Global $C^{2,\alpha}$ estimates for the Monge-Ampère equation on polygonal domains in the plane

Nam Q. Le
Indiana University

Geometric Analysis seminar
School of Mathematical Sciences
Peking University
October 23, 2019

Joint work with Ovidiu Savin

Funding from NSF grant DMS-1764248 is gratefully acknowledged.
Plan of the talk

- Recall some classical global $C^{2,\alpha}$ estimates for the Monge-Ampère equation
- Global $C^{2,\alpha}$ estimates for the Monge-Ampère equation on polygonal domains in the plane
- A Liouville type theorem for the Monge-Ampère in angles in the plane
- Ideas of the proofs
The Monge-Ampère equation

- Fully nonlinear, degenerate elliptic equation
- Classically, it consists of prescribing the determinant of the Hessian of a convex function \( u \) inside some domain \( \Omega \in \mathbb{R}^n \):

\[
\det D^2 u = f \text{ in } \Omega
\]

- Arising in several problems in analysis and geometry, such as
  - The prescribed Gaussian curvature equation

\[
\det D^2 u = K(1 + |Du|^2)^{\frac{n+2}{2}}
\]

  - affine geometry and complex geometry
  - Optimal transportation

\[
g(Du) \det D^2 u = f
\]
The Abreu equation in differential geometry

\[ U^{ij} w_{ij} = -A, \ w = [\det D^2 u]^{-1}. \]

The Abreu equation with suitable “stability” conditions appears in the constant scalar curvature problem for toric manifolds; cf. Abreu (IJM, 1998) and Donaldson (GAFA, 2009).

The affine maximal surface equation in affine geometry

\[ \sum_{i,j=1}^{n} U^{ij} w_{ij} = 0, \ w = [\det D^2 u]^{-\frac{n+1}{n+2}}. \]
Affine invariance

\[ \det D^2 u(x) = f(x) \]

- If \( T : \mathbb{R}^n \to \mathbb{R}^n \) is a linear transformation then for 
  \[ v(x) := u(Tx), \]
  we have 
  \[ \det D^2 v(x) = (\det T)^2 f(Tx) \]
- In particular, if \( \det T = 1 \) then 
  \[ \det D^2 v(x) = f(Tx). \]
Global $C^{2,\alpha}$ estimates for Monge-Ampère: convex domains

\[
\begin{aligned}
\begin{cases}
\det D^2 u = f \text{ in } \Omega, \\
u = \varphi \text{ on } \partial \Omega.
\end{cases}
\end{aligned}
\]

non degenerate RHS:

\[
0 < \lambda \leq f \leq \Lambda < \infty.
\]

On smooth and strictly convex domains $\Omega$,

- When $\partial \Omega, \varphi \in C^{3,1}$,

- When $\partial \Omega, \varphi \in C^{3}$,
  - Wang (Analysis, 1996): global $C^{2,\alpha}$ estimates, $f \in C^{0,1}(\Omega)$
  - Trudinger-Wang (Annals, 2008): global $C^{2,\alpha}$ estimates, $f \in C^{\alpha}(\Omega)$
  - Savin (JAMS, 2013): pointwise $C^{2,\alpha}$ estimates at the boundary, $f$
  - pointwise $C^{\alpha}$
Pointwise $C^\alpha$ and pointwise $C^{2,\alpha}$

- $u$ is pointwise $C^\alpha$ at $x_0$ (write $u \in C^\alpha(x_0)$) if, in the domain of definition of $u$,
  \[ u(x) = u(x_0) + O(|x - x_0|^\alpha). \]

- $u$ is pointwise $C^{2,\alpha}$ at $x_0$ (write $u \in C^{2,\alpha}(x_0)$) if there exists a quadratic polynomial $P_{x_0}$ such that, in the domain of definition of $u$,
  \[ u(x) = P_{x_0}(x) + O(|x - x_0|^{2+\alpha}). \]

