Singular Abreu equations and minimizers of convex functionals with a convexity constraint

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Plan of the talk

- Singular Abreu equations
- Second boundary value problems
- Minimization problems subject to a convexity constraint
- Approximation of 2D Rochet-Choné model in economics by singular Abreu equations
- Ingredients of the proofs
Theorem

(L-, 2018) Let $\Omega \subset \mathbb{R}^2$ be an open, smooth, bounded and uniformly convex domain. Assume: $\varphi \in C^\infty(\overline{\Omega})$, and $\psi \in C^\infty(\overline{\Omega})$ with $\inf_{\partial \Omega} \psi > 0$. Then there exists a unique smooth, uniformly convex solution $u \in C^\infty(\overline{\Omega})$ to the following second boundary value problem:

\[
\begin{cases}
\sum_{i,j=1}^{2} U^{ij} \frac{\partial^2 w}{\partial x_i \partial x_j} = -\Delta u & \text{in } \Omega, \\
w = (\det D^2 u)^{-1} & \text{in } \Omega, \\
u = \varphi & \text{on } \partial \Omega, \\
w = \psi & \text{on } \partial \Omega.
\end{cases}
\]

$(U^{ij})$ is the cofactor matrix of $D^2 u$, i.e., $(U^{ij}) = (\det D^2 u)(D^2 u)^{-1}$.

Notable features: Abreu type equation, and second boundary condition.
Abreu type equations

Notation:

\[ u_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}; \quad (U^{ij})_{1 \leq i, j \leq n} = (\det D^2 u) (D^2 u)^{-1}. \]

- Affine maximal surface equation in affine geometry

\[ \sum_{i,j=1}^{n} U^{ij} w_{ij} = 0, \quad w = [\det D^2 u]^{-\frac{n+1}{n+2}}. \]

- Abreu equation in differential geometry

\[ \sum_{i,j=1}^{n} U^{ij} w_{ij} = -A, \quad w = [\det D^2 u]^{-1}. \]
The constant scalar curvature problem for toric manifolds

- Reduces to the Abreu equation on a polytope
  \[ P = \{ x \in \mathbb{R}^n : \delta_k(x) > 0, k = 1, \ldots, m \}, \delta_k \text{ affine} \]

- Find a convex function \( u \) solving the Abreu equation
  \[ \sum_{i,j=1}^{n} U_{ij} w_{ij} = -A, \quad w = [\det D^2 u]^{-1} \text{ in } P \]

  with the Guillemin boundary condition
  \[ u(x) - \sum_{k=1}^{m} \delta_k(x) \log \delta_k(x) \in C^\infty(\overline{P}). \]

- Solved by Donaldson (GAFA, 2009) when \( n = 2 \).
The affine Bernstein problem

- The affine Bernstein problem in PDE language:
  
  If $u$ is a smooth, locally uniformly convex solution in $\mathbb{R}^n \ (n \geq 2)$ of

  $$
  \sum_{i,j=1}^{n} U^{ij} w_{ij} = 0, \ w = [\det D^2 u]^{-\frac{n+1}{n+2}}
  $$

  then $u$ is a quadratic polynomial.

The affine Bernstein problem

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  If \( u \) is a smooth, locally uniformly convex solution in \( \mathbb{R}^n (n \geq 2) \) of

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  \]

  then \( u \) is a quadratic polynomial.

- \( n=2 \): Chern’s conjecture (1977).
- Proved by Trudinger and Wang (Inventiones, 2000) for \( n = 2 \).
- Singular counterexamples for \( n \geq 10 \) by Trudinger-Wang.
Abreu equations as systems of two second order PDEs

\[ \sum_{i,j=1}^{n} U_{ij} w_{ij} = f, \quad w = [\det D^2 u]^{\theta-1}. \]

Fully nonlinear, fourth order, geometric PDEs. Viewed as systems of two second order PDEs:

- one is a Monge-Ampère equation
Abreu equations as systems of two second order PDEs

\[ \sum_{i,j=1}^{n} U^{ij} w_{ij} = f, \ w = [\det D^2 u]^\theta - 1. \]

Fully nonlinear, fourth order, geometric PDEs. Viewed as systems of two second order PDEs:

- one is a Monge-Ampère equation
- the other one is a linearized Monge-Ampère equation because

\[ U^{ij} = \frac{\partial \det D^2 u}{\partial u_{ij}}. \]

Example: in 2D, \( \det D^2 u = u_{11}u_{22} - u_{12}u_{21} \) and the matrix of cofactors is

\[
(U^{ij}) = \begin{pmatrix}
  u_{22} & -u_{21} \\
  -u_{12} & u_{11}
\end{pmatrix}
\]
Among all boundary value problems for 4th order equation of Abreu type:
- It is the problem we understand most
Convex functionals with a convexity constraint

- $\Omega_0$: bounded, open, smooth, convex domain in $\mathbb{R}^n$ ($n \geq 2$).
- $F(x, z, p) : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$: smooth Lagrangian, usually convex in each of $z \in \mathbb{R}$ and $p = (p_1, \cdots, p_n) \in \mathbb{R}^n$.
- $\varphi$: convex, smooth function defined in a neighborhood of $\Omega_0$.

