RESCALING LIMITS IN NON-ARCHIMEDEAN DYNAMICS

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Abstract. Suppose \( \{ f_t \} \) is an analytic one-parameter family of rational maps defined over a non-Archimedean field \( K \). We prove a finiteness theorem for the set of rescalings for \( \{ f_t \} \). This complements results of J. Kiwi.

1. Introduction

Let \( K \) be an algebraically closed field. For \( d \geq 1 \), let \( \text{Rat}_d(K) \) be the space of degree \( d \) rational maps over \( K \), thought of as dynamical systems on \( \mathbb{P}^1(K) \). The group \( \text{PGL}_2(K) \) acts on \( \text{Rat}_d(K) \) by conjugation. The moduli space of degree \( d \) rational maps on \( \mathbb{P}^1(K) \) is the quotient space \( M_d(K) := \text{Rat}_d(K)/\text{PGL}_2(K) \). Milnor [19] considered the moduli space \( M_2(\mathbb{C}) \) of quadratic complex rational maps and gave a dynamically natural compactification of \( M_2(\mathbb{C}) \). Then using geometric invariant theory, Silverman [21,22] compactified the moduli space \( M_d(K) \) in general. DeMarco [8] also considered different compactifications of the moduli space \( M_2(\mathbb{C}) \).

To study the dynamics of complex rational maps approaching the boundary of \( \text{Rat}_d(\mathbb{C}) \) (or \( M_d(\mathbb{C}) \)), Kiwi [18] considered rescaling limits for a holomorphic family \( \{ f_t \} \) (resp. a sequence \( \{ f_n \} \)) in \( \text{Rat}_d(\mathbb{C}) \). These arise as limits \( M_t^{-1} \circ f_t^q \circ M_t \to g \) (resp. \( M_n^{-1} \circ f_n^q \circ M_n \to g \)) of rescaled iterates where the convergence is locally uniform outside some finite subset of \( \mathbb{P}^1(\mathbb{C}) \). By regarding a holomorphic family as a rational map with coefficients in the field of formal Puiseux series, and by studying its induced action on the corresponding Berkovich space, Kiwi proved for any given holomorphic one-parameter family of degree \( d \geq 2 \) rational maps, there are at most \( 2d - 2 \) dynamically independent rescalings such that the corresponding rescaling limits are not postcritically finite [18, Theorem 1, Theorem 2]. Later, Arfeux [1] proved the same results using the Deligne-Mumford compactifications of the moduli spaces of the stable punctured spheres.

An algebraically closed complete valued field is isomorphic to either the field of complex numbers \( \mathbb{C} \) or a non-Archimedean field [9]. Our main result translates Kiwi’s finiteness result to non-Archimedean algebraically closed fields, subject to a natural tameness hypothesis. We now set up the statement.

Throughout this paper, \( K \) will denote an algebraically closed field which is complete with respect to a nontrivial non-Archimedean absolute value \( | \cdot |_K \). Let \( \phi(z) \in K(z) \) be a rational map. We can write \( \phi = \phi_1 \circ \phi_2 \), where \( \phi_1 \) is a separable rational map and \( \phi_2(z) = z^p \) for some \( j \geq 0 \) if the field \( K \) has positive characteristic \( p > 0 \) or \( \phi_2(z) = z \) if the field \( K \) has characteristic zero. The rational map \( \phi_1 \) is called the separable part of \( \phi \). The degree of \( \phi_1 \) is called the nontrivial
degree of $\phi$ and the preimages of critical points of $\phi_1$ under $\phi_2$ are called the non-trivial critical points of $\phi$. We say rational map $\phi \in K(z)$ is postcritically finite at nontrivial critical points if each nontrivial critical point of $\phi$ has a finite forward orbit; equivalently, each critical value of $\phi_1$ has a finite forward orbit under $\phi$.

