Noah Snyder: Annotated Publication List

I have grouped my papers into four main research programs and a miscellaneous category. The first of these programs is largely complete (though I am still doing some related work). The second program has achieved its main goals, but we are still finishing a few papers giving some applications. The final two are in earlier stages and are my main current areas of activity.

1. Classification of small index subfactors

Subfactors of the hyperfinite II$_1$ factor of index less than or equal to 4 were classified in the late 80s and early 90s through work of Jones, Ocneanu, Popa, and many others. There is an ADE classification, and all of the examples are closely related to quantum or classical $SU(2)$. When the index is larger than 4, there is one family of $A_\infty$ subfactors related to quantum $SU(2)$ which are difficult to classify, but once those are excluded Haagerup observed that a similar classification should be possible for index somewhat larger than 4. The goals of this research program (joint with Scott Morrison, Emily Peters, and others) were to complete Haagerup’s classification on subfactors of index less than $3 + \sqrt{3}$ (building on work of Asaeda, Bisch, Haagerup, and Yasuda) and to extend this classification to index 5. More recently work of Liu, Penneys, and Afzaly–Morrison–Penneys has extended these results to index 5.25.


At the time we began our project, the classification of subfactors of index below $3 + \sqrt{3}$ was almost finished, except that there was a single candidate which was neither known to exist nor was proven not to exist. In this paper we constructed this example (which we called the extended Haagerup planar algebra), solving a problem that had been open for 15 years since Haagerup’s original paper on the subject. The construction uses Jones’s theory of planar algebras and followed an outline developed by Jones and by Peters in her thesis. A key step was a new skein-theoretic “evaluation algorithm” we called the Jellyfish algorithm. This construction was computer assisted, and a key point was to better understand the number theoretic properties of this planar algebra in order to be able to do exact arithmetic in a reasonable amount of computer time.


In this paper we used the planar algebraic description of the Haagerup and extended Haagerup subfactors together with our improved number theoretic understanding of these planar algebras to show that the associated fusion categories give a negative answer to a question of Etingof, Nikshych and Ostrik. Namely, every representation of every finite group has a field of definition that’s a cyclotomic extension of $\mathbb{Q}$ and the same is true of representations of quantum groups at roots of unity, and ENO had asked whether all fusion categories have this property.


In these papers (and a short fourth paper which I was not a coauthor on), we extended the classification from index less than or equal to $3 + \sqrt{3}$ to index less than 5. Subfactors of index less than 4 were classified via their principal graphs which must be ADE Dynkin diagrams. Haagerup’s classification still went through principal graphs, but since there are now too many candidate graphs of norm squared above 4 he needed to leverage two techniques to restrict the possible graphs: Ocneanu’s triple point obstruction and an associativity obstruction. These two techniques work well together, since the triple point obstruction says that the principal graph and dual principal graph must be somewhat different from each other, while the associativity obstruction is difficult to satisfy when the graphs aren’t the same. In the first paper, we gave some modifications of Haagerup’s argument which allowed us to use a computer to do many more calculations than Haagerup could do by hand. In the second paper we used some stronger triple point obstructions (due to Jones and Ocneanu) to rule out some additional graphs. In the third paper we developed some similar obstructions for quadruple points.


The dimensions of objects in a fusion category must be cyclotomic integers by a result of Etingof–Nikshych–Ostrik (who prove this result by reducing it to an earlier theorem for modular tensor categories due to Coste–Gannon). Asaeda and Yasuda showed that this property is very useful in ruling out candidate families of principal graphs of a particular form (where you start with a fixed graph and add on a single long tail). However, their technique was somewhat ad hoc and difficult to apply uniformly to rule out many families of principal graphs. In this paper we showed that this cyclotomicity property always rules out all but finitely many of the graphs in such a family, and furthermore that this bound is effective and efficient in practice. Our argument does not follow Asaeda–Yasuda, but instead builds on ideas of Cassels, Loxton, and Gross–Hironaka–McMullen. In addition to this result, we also prove a purely number theoretic analogue of the classification of small index subfactors, which classifies all numbers smaller than 2.3 which are real algebraic integers lying in a cyclotomic extension of the rational numbers and which are maximal among their Galois conjugates.

