it spent only the first half of the course on whole numbers, and the second half on the real numbers (for example, introducing a rigorous foundations for calculus and proving that there’s a real number whose square is 2). The authors argue forcefully in their introduction that based on their experience teaching this course over many years, it’s important to cover both topics. I decided to follow their suggestion.

The next challenge after picking the material and the textbook, was to figure out how to structure the class and the homework so that the students could get practice really doing proofs in addition to watching me do them. One typical inquiry-based approach would be to have students come up with proofs outside of class and present them during class. This seemed especially promising for a way to teach math education majors who are going to be communicating math with their students in the classroom rather than in writing. Another typical approach would be to have groups of students come up with proofs during class, as we did in Calculus classes at Berkeley. At Berkeley, we had rooms with many blackboards, so that groups could work at the board and the teacher could easily scan the room. Unfortunately, since M391 was a 40-person class, I did not think that either of these approaches were workable. Only spending 1/40th of the time presenting is not sufficiently high impact for the students, and, one teacher can’t really coach 15 different groups through a proof simultaneously, especially without having enough blackboards for all students to work at boards. Nonetheless, I wanted to incorporate some student presentations in addition to them writing proofs on their homework. Another dilemma I had in structuring the class was that the model for homework that the textbook suggested (and also how the Ross program worked), was that problems were graded as either correct or as a “redo” and then students would keep writing them up until they’re correct. I like this approach, but thought it was too risky because it could well result in students talking to each other and copying from their friends a proof that was deemed correct. In the end, I decided on the following. There would be weekly homework, quizzes every other week, an in-class midterm and final, and once during the semester students in groups of three would come to my office to present a more difficult proof orally. Furthermore, the homework was given two scores: one for the correctness, and another for clarity. I hoped that this would
result in students learning not just how to write something technically correct, but also to
write in a clear and readable fashion.

After I finished teaching the course I reassessed what had worked and what had not.
The most important realization was that the class had been too difficult and too many of
the students had struggled with it. In particular, the second half of the course which cov-
ered the real numbers had been too challenging for the majority of the class. To a large
extent this was caused by my failure to understand the level of mathematical skills that
the students had coming in to the class, relative to a class consisting entirely of math ma-
jors. Naturally the higher difficulty led to lower grades for the students and to lower stu-
dent evaluations than I’d have hoped for (though still okay for a first-time teacher in this
particular course). There were a few other smaller problems. The “clarity” score for home-
work had really frustrated the students and I didn’t think that it had led to outcomes that
made it worth having frustrated students. Presentations in my office turned out to be too
time consuming for me to justify the relatively small value for the students. In part, these
presentations were less valuable than I’d expected because I was used to teaching proofs to
students for whom mathematics was easier than clear communication, whereas for most of
the students in this class the opposite was true. This meant that once they knew how to
prove a result, they typically were easily able to give a great presentation about it. Finally,
I thought the quizzes had been too much logistical work for me, and I wanted to find a
more efficient way to get the same value for the students.

The second time I taught the class, I made some changes. In addition, the class was 10
students smaller, due largely to the rapid decline in enrollment in the education school.
Although I kept the same textbook, I decided I would try to cover only the first half of the
material, but I would do so more slowly and thoroughly. As a result I also introduced a bit
of extra related material. First, I discussed other “commutative rings” in addition to the
integers (e.g. the rational numbers, real numbers, complex numbers, and polynomials are
all commutative rings). This is a relatively mild change since many of the theorems they
proved actually apply to any commutative ring, but it had a few advantages. Considering
more general commutative rings allowed me to use some additional examples, and allowed
me to explain how you knew that certain theorems needed to use certain axioms because
the theorem was false for other commutative rings. For example, I could state the theorem
that cancellation \((ab = ac \text{ for } a \neq 0 \text{ implies } b = c)\) holds if and only if there are no zero
divisors \((xy = 0 \text{ implies } x = 0 \text{ or } y = 0)\). I also discussed two other ways to think of the
integers, in addition to Beck and Geoghegan approach where it is thought of as a nontrivi-
al commutative ring with trichotomy and induction. First, I talked about how you can
think of non-negative numbers as sizes of sets, addition as disjoint union, and multipli-
cation as Cartesian product. Second, I included a section on Peano arithmetic, which is an
axiomatization of the natural numbers where + and − aren’t fundamental notions, but are
instead defined in terms of the successor. So \(m+n\) means “start at \(n\) and move to the next
time,” while \(n+m\) means “start at \(m\) and move to the next number \(n\) times,”
which means that \(m+n = n+m\) becomes a nontrivial theorem.