As in Kiwi’s study of degenerating rational maps over $\mathbb{C}$, we now study families $\phi_t$ approaching boundary of $\text{Rat}_a(K)$ (or $\mathcal{M}_a(K)$). Since the field $K$ is not locally compact with respective to the absolute value $|\cdot|_K$, the definition of rescaling limits in [18] needs to be slightly modified. The non-Archimedean property turns out to make pointwise convergence suitable; see Definition 2.4 and Proposition 3.5. For instance, let $f_t(z) = (z^3 + t)/z \in K(z)$, then, as $t \to 0$, $f_t(z)$ converges to $z^2$ pointwise on $\mathbb{P}^1(K) \setminus \{0\}$, but $f_t(0) = \infty$ for all $t \neq 0$. Since the field of formal Puiseux series over $K$ is not algebraically closed if $\text{char} \ K = p > 0$, we work on the field $L$ of Hahn series over $K$. For Puiseux series and Hahn series, we refer [13,15–17]. For an analytic family $L_1 \subset \text{Rat}_a(K)$, we can associate to $\{f_t\}$ a rational map $f : \mathbb{P}^1(L) \to \mathbb{P}^1(L)$. The space $\mathbb{P}^1(L)$ is naturally a subset of the corresponding Berkovich space $\mathbb{P}^1_{\text{Ber}}(L)$; see [3,14] for details. The rational map $f$ induces a map on $\mathbb{P}^1_{\text{Ber}}(L)$ extending its action on $\mathbb{P}^1(L)$, so we also use notation $f$ for the induced map. Let $\mathcal{R}_f$ be the Berkovich ramification locus, that is the set of points in $\mathbb{P}^1_{\text{Ber}}(L)$ such that the local degrees of $f$ at these points are at least 2, i.e $\mathcal{R}_f = \{\xi \in \mathbb{P}^1_{\text{Ber}}(L) : \text{deg}_f \geq 2\}$. It is a closed subset of $\mathbb{P}^1_{\text{Ber}}(L)$ with no isolated points and has at most $\text{deg}\ f - 1$ connected components [10, Theorem A], each of which has tree structure. Following Trucco [23], we say the rational map $f : \mathbb{P}^1_{\text{Ber}}(L) \to \mathbb{P}^1_{\text{Ber}}(L)$ is tame if $\mathcal{R}_f$ has only finitely many points with valence at least 3 in $\mathcal{R}_f$.

We will prove

**Theorem 1.1.** Let $\{f_t(z)\} \subset K(z)$ be an analytic family of rational maps of nontrivial degree $d \geq 2$ and let $f_1$ be the separable part of the associated rational map $f$ of $\{f_t\}$. Assume $f_1$ is tame. Then there are at most $2d - 2$ pairwise dynamically independent rescalings for $\{f_t\}$ such that the corresponding rescaling limits are not postcritically finite at nontrivial critical points.

In section 5 we give some examples of analytic families with rescaling limits that are not postcritically finite at nontrivial critical points.

The tameness of the separable part $f_1$ is needed in order to prove a non-Archimedean Rolle’s theorem in positive characteristic, see Lemma 5.1. If $K$ has characteristic zero or positive characteristic $p > \text{deg} \ f_1$, then the rational map $f_1 : \mathbb{P}^1_{\text{Ber}}(L) \to \mathbb{P}^1_{\text{Ber}}(L)$ is tame [10, Corollary 6.6]. Thus

**Corollary 1.2.** Suppose the field $K$ has characteristic zero. Let $\{f_t(z)\} \subset K(z)$ be an analytic family of degree $d \geq 2$ rational maps. Then there are at most $2d - 2$ pairwise dynamically independent rescalings for $\{f_t\}$ such that the corresponding rescaling limits are not postcritically finite.