In this paper we extend our classification to the case of index equal to five. All the new examples come from actions of finite groups on five element sets, and the main new input is to use several theorems of Izumi which show that subfactors that look like they come from certain finite group actions must actually come from finite group actions.


This paper uses some of the same ideas as part 3 in order to improve on the triple point obstructions of Jones and Ocneanu. The argument is very short and is a mix of planar algebraic techniques (specifically, rotating planar diagrams) and Ocneanu-style connection techniques. The results of this paper have subsequently been improved further by Penneys using purely planar algebraic techniques.


This is a survey article summarizing the major developments in the classification of small index subfactors.

2. **Brauer–Picard groupoids and exceptional subfactors**

A major motivation for classifying mathematical objects is to produce interesting new examples which would not have been found without an extensive search. For example, it was the classification of simple Lie algebras which uncovered $E_8$ and the classification of finite simple groups which revealed the Monster group. Similarly, a major motivation in the classification of small index subfactors is to find new interesting examples. These new examples can then drive later theoretical developments, as the ADE classification did in the 80s and 90s.

There are three main examples which appear in the classification of subfactors of index less than 5 which do not appear to come from groups or quantum groups: the Haagerup subfactor, the Asaeda–Haagerup subfactor, and the extended Haagerup subfactor. The best understood of the three exceptional small index subfactors is the Haagerup subfactor. Its principal graph has a 3-fold symmetry, and it appears to live in an infinite family of examples, with $\mathbb{Z}/3$ replaced by other finite abelian groups, following results of Izumi and Evans–Gannon.

Together with Pinhas Grossman, we developed a program to understand these exceptional subfactors by studying all the fusion categories in their (higher) Morita equivalence class. Taken together, all these fusion categories have a rich structure some aspects of which were described as the “maximal atlas” by Ocneanu and others as the “Brauer–Picard groupoid” by Etingof–Nikshych–Ostrik. For finite groups, the Morita equivalence class is described by the subgroups of the group you’re considering, so one could think of this approach as studying “finite quantum groups” by their “quantum subgroups” (though this language can be somewhat misleading, so we dropped it in our later papers).

In this paper we find all fusion categories Morita equivalent to the Haagerup fusion category, and describe all Morita equivalences between them. Namely, there are exactly three such categories (the two even parts of the Haagerup subfactor plus one additional new fusion category) and exactly one Morita equivalence between each of them. In addition to a new fusion category, this also gave many new subfactors, a complete classification of all intermediate subfactors of each of these new subfactors, and a much simpler construction of the “Haagerup plus one” subfactor constructed by Grossman-Izumi. The key ideas were that one can understand a module category just by looking at the internal endomorphisms of the smallest simple object, and that certain intermediate subfactor arguments can rule out some candidate Morita equivalences.


In this paper we give a partial classification of the Morita equivalence class of the Asaeda–Haagerup fusion categories. We show that there are at least 3 fusion categories in the Morita equivalence class, and exactly four Morita equivalences between each of them. Furthermore we show that the self-Morita equivalences of each form a Klein 4-group, and that any other fusion category in the Morita equivalence class must be of one of four specific forms. As a consequence we get a large number of new subfactors. The main techniques are combinatorial. Essentially, even though the fusion rules of a single fusion category usually aren’t very restrictive, by looking at several fusion categories and their Morita equivalences all together as a single object, the combinatorics become quite restrictive. A computer calculation was required to keep track of all of this combinatorics.


Etingof–Nikshych–Ostrik showed that extensions of fusion categories by a group $G$ can be classified algebro-topologically, by finding the space of maps from $BG$ to the classifying space of the Brauer–Picard 3-groupoid. Hence, our calculations of the Brauer–Picard groupoid in the previous paper should have applications to $G$-extensions. In order to get such applications, we need to understand the higher parts of the Brauer–Picard groupoid of the Asaeda–Haagerup category. We do so by introducing an interesting fibration between higher groups which allows one to often calculate $\pi_2$ of the Brauer–Picard groupoid. Since $H^4(G, \mathbb{C}^\times)$ vanishes for cyclic groups $G$, obstruction theory together with our calculation of $\pi_2$ is enough to classify all $\mathbb{Z}/2$ extensions of the three Asaeda–Haagerup fusion categories from the previous paper. One of these new fusion categories has some interesting properties.