In different scientific disciplines, one usually encounters the following variational problem with a convexity constraint:

$$\inf_{u \in \bar{S}[\varphi, \Omega_0]} \int_{\Omega_0} F(x, u(x), Du(x)) dx$$

where

$$\bar{S}[\varphi, \Omega_0] = \{ u : \Omega_0 \to \mathbb{R} \mid u \text{ is convex,}$$

$$u \text{ admits a convex extension } \varphi \text{ in a neighborhood of } \Omega_0 \}.$$
Two issues:

- The first boundary conditions
  \[ u = \varphi, \quad \partial_\nu u \leq \partial_\nu \varphi \text{ on } \partial \Omega_0. \]

- The Euler-Lagrange equation
Minimize

$$\int_{\Omega_0} \frac{1}{1 + |Du|^2} \, dx$$

over the set

$$\{ u \text{ concave in } \Omega_0, \ 0 \leq u \leq M \}.$$. 
Rochet-Choné (Econometrica, 1998) modeled the monopolist problem in product line design with quadratic cost using

$$\Phi(u) = \int_{\Omega_0} \{x \cdot Du(x) - \frac{1}{2}|Du(x)|^2 - u(x)\} \gamma(x) dx.$$ 

- $\Phi(u)$ is the monopolist’s profit
- $u$ is the buyers’ indirect utility function
- $\Omega_0 \subset \mathbb{R}^n$ is the collection of types of agents; $\gamma$ is the relative frequency of different types of agents in the population.
The Rochet-Choné model for monopolist’s problem

- Rochet-Choné (Econometrica, 1998) modeled the monopolist problem in product line design with quadratic cost using

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\Phi(u) = \int_{\Omega_0} \left\{ x \cdot Du(x) - \frac{1}{2} |Du(x)|^2 - u(x) \right\} \gamma(x) dx.
$$

- $\Phi(u)$ is the monopolist’s profit
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- $\Omega_0 \subset \mathbb{R}^n$ is the collection of types of agents; $\gamma$ is the relative frequency of different types of agents in the population.

- For a consumer of type $x \in \Omega_0$,

$$
u(x) = \max_{q \in Q} \{ x \cdot q - p(q) \}
$$

where $Q \subset \mathbb{R}^n$ is the product line and $p : Q \rightarrow \mathbb{R}$ is a price schedule that the monopolist needs to both design to maximize her profit.

- Clearly, $u$ is convex.
Analysis of the Rochet-Choné model

We are led to minimizing

\[ J_0(u) = \int_{\Omega_0} F^{RC}(x, u(x), Du(x)) \, dx \]

over convex functions \( u \) where

\[ F^{RC}(x, z, p) = \frac{1}{2} |p|^2 \gamma(x) - x \cdot p \gamma(x) + z \gamma(x). \]

Even in this simple looking variational problem, the convexity is not easy to handle from a numerical standpoint. Interesting progress: Mérigot and Oudet (SINUM, 2014)

We analyze this problem, and its generalization, from an asymptotic analysis standpoint.
Euler-Lagrange equation

\[
\inf_{u \in \bar{S}[\varphi, \Omega_0]} \int_{\Omega_0} F(x, u(x), Du(x)) \, dx \tag{1}
\]

where

\[
\bar{S}[\varphi, \Omega_0] = \{ u : \Omega_0 \to \mathbb{R} \mid u \text{ is convex, } u \text{ admits a convex extension } \varphi \text{ in a neighborhood of } \Omega_0 \}.
\]

- Difficult to write down a tractable E-L eqn for the minimizers \( u \).
- For any convex \( v \) in \( \Omega_0 \) with \( v = 0 \) on \( \partial \Omega_0 \)

\[
\int_{\Omega_0} \left[ \frac{\partial F}{\partial z}(x, u(x), Du(x)) - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left( \frac{\partial F}{\partial p_i}(x, u(x), Du(x)) \right) \right] v(x) \, dx \geq 0.
\]