**Outline.** In section 2 we recall the relevant backgrounds of Berkovich space and define the rescaling limits for an analytic family of rational maps over $K$. The goal of section 3 is to discuss the reduction map and show the relations between reductions and rescaling limits. Section 4 is devoted to restating Kiwi’s results which are still true for the case when $K$ has characteristic zero. Finally, we prove Theorem 1.1 in section 5 and give examples to illustrate it in section 6.
2. Preliminaries

2.1. Non-Archimedean fields. For the field $K$, let $|K^\times|_K \subset (0, \infty)$ be the set of absolute values attained by nonzero elements of $K$, which is called the value group of $K$. Then $|K^\times|_K$ is dense in $(0, \infty)$ since $K$ is algebraically closed, and hence $K$ can not be locally compact. Let $O_K = \{ z \in K : |z|_K \leq 1 \}$ be the ring of integers of $K$ and let $\mathcal{M}_K = \{ z \in K : |z|_K < 1 \}$ be the unique maximal ideal of $O_K$. Let $k = O_K/\mathcal{M}_K$ be the residue field. Note if $\text{char } K = p > 0$ then $\text{char } k = p$, but if $\text{char } K = 0$, then $k$ could have any characteristic. For instance, for a prime number $p \geq 2$, if $K$ is the completion of the algebraic closure of the formal power series field $\mathbb{F}_p[[t]]$ with its natural absolute value, then $\text{char } K = \text{char } k = p$; if $K$ is the complex $p$–adic field $\mathbb{C}_p$, then $k = \mathbb{F}_p$, the algebraic closure of $\mathbb{F}_p$, and $\text{char } K = 0$ but $\text{char } k = p$.

Given $a \in K$ and $r > 0$, define

$$D(a, r) := \{ z \in K : |z - a|_K < r \} \quad \text{and} \quad \overline{D}(a, r) := \{ z \in K : |z - a|_K \leq r \}.$$ 

If $r \in |K^\times|_K$, we say that $D(a, r)$ is an open rational disk in $K$ and $\overline{D}(a, r)$ is a closed rational disk in $K$. If $r \notin |K^\times|_K$, we call $D(a, r) = \overline{D}(a, r)$ an irrational disk.

Let $U(a, r) \subset K$ be a disk centered at $a \in K$ with radius $r > 0$, that is, $U$ has the form $D(a, r)$ or $\overline{D}(a, r)$. Then if $b \in U(a, r)$, we have $U(a, r) = U(b, r)$. Moreover, the radius $r$ is the same as the diameter of $U(a, r)$, that is $r = \sup\{|z - w|_K : z, w \in U(a, r)\}$. Furthermore, if two disks have a nonempty intersection, then one must contain the other. Finally, we should mention here every disk in $K$ is both open and closed under the topology of $K$.

Let $\mathbb{L} := K[[t^\mathbb{Q}]]$ be the field of Hahn series over $K$. It consists of all formal sums of the form $\sum_{n \geq 0} a_n t^{q_n}$, where $\{q_n\}$ is an increasing sequence of rational numbers and $a_n \in K$. Since $\mathbb{Q}$ is divisible under addition, the field $\mathbb{L}$ is algebraically closed. It can be equipped with a non-Archimedean absolute value $|\cdot|_\mathbb{L}$ by fixing a number $\epsilon \in (0, 1)$ and defining $|\sum_{n \geq 0} a_n t^{q_n}|_\mathbb{L} = \epsilon^{n_0}$, where $n_0$ is the smallest positive integer such that $a_n \neq 0$. With respect to $|\cdot|_\mathbb{L}$, the field $\mathbb{L}$ is complete. Then the ring of integer of the field $\mathbb{L}$ is

$$O_\mathbb{L} = \{ z \in \mathbb{L} : |z|_\mathbb{L} \leq 1 \} = \left\{ \sum_{n \geq 0} a_n t^{q_n} : q_n \geq 0 \right\}$$

and the unique maximal ideal $\mathcal{M}_\mathbb{L}$ of $O_\mathbb{L}$ consists of series with zero constant term, i.e.

$$\mathcal{M}_\mathbb{L} = \{ z \in \mathbb{L} : |z|_\mathbb{L} < 1 \} = \left\{ \sum_{n \geq 0} a_n t^{q_n} : q_n > 0 \right\}.$$

The residue field $O_\mathbb{L}/\mathcal{M}_\mathbb{L}$ is canonically isomorphic to $K$.