- $u$ is pointwise $C^2$ at $x_0$ (write $u \in C^2(x_0)$) if there exists a quadratic polynomial $P_{x_0}$ such that, in the domain of definition of $u$,
  \[ u(x) = P_{x_0}(x) + o(|x - x_0|^2). \]
Global $C^{2,\alpha}$ estimates for MA: non-convex domains

\[
\begin{cases}
\det D^2 u = f \text{ in } \Omega, \\
u = \varphi \text{ on } \partial \Omega.
\end{cases}
\]

non degenerate RHS:

\[0 < \lambda \leq f \leq \Lambda < \infty.\]

On bounded smooth domains $\Omega$ that are not necessarily convex,

- global $C^{2,\alpha}$ estimates by Guan-Spruck (Annals, 1993), when $f \in C^{1,1}(\Omega)$, $\partial \Omega$, $\varphi \in C^{3,1}$, and there exists a convex strict subsolution $u \in C^2(\Omega)$ taking the boundary values $\varphi$,
- convex strict subsolution: $\det D^2 u > f$ in $\bar{\Omega}$ and $u = \varphi$ on $\partial \Omega$
- Guan (TAMS, 1998) removed the strictness of the subsolution $u$
Global $C^{2,\alpha}$ estimates for MA: non-convex domains

\[ \begin{cases} \det D^2 u = f \text{ in } \Omega, \\ u = \varphi \text{ on } \partial \Omega. \end{cases} \]

non degenerate RHS:

\[ 0 < \lambda \leq f \leq \Lambda < \infty. \]

On bounded smooth domains $\Omega$ that are not necessarily convex,

- global $C^{2,\alpha}$ estimates by Guan-Spruck (Annals, 1993), when $f \in C^{1,1}(\Omega)$, $\partial \Omega$, $\varphi \in C^{3,1}$, and there exists a convex strict subsolution $u \in C^2(\Omega)$ taking the boundary values $\varphi$,
- convex strict subsolution: $\det D^2 u > f$ in $\overline{\Omega}$ and $u = \varphi$ on $\partial \Omega$,
- Guan (TAMS, 1998) removed the strictness of the subsolution $u$.

**Question:**

What happen if we relax the smoothness of the domains $\Omega$?
Global $C^{2,\alpha}$ estimates for Monge-Ampère on 2D polygons

**Theorem (L-Savin, 2019)**

Let $\Omega$ be a bounded convex polygonal domain in $\mathbb{R}^2$; $u$ convex solving

\[
\begin{aligned}
\det D^2 u &= f \quad \text{in } \Omega, \\
u &= \varphi \quad \text{on } \partial \Omega.
\end{aligned}
\]

Assume that

- $f \in C^\beta(\overline{\Omega})$, $f > 0$, and $\varphi \in C^{2,\beta}(\partial \Omega)$ for some $\beta \in (0,1)$,
- there is a globally $C^2$, convex, strict subsolution $u \in C^2(\overline{\Omega})$ to (1).

Then

\[
u \in C^{2,\alpha}(\overline{\Omega}),
\]

for some $\alpha > 0$. The constant $\alpha$ and the global $C^{2,\alpha}$ norm $\| u \|_{C^{2,\alpha}(\overline{\Omega})}$ depend on $\Omega$, $\beta$, $\min_{\Omega} f$, $\| f \|_{C^\beta(\overline{\Omega})}$, $\| \varphi \|_{C^{2,\beta}(\partial \Omega)}$, $\| u \|_{C^2(\overline{\Omega})}$ and the differences $\det D^2 u - f$ at the vertices of $\Omega$. 
Optimality of the assumptions

- If we relax the assumption on $u$ in Theorem 1 to be a subsolution (not necessarily strict), then $u \in C^2(\overline{\Omega})$.
- Without the existence of a subsolution $u \in C^2(\overline{\Omega})$ to (1), solutions might develop conical singularities at the corners where the Hessian matrix becomes unbounded.
Optimality of the assumptions

- If we relax the assumption on $u$ in Theorem 1 to be a subsolution (not necessarily strict), then $u \in C^2(\Omega)$.