Lions (CRAS, 1998) showed that, in the sense of distributions,

\[
\frac{\partial F}{\partial z}(x, u(x), Du(x)) - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left( \frac{\partial F}{\partial p_i}(x, u(x), Du(x)) \right) = \sum_{i, j=1}^{n} \partial^2 \frac{\partial F}{\partial x_i \partial x_j} \mu_{ij}
\]

for some symmetric non-negative matrix \( \mu = (\mu_{ij}) \) of Radon measures.
Euler-Lagrange equation

\[
\inf_{u \in \bar{S}[\varphi, \Omega_0]} \int_{\Omega_0} F(x, u(x), Du(x)) \, dx
\]

(1)

where

\[
\bar{S}[\varphi, \Omega_0] = \{ u : \Omega_0 \to \mathbb{R} \mid u \text{ is convex,} \]

\[ u \text{ admits a convex extension } \varphi \text{ in a neighborhood of } \Omega_0 \}.
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- Difficult to write down a tractable E-L eqn for the minimizers \( u \).
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\int_{\Omega_0} \left[ \frac{\partial F}{\partial z}(x, u(x), Du(x)) - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left( \frac{\partial F}{\partial p_i}(x, u(x), Du(x)) \right) \right] v(x) \, dx \geq 0.
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Approximate the Euler-Lagrange equation

\[ \frac{\partial F}{\partial z}(x, u(x), Du(x)) - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left( \frac{\partial F}{\partial p_i}(x, u(x), Du(x)) \right) = \sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j} \mu_{ij} \]

- For any \( v \in C^2_0(\Omega_0) \), we have

\[ \int_{\Omega_0} \text{LHS} v dx = \int_{\Omega_0} v_{ij} d\mu_{ij}. \]

- The structure of the matrix \( \mu \) is still mysterious.

\[ \implies \] the need to find suitably explicit approximations of \( \mu \) and minimizers of (1).

- **Earlier work:** Carlier-Radice (2018) when \( F \) does not depend on \( p \);
  - works for all dimensions
  - uses the regular Abreu equations
Approximate the Euler-Lagrange equation

\[
\frac{\partial F}{\partial z}(x, u(x), Du(x)) - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left( \frac{\partial F}{\partial p_i}(x, u(x), Du(x)) \right) = \sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j} \mu_{ij}
\]

- For any \( v \in C_0^2(\Omega_0) \), we have
  \[
  \int_{\Omega_0} \text{LHS} v dx = \int_{\Omega_0} v_{ij} d\mu_{ij}.
  \]

- The structure of the matrix \( \mu \) is still mysterious.
- \( \Rightarrow \) the need to find suitably explicit approximations of \( \mu \) and minimizers of (1).

**Earlier work**: Carlier-Radice (2018) when \( F \) does not depend on \( p \);
  - works for all dimensions
  - uses the regular Abreu equations

**Today’s talk**: the more challenging case when \( F \) depends on the gradient variable.
  - works for two dimensions
  - uses the singular Abreu equations
Let $\Omega$ be a bounded, open, smooth, uniformly convex domain containing $\overline{\Omega_0}$ and let $\varepsilon > 0$.

Approximate the original functional

$$J[u] = \int_{\Omega_0} F(x, u(x), Du(x))dx$$

under

$$\tilde{S}[\varphi, \Omega_0] = \{ u : \Omega_0 \to \mathbb{R} \mid u \text{ is convex, }$$
$$u \text{ admits a convex extension } \varphi \text{ in a neighborhood of } \Omega_0 \}.\$$

by

$$\int_{\Omega_0} F(x, u(x), Du(x))dx + \frac{1}{2\varepsilon} \int_{\Omega\setminus\Omega_0} (u - \varphi)^2 dx - \varepsilon \int_{\Omega} \log \det D^2 u dx.$$
Approximate Lions’ Euler-Lagrange equation by the second boundary value problem of a fourth order equation of Abreu type for a uniformly convex function $u$:

$$
\begin{aligned}
\varepsilon \sum_{i,j=1}^{n} U^{ij} w_{ij} &= f_{\varepsilon}(\cdot, u, Du, D^2 u) \text{ in } \Omega, \\
w &= (\det D^2 u)^{-1} \text{ in } \Omega, \\
u &= \varphi \text{ on } \partial \Omega, \\
w &= \psi \text{ on } \partial \Omega.
\end{aligned}
$$

(2)