2.2. The Berkovich projective line. In this subsection, we summarize some fundamental properties of the Berkovich projective line, for details we refer [2][9][0].

The Berkovich affine line $A^1_{\text{Ber}}(K)$ is the set of all multiplicative seminorms on the ring $K[z]$ of polynomials over $K$, whose restriction to the field $K \subset K[z]$ is equal to the given absolute value $|\cdot|_K$. For $a \in K$ and $r \geq 0$, let $\xi_{a,r}$ be the seminorm defined by $|f|_{\xi_{a,r}} = \sup_{z \in \overline{D}(a, r)} |f(z)|_K$. Then there are 4 types of points in $A^1_{\text{Ber}}(K)$:

1. Type I. $\xi_{a,0}$ for some $a \in K$. 

2. Type II. $\xi_{a,r}$ for some $a \in K$ and $r \in [K^\times]$.
3. Type III. $\xi_{a,r}$ for some $a \in K$ and $r \notin [K^\times]$.
4. Type IV. A limit of seminorms $\{\xi_{a,r}\}_{i \geq 0}$, where the corresponding sequence of closed disks $\{\overline{D}_{a_i,r_i}\}_{i \geq 0}$ satisfies $\overline{D}_{a_{i+1},r_{i+1}} \subset \overline{D}_{a_i,r_i}$ and $\cap \overline{D}_{a_i,r_i} = \emptyset$.

We can identify $K$ with the type I points in $\mathbb{A}_{\text{Ber}}^1(K)$ via $a \rightarrow \xi_{a,0}$. The point $\xi_{0,1} \in \mathbb{A}_{\text{Ber}}^1(K)$ is called the Gauss point and denoted by $\xi_G$. We put the weak topology on $\mathbb{A}_{\text{Ber}}^1(K)$, which makes the map $\mathbb{A}_{\text{Ber}}^1(K) \rightarrow [0, +\infty)$ sending $\xi$ to $|f|_\xi$ continuous for each $f \in K[z]$. Then $\mathbb{A}_{\text{Ber}}^1(K)$ is locally compact, Hausdorff and uniquely path-connected.

The Berkovich projective line $\mathbb{P}_{\text{Ber}}^1(K)$ is obtained by gluing two copies of $\mathbb{A}_{\text{Ber}}^1(K)$ along $\mathbb{A}_{\text{Ber}}^1(K) \setminus \{0\}$ via the map $\xi \rightarrow 1/\xi$. Then we can associate the Gelfand topology on $\mathbb{P}_{\text{Ber}}^1(K)$. The Berkovich projective line $\mathbb{P}_{\text{Ber}}^1(K)$ is a compact, Hausdorff, uniquely path-connected topological space and contains $\mathbb{P}^1(K)$ as a dense subset.

The space $\mathbb{P}_{\text{Ber}}^1(K)$ has tree structure. For a point $\xi \in \mathbb{P}_{\text{Ber}}^1(K)$, we can define an equivalence relation on $\mathbb{P}_{\text{Ber}}^1(K) \setminus \{\xi\}$, that is, $\xi'$ is equivalent to $\xi''$ if $\xi'$ and $\xi''$ are in the same connected component of $\mathbb{P}_{\text{Ber}}^1(K) \setminus \{\xi\}$. Such an equivalence class $\bar{v}$ is called a direction at $\xi$. We say that the set $T_\xi \mathbb{P}_{\text{Ber}}^1(K)$ formed by all directions at $\xi$ is the tangent space at $\xi$. For $\bar{v} \in T_\xi \mathbb{P}_{\text{Ber}}^1(K)$, denote by $B_\xi(\bar{v})^{-}$ the component of $\mathbb{P}_{\text{Ber}}^1(K) \setminus \{\xi\}$ corresponding to the direction $\bar{v}$. If $\xi \in \mathbb{P}_{\text{Ber}}^1(K)$ is a type I or IV point, $T_\xi \mathbb{P}_{\text{Ber}}^1(K)$ consists of a single direction. If $\xi \in \mathbb{P}_{\text{Ber}}^1(K)$ is a type II point, the directions in $T_\xi \mathbb{P}_{\text{Ber}}^1(K)$ are in one-to-one correspondence with the elements in $\mathbb{P}^1(K)$. If $\xi \in \mathbb{P}_{\text{Ber}}^1(K)$ is a type III point, $T_\xi \mathbb{P}_{\text{Ber}}^1(K)$ consists of two directions. Note the Gauss point $\xi_G$ is a type II point. We can identify $T_{\xi_G} \mathbb{P}_{\text{Ber}}^1(K)$ to $\mathbb{P}^1(K)$ by the correspondence $T_{\xi_G} \mathbb{P}_{\text{Ber}}^1(K) \rightarrow \mathbb{P}^1(k)$ sending $\bar{v}_x$ to $x$, where $\bar{v}_x$ is the direction at $\xi_G$ such that $B_{\xi_G}(\bar{v}_x)^-$ contains all the type I points whose images are $x$ under the canonical reduction map $\mathbb{P}^1(K) \rightarrow \mathbb{P}^1(k)$.