In this paper we complete our understanding of the Morita equivalence class of the Asaeda–Haagerup fusion categories. Namely, we show that there are exactly six fusion categories in the higher Morita equivalence class, exactly four equivalences between each of them, and the self-equivalences form a Klein 4-group. As in the Haagerup case, this has strong applications to understanding intermediate subfactor lattices.
The key step is to construct a new $\mathbb{Z}/4 \times \mathbb{Z}/2$ analogue of the Haagerup fusion category and to relate this new fusion category to the Asaeda–Haagerup subfactor. This new construction gives a large breakthrough in understanding the Asaeda–Haagerup subfactor. This construction is much easier to work with than Asaeda–Haagerup’s approach and thus has many applications including the first calculation of the Drinfeld center of Asaeda–Haagerup, a 15 year old open problem. We are also able to use these results to completely describe the Brauer–Picard 3-groupoid and classify extensions of the Asaeda–Haagerup subfactor by any (not necessarily cyclic) group.

3. Tensor categories and local topological field theories

Together with Chris Douglas and Chris Schommer-Pries, we have been studying local 3-dimensional topological field theories valued in tensor categories using the insight provided by the Baez–Dolan–Hopkins–Lurie cobordism hypothesis. The goals of this project are twofold: to use topology to better understand tensor categories, and to use the example of fusion categories to better understand local topological field theories. We aim to understand 3-dimensional local topological field theories with values in the 3-category of tensor categories in complete detail. This involves an interplay between the general theory given by the cobordism hypothesis, and special results about the particular case of tensor categories which do not apply to all 3-categories.

Recall that a local TFT is a functor of symmetric monoidal $n$-categories from the bordism category to an algebraic target category (we will consider the Morita 3-category of finite tensor categories, finite bimodule categories, bimodule functors, and bimodule natural transformations as the target). At first glance it appears that a local topological field theory is much more complex than an ordinary TFT. Instead of just assigning vector spaces and linear maps, we need to assign tensor categories, bimodule categories, etc. But, local TFTs are also much simpler because although $n$-manifolds are globally quite complicated, locally they are just euclidean space. This intuition is made precise by the cobordism hypothesis, which says that a framed local TFT is characterized by the image of the positively framed point, and that the value on this point can be any “fully dualizable object.” Thus, we can classify local TFTs by classifying fully dualizable objects. Furthermore, oriented TFTs (or TFTs with other topological structures) are given by certain homotopy fixed points of actions of groups on the space of fully dualizable objects.


The main result of this paper is that the fully dualizable objects in the Morita 3-category of finite tensor categories are the “separable tensor categories.” In characteristic zero this says that they’re exactly the fusion categories. This shows that to any fusion category there is a framed 3-dimensional local TFT. These theories should be closely related to Turaev–Viro theories but with two key differences. First, they’re fully local (Turaev–Viro had only been proved to be 321 extended previously). Second, following Barrett and Westbury, Turaev–Viro theories depend on the additional data of a spherical structure. This relationship can be explained (and we will do so in future work) by seeing that a spherical structure endows a fusion category with a (special kind of) $SO(3)$ homotopy fixed point structure.
We also show that finite tensor categories satisfy most (but not all) of the conditions of being fully dualizable. In particular, they give 2-dimensional local TFTs which are defined on certain 3-manifolds with boundary. We give an application of this TFT to show that Radford’s theorem on the quadruple dual of a finite tensor category (as generalized by ENO) follows from the fact that \( \pi_1(\text{SO}(3)) \cong \mathbb{Z}/2 \) by the belt trick. Namely, there is a belt bordism and applying the TFT to this bordism outputs a proof of Radford’s theorem. This is an important step in understanding the homotopy coherent \( \text{SO}(3) \) action on the space of finite tensor categories.