Here $(U^{ij})_{1 \leq i, j \leq n}$ is the cofactor matrix of $D^2 u = (u_{ij})$, and

$$
f_{\varepsilon} = \begin{cases}
\frac{\partial}{\partial z} F(x, u(x), Du(x)) - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left( \frac{\partial F}{\partial p_i} (x, u(x), Du(x)) \right) & x \in \Omega_0, \\
\frac{1}{\varepsilon} (u(x) - \varphi(x)) & x \in \Omega \setminus \Omega_0.
\end{cases}
$$
How is the matrix $\mu$ approximated?

\[
\frac{\partial F}{\partial z}(x, u(x), Du(x)) - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left( \frac{\partial F}{\partial p_i}(x, u(x), Du(x)) \right) = \sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j} \mu_{ij}
\]

- Assume $u_\varepsilon$ is a solution to the previous approximation problem and that $u_\varepsilon$ converges to $u$ locally uniformly.
- Then $\varepsilon(D^2 u_\varepsilon)^{-1} \equiv \varepsilon(u_\varepsilon^{ij})$ can be viewed as an approximation of $\mu$. 

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How is the matrix \( \mu \) approximated?

\[
\frac{\partial F}{\partial z}(x, u(x), Du(x)) - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left( \frac{\partial F}{\partial p_i}(x, u(x), Du(x)) \right) = \sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j} \mu_{ij}
\]

- Assume \( u_\varepsilon \) is a solution to the previous approximation problem and that \( u_\varepsilon \) converges to \( u \) locally uniformly.
- Then \( \varepsilon (D^2 u_\varepsilon)^{-1} \equiv \varepsilon (u_\varepsilon^{ij}) \) can be viewed as an approximation of \( \mu \).
- Explanation:
  - \( \sum_{j=1}^{n} \frac{\partial}{\partial x_j} U_\varepsilon^{ij} = 0 \) for all \( i \).
  - \( U_\varepsilon^{ij} w_\varepsilon = (\det D^2 u_\varepsilon) u_\varepsilon^{ij} (\det D^2 u_\varepsilon)^{-1} = u_\varepsilon^{ij} \).
  - we can write the LHS of \( \varepsilon \sum_{i,j=1}^{n} U_\varepsilon^{ij} (w_\varepsilon)^{ij} = f_\varepsilon(\cdot, u_\varepsilon, Du_\varepsilon, D^2 u_\varepsilon) \) as

\[
\varepsilon \sum_{i,j=1}^{n} U_\varepsilon^{ij} \frac{\partial^2 w_\varepsilon}{\partial x_i \partial x_j} = \sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j} (\varepsilon U_\varepsilon^{ij} w_\varepsilon) = \sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j} \varepsilon u_\varepsilon^{ij}.
\]
PDE questions

\[
\begin{aligned}
\sum_{i,j=1}^{n} U_{ij} w_{ij} &= f_{\delta}(\cdot, u, Du, D^2 u) \text{ in } \Omega, \\
\end{aligned}
\]

\[
\begin{aligned}
\quad w &= (\det D^2 u)^{-1} \quad \text{in } \Omega, \\
\quad u &= \varphi \quad \text{on } \partial \Omega, \\
\quad w &= \psi \quad \text{on } \partial \Omega.
\end{aligned}
\]

(3)

\[
f_{\delta} = \begin{cases} 
\frac{\partial}{\partial z} F(x, u(x), Du(x)) - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left( \frac{\partial F}{\partial p_i}(x, u(x), Du(x)) \right) & x \in \Omega_0, \\
\frac{1}{\delta} (u(x) - \varphi(x)) & x \in \Omega \setminus \Omega_0.
\end{cases}
\]

Question 1:

Given \( \varphi, \psi, \) and \( F, \) can we solve the above problem?

Question 2:

Are minimizers of \( J \) well approximated by solutions of (3)?
The heart of the analysis is reduced to $F(x, z, p) = \frac{1}{2} |p|^2 + \frac{\rho}{2} |z|^2$

\[
\begin{align*}
U^{ij} w_{ij} &= f_{\delta}(\cdot, u, Du, D^2 u) \text{ in } \Omega, \\
w &= (\det D^2 u)^{-1} \text{ in } \Omega, \\
 u &= \varphi \text{ on } \partial\Omega, \\
w &= \psi \text{ on } \partial\Omega.
\end{align*}
\]

Here $U = (U^{ij})_{1 \leq i, j \leq n}$ is the cofactor matrix of $D^2 u$ and

\[
f_{\delta} = \begin{cases} 
\rho u(x) - \Delta u(x) & x \in \Omega_0, \\
\frac{1}{\delta}(u(x) - \varphi(x)) & x \in \Omega \setminus \Omega_0.
\end{cases}
\]

**Theorem**

(L-, 2018) Assume that $\rho \geq 0$, $\delta > 0$, $n = 2$, $\Omega$ is uniformly convex, $\varphi \in C^{3,1}(\overline{\Omega})$ and $\psi \in C^{1,1}(\overline{\Omega})$ with $\inf_{\partial\Omega} \psi > 0$. Then, there is a uniformly convex solution $u \in W^{4,q}(\Omega)$ to the above system for all $q \in (n, \infty)$. 