Logistically I also made some changes: I got rid of quizzes and presentations and in
their place I put a classroom participation grade. Classroom participation had two com-
ponents. First, roughly once per class I would have a student present a proof at the board,
and both the rest of the class and I would give feedback. Each student was expected to do
this at least once during the semester. This element of class participation was the main substitute for the office presentations in the previous incarnation of the course. Second, I had the students buy a stack of index cards, and then I would ask mini-quizzes of a question or two during class which they would answer on the cards. After collecting the index cards, we would immediately discuss the answers or have one of the students present an answer. Rather than grading each quiz individually, I just expected students to give serious answers even if they were incorrect. This was better for the students because they got immediate feedback, and better for me because it avoided the work of typing, copying, supervising grading, and returning quizzes. Furthermore, this system allowed me to also ask the students for anonymous feedback weekly about things like whether I was moving too quickly and whether the material was too difficult. This let me adapt on a weekly or monthly basis, rather than waiting for the final evaluations to get good student feedback.

All of these changes made a significant improvement in the course. Since the students were struggling less with the material and since we could go more slowly, they were more confident, which lead to better outcomes in terms of their proof skills and course grades. The students were also less frustrated and happier. I think the second version of this course was quite successful and had little in the way of clear flaws.

Since I last taught this course two years ago, I have continued to think about what changes I would consider making when I teach this course in the future. Last year one of our Zorn postdocs, Corrin Clarkson, taught this class and we discussed her thoughts on the course. She took a more thoroughgoing inquiry-based approach where nearly all classroom time was spent on in-class presentations. Based on her feedback, I think this approach is more viable than I had originally thought. I would like to try teaching a fully inquiry-based version of this class in the future, so long as the class size is manageable. I think were I to teach in this fashion, I might want to find material that’s less technical than some of the number theory that I used. The textbook that Corrin taught from took its examples from graph theory and group theory. Those both seem promising, but I have not yet fully investigated them as options. Another change I would like to try doing in the future, is to incorporate the material from a course I’ve taught many times at Mathcamp on using geometry to understand rational approximations of numbers and continued fractions. I have developed a lot of material myself for teaching this in an inquiry-based fashion, and I think it might be appropriate for a class like M391.

A third direction I’ve considered for this course, is to incorporate a computer proof assistant that verifies student proofs. I recently took a class from Amr Sabry in the IU computer science department on homotopy type theory. As part of this class we wrote complete proofs in code, which the computer checked. Although some of the proofs in this class were quite difficult, we started out by proving easy results in Peano arithmetic that were totally appropriate for M391. I found the process of having a computer-checked proof valuable for two reasons. First, it makes it more objective whether the proof is right or wrong, and doesn’t involve arbitrary decisions by the grader of how much rigor is expected. Second, it was a lot of fun, because getting immediate feedback on whether your proof is right turns proving things into a kind of a game where you could just try things until you find something that works. In a sense, a computer proof checker gamifies proofs, much as DragonBox gamifies high school algebra (a perspective I learned from a blog post https://golem.ph.utexas.edu/category/2012/06/the_gamification_of_higher_cat.html).
Unfortunately, languages like Agda and Coq that are appropriate at the graduate level are much too difficult to use for an introductory level class. An alternative which is directly aimed at the undergraduate level, which I learned about reading the mathoverflow sister site matheducators.stackexchange.net, is called Lurch (see http://lurchmath.org). I'm not yet sure whether Lurch is sufficiently developed to really make it worth using in a class like M391, but I will keep my eye on developments in this direction to see if something sufficiently user friendly appears. I would especially be interested in proof assistant that had a touch interface which students could use directly on their phones.

Outside of the classroom, as part of my CAREER grant, Chuck Livingston and I have started running a math education club several times a semester. We had one meeting where we looked at the puzzles in the Slocum Puzzle Collection at the Lilly library, and had a guided tour of the mathematical sculptures displayed in Rawles Hall. In another meeting, two of my students from Math M391, who had graduated and were teaching at Bloomington North, came in to answer questions about their teaching experience. I had assumed that running a math ed club would not be so different than running math club, but there were some new challenges. For example, due to tutoring jobs and student teaching, many of the math ed majors are very tightly scheduled in the evenings, which made organization more challenging. The main upshot of this was that meeting once a month is a more reasonable goal than meeting once a week.