- Without the existence of a subsolution $u \in C^2(\Omega)$ to (1), solutions might develop conical singularities at the corners where the Hessian matrix becomes unbounded.

- A necessary condition for the $C^2$ estimates is the existence of a classical convex subsolution with the same boundary data. By the above results, this condition is sufficient.

- Contrast with the case of second order linear elliptic equations: regularity of solutions depends on the smallness of the angles at the vertices.
Consider an $\alpha$ angle domain in the plane:

$$A_\alpha := \{(r, \theta) : r > 0; 0 < \theta < \alpha\}.$$

The function

$$v(r, \theta) = r^\frac{\pi}{\alpha} \sin\left(\frac{\pi \theta}{\alpha}\right)$$

is harmonic in $A_\alpha$ and vanishes on $\partial A_\alpha$.

- $v \in C^{2,\beta}(A_\alpha \cap B_1)$ if $\alpha < \pi/2$.
- $v$ is only in $C^{1,\beta}(A_\alpha \cap B_1)$ if $\alpha > \pi/2$. 
Higher dimensional case

\( \Omega: \) bounded convex polytope in \( \mathbb{R}^n \); \( u \) convex solving

\[
\begin{cases}
\det D^2 u = f & \text{in } \Omega, \\
u = \varphi & \text{on } \partial \Omega.
\end{cases}
\]

Theorem 1 cannot hold in \( n \geq 3 \) dimensions.

- **Counterexample.** Take \( \Omega = [0,1]^3 \subset \mathbb{R}^3 \), \( f \equiv c < 1 \), and \( \varphi = \frac{|x|^2}{2} \) on \( \partial \Omega \).
  - Then, \( u := \frac{|x|^2}{2} \) is a globally \( C^2 \), convex, strict subsolution.
  - But \( u \) cannot be \( C^2 \) at the origin; since otherwise the boundary data imposes \( D^2 u(0) = I \) hence \( \det D^2 u(0) = 1 \neq f(0) \).
Higher dimensional case

\[ \Omega: \text{bounded convex polytope in } \mathbb{R}^n; \text{ } u \text{ convex solving} \]

\[
\begin{align*}
\begin{cases}
\det D^2 u &= f \text{ in } \Omega, \\
u &= \varphi \text{ on } \partial \Omega.
\end{cases}
\end{align*}
\]

Theorem 1 cannot hold in \( n \geq 3 \) dimensions.

- **Counterexample.** Take \( \Omega = [0, 1]^3 \subset \mathbb{R}^3, f \equiv c < 1, \) and \( \varphi = \frac{|x|^2}{2} \) on \( \partial \Omega. \)
  - Then, \( u := \frac{|x|^2}{2} \) is a globally \( C^2 \), convex, strict subsolution.
  - But \( u \) cannot be \( C^2 \) at the origin; since otherwise the boundary data imposes \( D^2 u(0) = I \) hence \( \det D^2 u(0) = 1 \neq f(0). \)
  - **Computation.**
    - If \( u \) is \( C^2 \) at 0 then near 0,
      \[
u(x_1, x_2, x_3) = \text{affine function } + \sum_{i=1}^{3} a_i x_i^2 + \sum_{1 \leq i < j \leq 3} b_{ij} x_i x_j + o(|x|^2). \]
      - Use 3 edges of \( \partial \Omega \) starting from 0 to get \( a_i = 1/2. \)
      - Use 3 faces of \( \partial \Omega \) starting from 0 to get \( b_{ij} = 0. \)
Features of our global $C^{2,\alpha}$ estimates