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Convergence of singular Abreu equations

Theorem

(L-, 2018) \( n = 2; \rho > 0; \varphi \in C^{3,1}(\overline{\Omega}) \) uniformly convex and \( \psi \in C^{1,1}(\overline{\Omega}) \) with \( \inf_{\partial \Omega} \psi > 0 \). For \( \varepsilon > 0 \), let \( u_\varepsilon \in W^{4,q}(\Omega) \) be a uniformly convex solution to

\[
\begin{align*}
\varepsilon U^{ij}_\varepsilon (w_\varepsilon)_{ij} &= f_\varepsilon (\cdot, u_\varepsilon, Du_\varepsilon, D^2 u_\varepsilon) \text{ in } \Omega, \\
w_\varepsilon &= (\det D^2 u_\varepsilon)^{-1} \text{ in } \Omega, \\
u_\varepsilon &= \varphi \text{ on } \partial \Omega, \\
w_\varepsilon &= \psi \text{ on } \partial \Omega.
\end{align*}
\]

Here \( U_\varepsilon = (U^{ij}_\varepsilon)_{1 \leq i, j \leq n} \) is the cofactor matrix of \( D^2 u_\varepsilon \) and

\[
f_\varepsilon = (\rho u_\varepsilon - \Delta u_\varepsilon) \chi_{\Omega_0} + \frac{1}{\varepsilon} (u_\varepsilon - \varphi) \chi_{\Omega \setminus \Omega_0}.
\]

Then, \( u_\varepsilon \) converges locally uniformly to the unique minimizer \( u \in \tilde{S}[^{\varphi, \Omega_0}] \) of the problem

\[
\inf_{u \in \tilde{S}[^{\varphi, \Omega_0}]} \int_{\Omega_0} \left( \frac{1}{2} |Du|^2 + \frac{\rho}{2} |u|^2 \right) dx.
\]
In application to the Rochet-Chène model with

\[ F^{RC}(x, z, p) = \frac{1}{2} |p|^2 \gamma(x) - x \cdot p \gamma(x) + z \gamma(x), \]

\( \rho > 0 \) corresponds to adding \( \rho_0 \frac{1}{2} |z|^2 \) to \( F^{RC} \) for some large \( \rho_0 \).

\( \rightsquigarrow \) minimizers of the perturbed Rochet-Choné model in 2D can be approximated, in the uniform norm, by solutions of Abreu type equations.
In application to the Rochet-Chône model with

\[ F^{RC}(x, z, p) = \frac{1}{2} |p|^2 \gamma(x) - x \cdot p \gamma(x) + z \gamma(x), \]

\[ \rho > 0 \] corresponds to adding \( \frac{\rho_0}{2} |z|^2 \) to \( F^{RC} \) for some large \( \rho_0 \).

\( \rightsquigarrow \) minimizers of the perturbed Rochet-Choné model in 2D can be approximated, in the uniform norm, by solutions of Abreu type equations.

Open: the uniqueness of uniformly convex solutions \( u_\varepsilon \).

Surprising fact: full convergence of all solutions \( u_\varepsilon \) to the unique minimizer \( u \in \bar{S}[\varphi, \Omega_0] \) of the functional

\[ \int_{\Omega_0} \left( \frac{1}{2} |Du|^2 + \frac{\rho}{2} |u|^2 \right) dx \]
Strategies of the proof of convergence

For the convergence of $u_\varepsilon$ to the minimizer $u$ of

$$J[u] = \int_{\Omega_0} \left( \frac{1}{2} |Du|^2 + \frac{\rho}{2} |u|^2 \right) dx$$

- Uniform a priori estimates in $L^2(\Omega)$ for $u_\varepsilon$:
  $$\int_{\partial\Omega} \varepsilon(u_\varepsilon)_\nu^2 + \rho \int_{\Omega_0} |u_\varepsilon - \varphi|^2 dx + \int_{\Omega \setminus \Omega_0} \frac{1}{\varepsilon} |u_\varepsilon - \varphi|^2 dx \leq C.$$