2.3. Rational maps. In this subsection, we consider rational maps over the field $K$ and define an analytic family of rational maps over $K$. For rational maps over a non-Archimedean field, we refer [3, 5].

We first define analytic maps on a disk $U \subset K$.

**Definition 2.1.** Let $U \subset K$ be a disk and $z_0 \in U$. We say a map $f : U \rightarrow K$ is analytic if $f$ can be written as a power series

$$f(z) = \sum_{n=0}^{\infty} c_n(z - z_0)^n \in K[[z - z_0]],$$

which converges for all $z \in U$. The smallest $n$ such that $c_n \neq 0$ is called the order of $f$ at $z_0$ and denoted $\text{ord}_{z_0}(f)$.

It is easy to check that analytic property is independent of the choice of $z_0 \in U$. Moreover, if $U = \overline{D}(z_0, r)$ is a rational closed disk, then $\sum_{n=0}^{\infty} c_n(z - z_0)^n$ converges for each $z \in U$ if and only if $\lim_{n \rightarrow \infty} |c_n|_{K^n} = 0$. For rational open or irrational disks, $\lim_{n \rightarrow \infty} |c_n|_{K^n} = 0$ implies $\sum_{n=0}^{\infty} c_n(z - z_0)^n$ converges, but the converse is not true.

We denote by $\mathbb{P}^1(K) := K \cup \{\infty\}$ the projective line over $K$. We define the spherical metric on $\mathbb{P}^1(K)$ as follows: for points $z = [x : y]$ and $w = [u : v]$ in $\mathbb{P}^1(K)$,

$$\Delta(z, w) := \frac{|xv - yu|_K}{\max\{|x|_K, |y|_K\} \max\{|u|_K, |v|_K\}}.$$
Equivalently,

\[ \Delta(z, w) := \begin{cases} 
\frac{|z-w|_K}{\max\{1,|z|_K\} \max\{1,|w|_K\}}, & \text{if } z, w \in K, \\
1 & \text{if } z \in K, w = \infty.
\]

Recall a degree \( d \geq 1 \) rational map \( f : \mathbb{P}^1(K) \to \mathbb{P}^1(K) \) is represented by a pair \( f_1, f_2 \in K[X, Y] \) of degree \( d \) homogeneous polynomials with no common factors, that is, \( f([X : Y]) = [f_1(X, Y) : f_2(X, Y)] \) for all \( [X : Y] \in \mathbb{P}^1(K) \). Equivalently, the map \( f \) can be considered as the quotient of two relatively prime polynomials, of which the greatest degree is \( d \). Let \( \text{Rat}_d(K) \) denote the set of rational maps of degree \( d \) over \( K \). Then \( \text{Rat}_d(K) \) can be naturally identified with an open subset of \( \mathbb{P}^{2d+1}(K) \) via the map \( \text{Rat}_d(K) \to \mathbb{P}^{2d+1}(K) \) sending \( (a_d z^d + \cdots + a_0)/(b_d z^d + \cdots + b_0) \) to \( [a_d : \cdots : a_0 : b_d : \cdots : b_0] \).