- Establishing continuity estimates of $D^2u$ for the solutions to the Monge-Ampère equation near the vertices of a domain with corners.
- Our $C^{2,\alpha}$ estimates are not stable under small perturbations of the data $\varphi$ and $f$.
- The $C^{2,\alpha}$ norm of the solution $u$ depends crucially on the $C^2$ norm of the subsolution $u$ and on the differences $\det D^2u - f$ at the vertices of $\Omega$.
- We show that it is possible for $D^2u$ to oscillate of order 1 in an arbitrarily small neighborhood of a vertex when $\det D^2u$ and $f$ are allowed to be sufficiently close at that vertex.
- Proof of the global $C^{2,\alpha}$ estimates: Reduction to Liouville type theorem
The Jörgens-Calabi-Pogorelov theorem

Theorem (Jörgens-Calabi-Pogorelov)

Any classical convex solution of

$$\det D^2 u = 1 \text{ in } \mathbb{R}^n \ (n \geq 2)$$

must be a quadratic polynomial.

Proved by

- Calabi ($n \leq 5$, Michigan Math. J., 1958)
- Pogorelov (all $n \geq 2$, Geometriae Dedicata, 1972)
Savin’s theorem

Theorem (Savin, JDE, 2014)

Assume the convex function $u \in C(\mathbb{R}_+^n)$ satisfies

$$\det D^2 u = 1 \text{ in } \mathbb{R}_+^n \text{ and } u(x', 0) = \frac{1}{2} |x'|^2.$$ 

If there exists $\varepsilon > 0$ small such that $u = O(|x|^{3-\varepsilon})$ as $|x| \to \infty$, then

$$u(Ax) = bx_n + \frac{1}{2} |x|^2$$

for some sliding $A$ along $\{x_n = 0\}$, and some constant $b$.

- A linear map $A$ of the form $Ax = x + \tau x_n$, with $\tau \cdot e_n = 0$ is called a sliding along $\{x_n = 0\}$.
- Without the sub-cubic growth condition, there is another solution

$$\frac{x_1^2}{2(1 + x_n)} + \frac{1}{2}(x_2^2 + \cdots + x_n^2) + \frac{x_n^3}{6}.$$
A global problem on the first quadrant in the plane

- At a vertex of the polygon the solution $u$ to (1) is pointwise $C^{1,1}$: bounded above by the convex function generated by the boundary data $\varphi$ and bounded below by the tangent plane of $u$, which is also the tangent plane for the upper barrier.
- Using the affine invariance of the MA equation, we may assume after an affine transformation that $\Omega$ is given by the first quadrant
  \[ Q := \{ x = (x_1, x_2) \in \mathbb{R}^2 : x_1, x_2 > 0 \}, \]
in a neighborhood of the origin, and $\varphi_{x_1x_1}(0) = \varphi_{x_2x_2}(0) = 1$.
- Then a quadratic blow-up of the solution must converge to a global convex solution defined in the first quadrant $Q$ that satisfies
  \[ \det D^2 u = c, \quad \text{and} \quad u \geq 0, \quad \text{in } Q, \]
  for some constant $c > 0$, and
  \[ u(x) = \frac{|x|^2}{2} \quad \text{on } \partial Q. \]
Special solutions

Let \( c \in (0, 1) \). Consider

\[
\det D^2 u = c \quad \text{in} \ Q, \quad \text{and} \quad u(x) = \frac{|x|^2}{2} \quad \text{on} \ \partial Q.
\]

- If \( u \) is a solution then so is any of its quadratic rescaling

\[
u_\lambda(x) := \lambda^2 u\left(\frac{x}{\lambda}\right) \quad (\lambda > 0)
\]

- Two quadratic polynomials:

\[
P_c^\pm(x) := \frac{1}{2} x_1^2 + \frac{1}{2} x_2^2 \pm \sqrt{1 - c} \ x_1 x_2.
\]