- $u_\varepsilon$ is an almost minimizer of
  $$J_\varepsilon(v) = J[v] + \frac{1}{2\varepsilon} \int_{\Omega \setminus \Omega_0} (v - \varphi)^2 dx - \varepsilon \int_{\Omega} \log(\det D^2 v) dx.$$
For the solvability in $W^{4,q}(\Omega)$ of

$$
\begin{align*}
U_{ij} w_{ij} &= (\rho u - \Delta u) \chi_{\Omega_0} + \frac{1}{\delta} (u - \varphi) \chi_{\Omega\setminus\Omega_0} \quad \text{in } \Omega, \\
\varepsilon &= (\det D^2 u)^{-1} \quad \text{in } \Omega, \\
u &= \varphi \quad \text{on } \partial \Omega, \\
\psi &= \varphi \quad \text{on } \partial \Omega.
\end{align*}
$$

- we establish a priori estimates in $W^{4,q}(\Omega)$ and then use degree theory
- **Key step**: obtain estimates for $u$ in $C^{2,\alpha}(\overline{\Omega})$ and the Hessian determinant bounds

$$
C^{-1} \leq \det D^2 u \leq C.
$$
It is not known a priori if we can establish the lower bound and upper bound for \( \det D^2 u \). Thus, \( \Delta u \) can be only a measure. ~\( \Rightarrow \) singular equations
Fundamental difficulties

\[
\begin{aligned}
\sum_{i,j=1}^{n} U^{ij} w_{ij} &= -\Delta u \quad \text{in } \Omega, \\
w &= (\det D^2 u)^{-1} \quad \text{in } \Omega, \\
u &= \varphi \quad \text{on } \partial \Omega, \\
w &= \psi \quad \text{on } \partial \Omega.
\end{aligned}
\]

- It is not known a priori if we can establish the lower bound and upper bound for \( \det D^2 u \). Thus, \( \Delta u \) can be only a measure. \( \rightsquigarrow \) singular equations

- Even if we can prove \( \lambda_1 \leq \det D^2 u \leq \lambda_2 \) in \( \Omega \), then we can only obtain \( \Delta u \in L^{1+\varepsilon_0}(\Omega) \) where \( \varepsilon_0 = \varepsilon_0(n, \lambda_1, \lambda_2) > 0 \). (From regularity results for the Monge-Ampère equation by De Philippis-Figalli-Savin (2013), Schmidt (2013) and Savin (2013))

- From Wang's counter-example (1995), \( \varepsilon_0 \to 0 \) when \( \lambda_2/\lambda_1 \to \infty \). Hence, the RHS has low integrability, usually less than the dimension \( n \).

- Optimal results on the solvability of the second boundary value problem of the regular Abreu equation (with RHS independent of \( u \)) do not apply.
Counterexamples in the Monge-Ampère equation

- **Counterexample 1.** Given any $\Lambda > 0$, Mooney (Anal. PDE, 2016) constructed a convex function $u$ in $\mathbb{R}^2$ satisfying
  \[
  \det D^2 u \leq \Lambda \quad \text{in } B_1 \subset \mathbb{R}^2, \quad u = 0 \text{ on } \partial B_1
  \]
  such that $D^2 u \notin L^1_{\text{loc}}$.

- **Counterexample 2.** Given any $\Lambda > 0$ and $\varepsilon > 0$, Wang (PAMS, 1995) found a $\lambda_0 > 0$ and constructed a smooth, convex function satisfying
  \[
  \lambda_0 \leq \det D^2 u \leq \Lambda \quad \text{in } B_1 \subset \mathbb{R}^2, \quad u = 0 \text{ on } \partial B_1
  \]
  such that
  \[
  \int_{B_{1/2}} \| D^2 u \|^{1+\varepsilon} \, dx = \infty.
  \]
Solvability the second BVP of the regular Abreu equation

- Studied by Trudinger-Wang (Annals, 2008), Chau-Weinkove (MRL, 2015), L- (IMRN, 2013; JDE, 2016)