In this paper we prove some technical results constructing the balanced tensor product of module categories \( \mathcal{M} \boxtimes_c \mathcal{N} \) used in the previous paper. A different construction was given earlier by Davydov–Nikshych, but many of the arguments we’d already written in the previous paper use our construction.

4. **Classification of simple skein theories**

A subfactor \( N \subset M \) can be thought of as a very special \( N\)-\( N \) bimodule. Thus it is natural to ask whether the classification of small index subfactors can be generalized to other bimodules over factors. Again, one can define the standard invariant of \( _N X_N \) to be the unitary tensor category tensor generated by \( X \). Some results along these lines are relatively straightforward, for example, it is well-known that there is an ADET classification of self-dual bimodules of dimension less than 2. Similarly, our classification result up to index 5 shows that if \( X \) is a self-dual \( N\)-\( N \) bimodule with \( 2 < \dim X < \sqrt{5} \), then its standard invariant is Temperley-Lieb or the 2221 fusion category constructed by Ostrik.

Subfactors have additional structure, in particular if \( 1 + X \) gives a 2-supertransitive subfactor, then there’s a canonical map \( X \otimes X \rightarrow X \). So it’s natural to consider small index bimodules together with a map \( \lambda : X \otimes X \rightarrow X \). If this map generates all maps we call its standard invariant a trivalent category, since \( \lambda \) can be interpreted as a trivalent vertex in the planar algebra. In a program joint with Scott Morrison, Emily Peters, and Dylan Thurston we plan to generalize, strengthen, and automate skein theoretic techniques to classify such bimodules.

**Categories generated by a trivalent vertex.** Joint with S. Morrison and E. Peters. Accepted by *Selecta Mathematica*. arXiv:1501.06869.

In this paper we classify trivalent categories where the sequence \( \dim \text{Inv}_C(X^\otimes n) \) for \( 0 \leq n \leq 6 \) is bounded by \( (1, 0, 1, 1, 4, 11, 40) \). This strengthens a result of Kuperberg’s characterizing quantum \( G_2 \) as the only trivalent category where the dimensions of invariant spaces start \( (1, 0, 1, 1, 4, 10) \) and a certain obvious collection of diagrams forms a basis. In addition to \( G_2 \), several other interesting examples appear, including some related to the Haagerup subfactor and some coming from a Bisch–Jones style “free product” construction. We also prove a similar result about “twisted trivalent categories” (where rotating the trivalent vertex picks up a cube root of unity), where a candidate example appears that is completely mysterious. We also prove some stronger results for braided trivalent
categories. We expect that these results for braided tensor categories will give Wenzl-style recognition theorems for quantum $G_2$ and Deligne’s interpolated symmetric groups $S_t$.

The key graph-theoretic lemma underlying Kuperberg’s theorem, is that Euler characteristic shows that any planar 3-valent graph must contain a face that is a pentagon or smaller. To understand larger trivalent categories we need generalizations of this lemma. An enormous number of such generalizations can be proved using the discharging method, which was developed to prove the four color theorem. A key step in our theorem is to apply the simplest such application of the discharging method. The other key technique needed for some of these results is to rephrase the skein theoretic arguments in terms of certain linear algebraic calculations which can be done by a computer using Grobner basis techniques.

5. MISCELLANEOUS


Mason–Stothers theorem is the polynomial analogue of the famous abc conjecture. When I was a senior in High School, Serge Lang mentioned this result to me and asked me to think about it. I was able to come up with a short easily understandable proof of this result that is somewhat different from the standard one. (Note that although neither Lang nor I knew it at the time, this proof had also been found independently much earlier by Osterl´e who only published it in some lecture notes.) This paper attracted wide attention at the time since Lang talked about it in talks with undergraduates (it is discussed both in his “Lectures for Undergraduates” and his graduate “Algebra” text). More recently it was cited in the Princeton math department’s April Fools Day joke about someone finding a manuscript containing Fermat’s apocryphal proof of his famous Last Theorem (which is a slightly modified and incorrect version of my argument).