- There are at least two more non-quadratic solutions \( \bar{P}_c, P_c \) so that

\[
P_c^- < P_c^- < \bar{P}_c < P_c^+ \quad \text{in} \ Q, \quad \text{and} \ \bar{P}_c(1, 1) = 1, \quad P_c(1, 1) = 0.
\]
Construction of $\bar{P}_c$

- For each $R > 0$, we solve the Dirichlet problem on $Q \cap B_R$
  \[
  \begin{cases}
    \det D^2 P_R = c & \text{in } Q \cap B_R, \\
    P_R = P_c^- + t_R x_1 x_2 & \text{on } \partial (Q \cap B_R),
  \end{cases}
  \]
  where $t_R \in (0, 2\sqrt{1-c})$ is chosen such that $P_R(1,1) = 1$.
- The existence follows by continuity. When $t_R = 0$, we have $P_R = P_c^-$ with $P_c^-(1,1) = 1 - \sqrt{1-c}$, and when $t_R = 2\sqrt{1-c}$, we have $P_R = P_c^+$ with $P_c^+(1,1) = 1 + \sqrt{1-c}$.
- From $t_R \in (0, 2\sqrt{1-c})$, we have $P_c^- \leq P_c^- + t_R x_1 x_2 \leq P_c^+$ on $\partial (Q \cap B_R)$. By the comparison principle, $P_c^- \leq P_R \leq P_c^+$ in $Q \cap B_R$, hence the $P_R$'s are locally bounded independent on $R$.
- We let $R \to \infty$ and, by compactness extract a convergence subsequence of $P_R$ to $\bar{P}_c$ satisfying
  \[
  \det D^2 u = c \quad \text{in } Q, \quad \text{and} \quad u(x) = \frac{|x|^2}{2} \quad \text{on } \partial Q \text{ and } \bar{P}_c(1,1) = 1.
  \]
  Moreover, $P_c^- < \bar{P}_c < P_c^+$ in $Q$. 
Properties of $\tilde{P}_c$ and $P_c$

- At the origin
  - $\tilde{P}_c$ is pointwise $C^{2,\alpha}$ for some $\alpha = \alpha(c) \in (0, 1)$ and
    $\tilde{P}_c(x) = P_c^+(x) + O(|x|^{2+\alpha})$ near $x = 0$.
  - $P_c$ has a conical singularity.

- At infinity,
  $$\tilde{P}_c(x) - P_c^-(x) = O(|x|^{2-\alpha}).$$
  $$P_c(x) - P_c^-(x) = O(|x|^{2-\alpha}).$$
Let $c > 0$. Assume that $u$ is a convex solution to

\[ \det D^2 u = c \text{ in } Q, \quad u \geq 0 \text{ in } Q, \text{ and } u(x) = \frac{|x|^2}{2} \text{ on } \partial Q. \quad (2) \]

Then $c \leq 1$ and

(i) if $c = 1$ then the only solution is $u(x) = \frac{|x|^2}{2}$.

(ii) (not assuming $u \geq 0$) if $c < 1$ then either

$u = P_c^\pm$, or $u(x) = \lambda^2 \tilde{P}_c \left( \frac{x}{\lambda} \right)$, or $u(x) = \lambda^2 P_c \left( \frac{x}{\lambda} \right)$ for some $\lambda \in (0, \infty)$ where $\tilde{P}_c, P_c$ are particular solutions so that

$P_c^- < P_c^- < \tilde{P}_c < P_c^+ \text{ in } Q, \text{ and } \tilde{P}_c(1,1) = 1, \quad P_c(1,1) = 0.$
The quadratic polynomial $P_c^-$ is unstable for the $C^2$ norm.

Any small positive perturbation on $\partial B_1 \cap Q$ produces a jump of order 1 for $D^2 u(0)$.

A small negative perturbation on $\partial B_1 \cap Q$ produces a conical singularity at the origin, i.e., $\|D^2 u(x)\| \to \infty$ as $x \to 0$.