If $f \in L^q(\Omega)$ where $q > n$, and $\phi \in W^{4, q}(\Omega)$, $\psi \in W^{2, q}(\Omega)$ with $\inf_{\partial \Omega} \psi > 0$, and $G(t) = t^{\frac{\theta}{1 - \theta}}$ where $0 < \theta < \frac{1}{n}$ or $G(t) = \log t$,

then we have a unique uniformly convex $W^{4, q}(\Omega)$ solution to

\[
\begin{cases}
\sum_{i,j=1}^n U_{ij} w_{ij} = f \quad \text{in } \Omega, \\
w = \frac{d}{dt}(\det D^2 u) \quad \text{in } \Omega, \\
u = \phi \quad \text{on } \partial \Omega, \\
w = \psi \quad \text{on } \partial \Omega.
\end{cases}
\]
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- (L-, 2016) If $f \in L^q(\Omega)$ where $q > n$, and $\varphi \in W^{4,q}(\Omega)$, $\psi \in W^{2,q}(\Omega)$ with $\inf_{\partial \Omega} \psi > 0$, and $G(t) = \frac{t^{\theta} - 1}{\theta}$ where $0 < \theta < 1/n$ or $G(t) = \log t$, then we have a unique uniformly convex $W^{4,q}(\Omega)$ solution to

$$
\begin{cases}
\sum_{i,j=1}^{n} U^{ij} w_{ij} = f & \text{in } \Omega, \\
w = G'(\det D^2 u) & \text{in } \Omega, \\
u = \varphi & \text{on } \partial \Omega, \\
w = \psi & \text{on } \partial \Omega
\end{cases}
$$

(4)

- (L-, 2016) If $f$ is only in $L^q(\Omega)$ with $q < n$ then solutions to (4) might not be in $W^{4,q}(\Omega)$.
Proof of solvability 1

\[
\begin{aligned}
\sum_{i,j=1}^{n} U_{ij} w_{ij} &= -\Delta u & \text{in } \Omega, \\
\omega &= (\det D^2 u)^{-1} & \text{in } \Omega, \\
u &= \varphi & \text{on } \partial\Omega, \\
w &= \psi & \text{on } \partial\Omega.
\end{aligned}
\]

Assume \(\inf_{\partial\Omega} \psi > 0\). Recall \((U_{ij}) = (\det D^2 u)(D^2 u)^{-1}\).

- By the maximum principle, \(w \geq \inf_{\partial\Omega} \psi \geq C^{-1}\).
- If \(n = 2\), then \(\text{trace}(U_{ij}) = \Delta u\). Thus \(U_{ij}(w + \frac{1}{2}|x|^2)_{ij} = 0\).
- By the maximum principle,

\[
w + \frac{1}{2}|x|^2 \leq \max_{\partial\Omega}(\psi + \frac{1}{2}|x|^2) \leq C.
\]

Thus \(C^{-1} \leq w \leq C\) and hence \(C^{-1} \leq \det D^2 u \leq C\).
Tool 1: $W^{2,1+\varepsilon}$ regularity for the Monge-Ampère

**Theorem**

Let $u$ be a convex solution to the Monge-Ampère equation

$$\det D^2 u = f \quad \text{in } B_1 \subset \mathbb{R}^n, \quad u = 0 \text{ on } \partial B_1, \quad \text{where } \lambda \leq f \leq \Lambda.$$

(i) **De Philippis-Figalli** (*Inventiones*, 2013)

$$D^2 u \in L^1(B_{1/2}).$$
Tool 1: $W^{2,1+\varepsilon}$ regularity for the Monge-Ampère

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(i) **De Philippis-Figalli** (*Inventiones, 2013*)

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(ii) **De Philippis-Figalli-Savin** (*Math. Ann, 2013*), **Schmidt** (*Adv. Math, 2013*), **Savin** (*PAMS, 2013*): There are $\varepsilon(n, \lambda, \Lambda) > 0$ and $C(n, \lambda, \Lambda) > 0$ such that

$$\int_{B_1} \| D^2 u \|^{1+\varepsilon} \, dx \leq C.$$

In particular, $D^2 u \in L^{1+\varepsilon}(B_1)$. 
Proof of solvability 2

\[ \sum_{i,j=1}^{n} U_{ij} w_{ij} = -\Delta u \quad \text{in } \Omega, \]

\[ w = (\det D^2 u)^{-1} \quad \text{in } \Omega, \]

\[ u = \varphi \quad \text{on } \partial \Omega, \]

\[ w = \psi \quad \text{on } \partial \Omega. \]

Assume \( \inf_{\partial \Omega} \psi > 0 \) and \( n = 2 \).

- Thus \( C^{-1} \leq w \leq C \) and hence \( C^{-1} \leq \det D^2 u \leq C \).
- From the global \( W^{2,1+\varepsilon} \) regularity for the Monge-Ampère: \( \Delta u \in L^{1+\varepsilon_0}(\Omega) \) where \( \varepsilon_0 > 0 \) can be arbitrary small.
Proof of solvability 2

\[
\begin{aligned}
\sum_{i,j=1}^{n} U_{ij} w_{ij} &= -\Delta u & \text{in } \Omega, \\
w &= (\det D^2 u)^{-1} & \text{in } \Omega, \\
u &= \varphi & \text{on } \partial \Omega, \\
w &= \psi & \text{on } \partial \Omega.
\end{aligned}
\]

Assume \( \inf_{\partial \Omega} \psi > 0 \) and \( n = 2 \).