2. Graduate teaching

One of my main priorities in graduate teaching is to spend time in class focusing on how you might come up with arguments and what the main ideas of the arguments are, and not to spend too much time in class on low-level detail. Graduate students should be able to fill in the details on their own or by reading the textbook, and there’s very little value in watching someone else do a complicated calculation. Even before stating a theorem I try to first work out why one should expect a theorem in that direction and what the proof would look like. This the opposite direction from how math textbooks are usually written (where you first prove a bunch of lemmas which you’ll need in the theorem, then you state the theorem, and then you do the proof), but it more closely the order of actual mathematics research. This kind of motivation-first approach also affects how I structure my courses. For example, I recently taught M507-508 on Lie Groups and Lie Algebras. In this subject, Lie groups are very natural objects, whereas Lie algebras are less natural and were developed as a way of studying Lie groups. Nonetheless, since Lie algebras are more linear and since linear algebra is extremely powerful, many textbooks emphasize Lie algebras and may even start by studying Lie algebras first. I took the opposite approach, where I spent most of the first semester on compact Lie groups and their representations, before later introducing Lie algebras. For example, we studied the representation theory of $SU(2)$ via character theory and integrating over a 3-sphere, before considering the standard Lie algebraic approach by raising and lowering operators. This required both finding an appropriate textbook (Brian C. Hall’s Lie Groups, Lie Algebras, and Representations: An Elementary Introduction), and supplementing with additional sources (lecture notes from courses taught by Constantin Teleman and by Allen Knutson).
There can be a risk in graduate classes that one can focus too heavily on the overarching theory, and miss making sure that the students learn concrete calculational skills which are also needed in research. I have been making an effort to include more of this concrete approach. For example, in M507-508 there is a key theorem called the Weyl Character Formula. Often this is just stated as a formula, but it’s important to understand that this formula actually gives an easy by-hand algorithm for computing the characters via the method of undetermined coefficients. André Henriques has a particularly memorable way of teaching this technique via a large cardboard cutout he called the “cardboard denominator” and I borrowed his approach. I also included a homework where students used triangular graph paper to compute these characters for $SU(3)$. In order to illustrate the same kind of calculations for $SU(4)$, I brought to class some 3-dimensional models made of Zome tool, following instructions from the textbook. The graduate students really enjoyed having a bit of show-and-tell as a change of pace.

**Figure 1.** The Weyl group orbit of $\rho$ for $SU(4)$ made of Zome tool (taken from Brian C. Hall’s webpage).

Let me conclude by saying a few words about graduate advising. I currently have two official graduate students, one who passed his Tier 3 oral exam a year ago, and the other who passed his Tier 3 last spring. There is also a third student that I have agreed to work with. I have also done reading courses with 6 other graduate students. I have not yet graduated any students, though I expect my first student to graduate within the next two years.

My first priority as a graduate advisor is to be able to prepare my students for their own career goals. This means working hard to make clear to them that I am happy with them going into research, teaching, or industry, and also to make clear to them that there is a connection between what Ph.D. project is most appropriate and their career goals. To go into research, students need to work on a problem which opens the door to a rich world of related problems, so that they will be positioned to have a productive postdoc. To go into teaching at small liberal arts colleges, students should work on a project that is related to potential undergraduate research projects and they should seek out opportunities to work on research with undergraduates during their time as a graduate student. To go into industry, students should have a problem that has a heavy coding component. My first student is strong and was initially unsure whether he wanted to aim for a postdoc or for an industry job. So I initially suggested a topic (planar algebras for non-semisimple tensor categories and over characteristic $p$) where there are a lot of open problems some of them quite important (e.g. a suggestion of Etingof that one might be able to better understand the Haagerup subfactor by first reducing modulo 3 and then lifting). More recently he has decided that he wants to work in Silicon Valley and interviewed for a Google internship. As a result I am adjusting somewhat my plans for him, so that he can graduate more quickly and be better prepared for his chosen field. This change of direction will also let me save the “Haagerup modulo 3” question for a later student who is aiming for a postdoc position, but that student will hopefully be able to jump into this subject more quickly by reading my first student’s thesis. My second student graduated from a small
liberal arts college and is one of our best teachers. So I have been thinking about ways to position him to be a strong candidate for teaching jobs, for example by jointly advising an REU student.

My main interaction with my students is meeting with each individually for 45 minutes. Equally important is running a regular seminar where the students give presentations to each other, which I organized jointly with our Zorn postdoc Matt Hogancamp. During my time as a graduate student at Berkeley, I organized my own seminar on “Topological Invariants and Quantum Algebra,” where I chose the topic, gave half of the talks, and recruited speakers for the other talks. I also co-organized the student representation theory seminar. I gave over 30 talks in half a dozen different seminars on topics ranging from number theory to operator algebras, to representation theory, to topology. I found this experience incredibly valuable, and hope to give my students some of those opportunities. One of the reasons why I am jumping directly into having several students at once is so that they can have a large enough group that they can maintain an interesting seminar.