On the other hand, in Theorem 1 the existence of a global strict subsolution $u \in C^2$ prevents $D^2 u$ being close to $D^2 P_c^-$ near the origin.
We say that \( u \) is \( C^{2,\alpha} \) at \( x_0 \), and write \( u \in C^{2,\alpha}(x_0) \), if there exists a quadratic polynomial \( P_{x_0} \) such that, in the domain of definition of \( u \),

\[
u(x) = P_{x_0}(x) + O(|x - x_0|^{2+\alpha}).\]

Assume that the convex function \( u \) solves the following Dirichlet problem for the Monge-Ampère equation

\[
\det D^2 u = f \quad \text{in} \ Q, \quad u = \varphi \quad \text{on} \ \partial Q.
\]

Crucial to our analysis are the following pointwise \( C^{2,\alpha} \) estimates at \( 0 \) when \( f \) is close to \( c \) and \( \varphi \) to

\[
q(x) := \frac{|x|^2}{2}.
\]
Proposition

Let \( c \in (0, 1) \). Assume that \( u \) satisfies (3) and suppose that

\[
|u - P_c^+| \leq \varepsilon \quad \text{and} \quad |f - c| \leq \delta \varepsilon \quad \text{in} \ B_1 \cap Q, \quad \text{and} \quad |\varphi - q| \leq \delta \varepsilon \quad \text{on} \ B_1 \cap \partial Q,
\]

for some \( \varepsilon \leq \varepsilon_0(c) \) small and \( \delta(c) \) small. Then there exist \( \alpha \in (0, 1) \) and \( r \leq \frac{1}{2} \) depending only on \( c \) such that \( |u - P_c^+| \leq \varepsilon r^{2+\alpha} \) in \( B_r \cap Q \).
Key to pointwise $C^{2,\alpha}$ estimates for $u$ being close to $P^+_c$

\[
\det D^2 u = f \quad \text{in } Q, \quad u = \varphi \quad \text{on } \partial Q. \tag{3}
\]

**Proposition**

Let $c \in (0, 1)$. Assume that $u$ satisfies (3) and suppose that

\[
|u - P^+_c| \leq \varepsilon \quad \text{and} \quad |f - c| \leq \delta \varepsilon \quad \text{in } B_1 \cap Q, \quad \text{and} \quad |\varphi - q| \leq \delta \varepsilon \quad \text{on } B_1 \cap \partial Q,
\]

for some $\varepsilon \leq \varepsilon_0(c)$ small and $\delta(c)$ small. Then there exist $\alpha \in (0, 1)$ and $r \leq \frac{1}{2}$ depending only on $c$ such that $|u - P^+_c| \leq \varepsilon r^{2+\alpha}$ in $B_r \cap Q$.

Why true? Via affine transformations, can transform $P^+_c$ into $q(x) = \frac{|x|^2}{2}$ on angular domains $Q^+_c$ in $\mathbb{R}^2$. Then the linearized operator of $\det D^2 u$ around $q$ is the Laplace operator. At the vertex 0, the opening of $Q^+_c$ is an acute angle $\alpha^+_c \in (0, \frac{\pi}{2})$ while the opening of $Q^-_c$ is an obtuse angle $\alpha^-_c \in (\frac{\pi}{2}, \pi)$ where $\cos \alpha^\pm_c = \pm \sqrt{1 - c}$. 
Pointwise $C^{2,\alpha}$ estimates for $u$ being close to $P_c^+$

If $f$ and $\varphi$ are pointwise $C^\alpha$ and $C^{2,\alpha}$ respectively, then we can apply the above Proposition indefinitely and obtain the pointwise $C^{2,\alpha}$ estimate for $u$ at the origin.