- Thus \( C^{-1} \leq w \leq C \) and hence \( C^{-1} \leq \det D^2 u \leq C \).
- From the global \( W^{2,1+\varepsilon} \) regularity for the Monge-Ampère: \( \Delta u \in L^{1+\varepsilon_0}(\Omega) \) where \( \varepsilon_0 > 0 \) can be arbitrary small.
- **Critical step:** From L-Nguyen (2017), can bound \( w \) in \( C^\alpha(\overline{\Omega}) \).
- From Trudinger-Wang (Annals, 2008), can bound \( u \) in \( C^{2,\alpha}(\overline{\Omega}) \).
- Now, bootstrap.
Theorem (L-Nguyen (JGA, 2017))

**Ω**: bounded, uniformly convex in $\mathbb{R}^n$ with $\partial \Omega \in C^3$. Let $u \in C^{0,1}(\overline{\Omega}) \cap C^2(\Omega)$ be a convex function satisfying $0 < \lambda \leq \det D^2u \leq \Lambda < \infty$, and $u|_{\partial \Omega} \in C^3$. Denote by $(U_{ij})$ the cofactor matrix of $D^2u$. Let $v \in C(\overline{\Omega}) \cap W^{2,n}_{loc}(\Omega)$ be the solution to the LMA equation

$$
\begin{align*}
U_{ij}v_{ij} &= f \quad \text{in } \Omega, \\
v &= \varphi \quad \text{on } \partial \Omega,
\end{align*}
$$

where $\varphi \in C^\alpha(\partial \Omega)$ and $f \in L^q(\Omega)$ with $q > n/2$. Then,

$$
\|v\|_{C^\beta(\overline{\Omega})} \leq C \left( \|\varphi\|_{C^\alpha(\partial \Omega)} + \|f\|_{L^q(\Omega)} \right)
$$

where $\beta$ depends only on $\lambda, \Lambda, n, q, \alpha$, and $C$ depends only on $\lambda, \Lambda, n, q, \alpha$, $\text{diam}(\Omega)$, $\|u\|_{C^3(\partial \Omega)}$, $\|\partial \Omega\|_{C^3}$ and the uniform convexity of $\Omega$.

Best quantitative estimates for Hessian: $D^2u \in L^{1+\varepsilon}(\Omega)$. 

with right hand side having low integrability:
Based on the combined work of

- Caffarelli-Gutiérrez’s interior Harnack inequality for LMA (AJM, 1997)
- Savin’s localization theorem at the boundary for Monge-Ampère (JAMS, 2013)
- Boundary Harnack inequality for LMA (L-, TAMS, 2017)
- Optimal integrability estimate for the Green’s function of the LMA operator (L-, TAMS, 2017)
Alternative for critical step

\[
\begin{cases}
U_{ij}w_{ij} = \text{div } F & \text{in } \Omega, \\
w = \varphi & \text{on } \partial \Omega,
\end{cases}
\]

- \( C^{-1} \leq \det D^2 u \leq C; F \in L^\infty(\Omega) \)
- In the **critical step**: \( F = -Du \)
- (L-, CMP, 2018): Global Hölder for LMA equations with divergence RHS implies \( w \in C^\alpha(\overline{\Omega}) \).
A remark on solvability of singular Abreu equations

Our analysis extends to certain non-convex Lagrangians $F$ including the case of

$$F(x, z, p) = \frac{1}{4}(z^2 - 1)^2 + \frac{1}{2}|p|^2$$

arising from the study of Allen-Cahn functionals.

**Theorem**

Assume that $\varphi \in C^\infty(\overline{\Omega})$ and $\psi \in C^\infty(\overline{\Omega})$ with $\inf_{\partial\Omega}\psi > 0$. Then there exists a smooth, uniformly convex solution $u \in C^\infty(\overline{\Omega})$ to the following second boundary value problem:

$$\begin{cases}
\sum_{i,j=1}^{2} U_{ij} w_{ij} = u^3 - u - \Delta u \text{ in } \Omega, \\
 w = (\det D^2 u)^{-1} \text{ in } \Omega, \\
 u = \varphi \text{ on } \partial\Omega, \\
 w = \psi \text{ on } \partial\Omega.
\end{cases} \quad (5)$$
Thank you for your attention!