This was my first paper in graduate school, which developed out of questions I asked Hendrik Lenstra while I was taking his graduate course in representation theory. Recall that if $G$ is a finite group of order $n$, and $V$ is an irreducible complex representation of dimension $d$, then $d$ divides $n$ and $d^2 \leq n$. Hence there exists a non-negative integer $e$ such that $d(d + e) = n$. When $e$ is small, then $G$ has a character of surprisingly large degree. The goal of this paper was to understand when this was possible. When $e$ is 0, the group must be trivial. When $e = 1$ there is an interesting infinite family of examples called doubly transitive Frobenius groups. The main result of this paper is that when $e$ is any integer larger than 1 then there are only finitely many such groups. For example, when $e = 2$ the only such examples are the cyclic group of size 3 and the two non-abelian groups of size 8. The explicit bounds given in the paper are superexponential. This paper prompted a number of follow-ups papers by Isaacs, Durfee–Jensen, Larsen–Malle–Tiep, and Lewis. These papers substantially improved the bounds on $n$ in terms of $e$, eventually getting a polynomial (degree 6) bound without requiring the classification of finite simple groups and an almost sharp bound (degree 4) using the classification.

I wrote this paper early on in graduate school on classification of Cartan subalgebras of certain infinite dimensional lie algebras (like the Lie algebra of all matrices of finite support). It is part of Dan-Cohen and Penkov’s research program to study the structure of root-reductive Lie algebras.

Mednykh’s Formula via Lattice Topological Quantum Field Theories Accepted by Proceedings of the Centre for Mathematics and its Applications (special issue dedicated to V.F.R. Jones’s 60th birthday). arXiv:0703073

Mednykh proved that for any finite group $G$ and any orientable surface $S$, there is a formula for $\#\text{Hom}(\pi_1(S), G)$ in terms of the Euler characteristic of $S$ and the dimensions of the irreducible representations of $G$. A similar formula in the nonorientable case was proved by Frobenius and Schur. Both of these proofs use character theory and an explicit presentation for $\pi_1$. These results have been reproven using quantum field theory. This paper gives a greatly simplified proof of these results which uses only elementary topology and combinatorics. The main tool is an elementary invariant of surfaces attached to a semisimple algebra called a lattice topological quantum field theory. I wrote this paper during graduate school, but only recently submitted it for publication since it is perhaps more of expository interest than research interest.


In this paper we give a complete and explicit description of the planar algebra of the $D_{2n}$ subfactors of index below 4. Although the results are not highly original (these examples were already well-understood by work of Kawahigashi, and Jones had already sketched a planar algebraic description), this paper has been very popular as a starting point for graduate students trying to learn about planar algebras.


In this paper we use the $D_{2n}$ planar algebras together with level-rank duality (specifically the version developed by Beliakova and Blanchet), Kirby–Melvin symmetry, and coincidences of small Dynkin diagrams to prove some very strange identities between knot polynomials. For example, there is an interesting automorphism of the even part of the $D_{10}$ subfactor which is inherited via level-rank duality from the triality automorphism of $SO(8)$. These results are perhaps best thought of as proving some quantum group analogues of small coincidences of groups like $A_4 \cong PSL_2(\mathbb{F}_3)$.


Reshetikhin–Turaev and Joyal–Street showed that representations of quantum groups are closely related to ribbon tangles. Such ribbon tangles are required to always twist the whole way around, so that one never sees the “back side” of the ribbon on the boundary of a tangle. It is natural to wonder whether one can generalize these pictures to allow such
“half-twists”. We show that this is possible, and that it explains an interesting identity for the $R$-matrix.


This paper is a small part of Elias–Williamson’s program which they used to give a new proof of the Kazhdan–Lusztig conjectures by using diagrammatic descriptions of the Soergel bimodule categories. This particular paper proves some identities that hold in much larger generality than just Soergel bimodule categories, and which are very closely analogous to certain diagrammatic descriptions of intermediate subfactors that I developed and talked to Elias about while he was a PhD student. When they initially posted this paper to the arxiv they were unable to prove one of the main lemmas. I was able to give a short diagram proof of their missing result and so was added as a coauthor.