**Corollary (Pointwise $C^{2,\alpha}$ estimate; $P_c^+$ version)**

Let $c \in (0,1)$. Assume that $u$ satisfies

\[
\text{det } D^2 u = f \quad \text{in } Q, \quad u = \varphi \quad \text{on } \partial Q.
\]

and suppose that

\[
|u - P_c^+| \leq \varepsilon_0, \ |f - c| \leq \delta \varepsilon_0 |x|^{\alpha} \text{ in } B_1 \cap Q, \ \text{and} \ |\varphi - q| \leq \delta \varepsilon_0 |x|^{2+\alpha} \text{ on } B_1 \cap \partial Q
\]

for $\varepsilon_0(c)$ and $\delta(c)$ small. Then

\[
|u(x) - P_c^+(x)| \leq C \varepsilon_0 |x|^{2+\alpha} \text{ in } B_1 \cap Q.
\]
Key to pointwise $C^{2,\alpha}$ estimates when $u$ is close to $P_c^-$

Let $\beta_c^- := \frac{\pi}{\arccos(-\sqrt{1-c})} \equiv \frac{\pi}{\alpha_c \sqrt{1-c}} \in (1, 2)$. Then $\beta_c^-$ is the homogeneity of the harmonic function positive in $Q_c^-$ and vanishing on $\partial Q_c^-$. 

Proposition

Let $c \in (0, 1)$ and $\beta \in (\beta_c^-, 2]$. Assume $u$ satisfies

$$
det D^2 u = f \quad \text{in } Q, \quad u = \varphi \quad \text{on } \partial Q.
$$

and suppose that

$$
|u - P_c^-| \leq \varepsilon |x|^\beta, \quad \text{and} \quad |f - c| \leq \delta \varepsilon \quad \text{in } (B_{1/\rho} \setminus B_\rho) \cap Q,
$$

and

$$
|\varphi - q| \leq \delta \varepsilon \quad \text{on } (B_{1/\rho} \setminus B_\rho) \cap \partial Q
$$

with $\varepsilon \leq \varepsilon_0(c, \beta)$, $\delta = \delta(c, \beta)$, $\rho = \rho(c, \beta)$ small. Then

$$
|u - P_c^-| \leq \frac{\varepsilon}{2} \quad \text{on } \partial B_1 \cap Q.
$$
Pointwise $C^{2,\alpha}$ estimates when $u$ is close to $P_c^-$ at all scales less than 1

A consequence of the above result is that if $u$ is quadratically close to $P_c^-$ at all scales less than 1, then $u$ is pointwise $C^{2,\alpha}$ at the origin.

**Lemma (Pointwise $C^{2,\alpha}$ estimate; $P_c^-$ version)***

Assume that $u$ satisfies

$$
\det D^2 u = f \quad \text{in } Q, \quad u = \varphi \quad \text{on } \partial Q.
$$

where $c \in (0, 1)$. Furthermore, assume that

$$
|u - P_c^-| \leq \varepsilon_0 |x|^2, \quad |f - c| \leq \delta \varepsilon_0 |x|^\alpha \quad \text{in } Q \cap B_1
$$

and

$$
|\varphi - q| \leq \delta \varepsilon_0 |x|^{2+\alpha} \quad \text{on } \partial Q \cap B_1,
$$

where $\varepsilon_0, \delta, \alpha$ are small depending on $c$. Then

$$
|u(x) - P_c^-(x)| \leq C \varepsilon_0 |x|^{2+\alpha} \quad \text{in } Q \cap B_1.
$$
Global $C^{2,\alpha}$ estimates for Monge-Ampère on 2D polygons

Theorem (L-Savin, 2019)

$\Omega$: bounded convex polygonal domain in $\mathbb{R}^2$; $u$ convex solving

\[
\begin{align*}
det D^2 u &= f \text{ in } \Omega, \\
u &= \varphi \text{ on } \partial \Omega.
\end{align*}
\]

Assume that
- $f \in C^\beta(\overline{\Omega})$, $f > 0$, and $\varphi \in C^{2,\beta}(\partial \Omega)$ for some $\beta \in (0, 1)$,
- there is a globally $C^2$, convex, strict subsolution $u \in C^2(\overline{\Omega})$ to (1).

Then

\[u \in C^{2,\alpha}(\overline{\Omega}),\]

for some $\alpha > 0$. 
Proof of the global $C^{2,\alpha}$ estimates: Preparation

**Step 1: Show that $u$ is pointwise $C^{2,\alpha}$ at each vertex, say 0.**  
Can assume:

1. the local geometry of $\Omega$ at 0 is that of the first quadrant,  
   $$\Omega \cap B_\rho = Q \cap B_\rho$$  
   for some $\rho \in (0, 1)$.

2.  
   $$u(0) = 0, \quad \nabla u(0) = 0, \quad u_{11}(0) = u_{22}(0) = 1.$$  

- This implies that $u \geq u \geq 0$ and  
  $$\det D^2 u = f \quad \text{in} \quad Q \cap B_\rho, \quad u = \varphi \quad \text{on} \quad \partial Q \cap B_\rho$$  
  with  
  $$|f(x) - f(0)| \leq C|x|^{\beta} \quad \text{in} \quad Q \cap B_\rho, \quad |\varphi(x) - q(x)| \leq C|x|^{2+\beta} \quad \text{on} \quad \partial Q \cap B_\rho,$$
  for some $C > 0$ depending on $\|f\|_{C^\beta(\overline{\Omega})}$ and $\|\varphi\|_{C^{2,\beta}(\partial\Omega)}$.

- Define $c := f(0)$, and using that $u$ is a strict subsolution  
  $$c = f(0) < \det D^2 u(0) \leq 1.$$
Claim: there exists $r$ small depending on the data above and the $C^2$ norm of $u$ such that the rescalings

$$u_r(x) := \frac{1}{r^2} u(rx), \quad f_r(x) := f(rx), \quad \varphi_r(x) := \frac{1}{r^2} \varphi(rx),$$

satisfy the hypotheses of the Pointwise $C^{2,\alpha}$ estimate ($P_c^+$ version).

- We can always choose $\alpha \leq \beta$ if necessary in the Pointwise $C^{2,\alpha}$ estimate ($P_c^+$ version), so the only part that needs to be checked is
  
  $$|u_r(x) - P_c^+(x)| \leq \epsilon_0 |x|^2 \quad \text{in} \quad Q \cap B_1. \quad (4)$$

- This follows by compactness. Indeed, we have
  
  $$u \geq u = \frac{1}{2} x^T D^2 u(0) x + o(|x|^2),$$

  and any blow-up limit $\bar{u}$ of a sequence of $u_r$’s must be one of the global solutions characterized in the Liouville type Theorem.

- Since $\bar{u}$ is above the quadratic tangent polynomial of $u$ at the origin, which in turn separates quadratically above $P_c^-$ we find $\bar{u} = P_c^+$, proving the claim.
Step 2: \( u \) is \( C^{2,\alpha} \) in a neighborhood of each vertex.

- Assume that we are in the setting of Step 1.
- We have that \( u \) separates quadratically from its tangent plane at the boundary points on \( \partial Q \) in annular domains \( Q \cap (B_{4r} \setminus B_r) \) for all \( r > 0 \) small. We can apply the results in Savin (JAMS, 2013) and conclude that

  \[
  \| u_r - P_c^+ \|_{C^{2,\alpha}} \leq Cr^\alpha \quad \text{in} \quad Q \cap (B_{3r} \setminus B_{2r}),
  \]

  for all \( r \) small.
- This implies that \( u \) is \( C^{2,\alpha} \) in a neighborhood of the origin.

Step 3: Conclusion. Combine the result in Step 2 with \( C^{2,\alpha} \) estimates for the Monge-Ampère equation at the boundary by Savin (JAMS, 2013) and in the interior by Caffarelli (Annals, 1990) to conclude that \( u \in C^{2,\alpha}(\Omega) \).
Thank you for your attention!