Let \( \phi(z) \in K(z) \) be a rational map. Suppose \( z_0 \in \mathbb{P}^1(K) \) and set \( w_0 = \phi(z_0) \). Pick \( \psi_1, \psi_2 \in \text{PGL}_2(K) \) such that \( \psi_1(0) = z_0 \) and \( \psi_2(w_0) = 0 \), and define \( \Phi = \psi_2 \circ \phi \circ \psi_1 \). The multiplicity \( m_{\phi}(z_0) \) of \( \phi \) at \( z_0 \) is the order of \( \Phi \) at 0. The weight \( w_{\phi}(z_0) \) of \( \phi \) at \( z_0 \) is the order of \( \Phi' \) at 0. If \( \Phi'(z) \equiv 0 \), we set \( w_{\phi}(z_0) = \infty \). This can happen: for example, if \( \text{char } K = p \) and \( \phi(z) = z^p \), then \( \phi'(z) = 0 \) for each \( z \in K \). A point \( z_0 \in \mathbb{P}^1(K) \) is called a critical point of \( \phi \) if \( w_{\phi}(z_0) > 0 \). Denote \( \text{Crit}(\phi) \subset \mathbb{P}^1(K) \) for the set of all critical points of \( \phi \). If every point \( z \in \mathbb{P}^1(K) \) is a critical point of \( \phi \), then we say \( \phi \) is inseparable. Otherwise, \( \phi \) is called separable.

Recall that for every rational map \( f \in K(z) \), we can write \( \phi(z) = \phi_1 \circ \phi_2(z) \), where \( \phi_1 \) is a separable rational map and \( \phi_2(z) = z^{p'} \) for some \( j \geq 0 \) if the field \( K \) has positive characteristic \( p > 0 \) or \( \phi_2(z) = z \) if the field \( K \) has characteristic zero. It is called the (in)separable decomposition of \( \phi \). The rational map \( \phi_1 \) is called the separable part of \( \phi \). We define the nontrivial degree \( \deg_0 \phi := \deg \phi_1 \) and the nontrivial critical set \( \text{Crit}_0(\phi) := \phi^{-1}(\text{Crit}\phi_1) \) of \( \phi \).

**Definition 2.2.** Let \( U \subset K \) be a disk containing 0. A collection \( \{f_t\}_{t \in U} \subset \mathbb{P}^{2d+1}(K) \) is a 1-dimensional separable analytic family of degree \( d \geq 1 \) rational maps if the map \( F : U \to \mathbb{P}^{2d+1}(K) \) sending \( t \) to \( f_t \) is an analytic map such that \( f_t \in \text{Rat}_d(K) \) is separable for all \( t \neq 0 \). If \( \{f_t\}_{t \in U} \) is a 1-dimensional separable analytic family of degree 1 rational maps, we call it a moving frame.

Let \( U \subset K \) be a disk containing 0. We say \( \{f_t\}_{t \in U} \subset \mathbb{P}^{2d+1}(K) \) is a 1-dimensional analytic family of nontrivial degree \( d' \geq 1 \) rational maps if we can write \( f_t = g_t \circ h \), where \( \{g_t\}_{t \in U} \subset \mathbb{P}^{2d'+1}(K) \) is a 1-dimensional separable analytic family of degree \( d' \geq 1 \) rational maps and \( h(z) = z^{p'} \) for some \( j \geq 0 \) if \( \text{char } K = p > 0 \) or \( h(z) = z \) if \( \text{char } K = 0 \).

**Remark 2.3.**

1. We are really interested in the germ defined by an analytic family, so considering a small disk \( V \subset U \) containing 0 if necessary, we can always assume \( U = \mathbb{D}(0, r) \) is a rational closed disk.

2. For an analytic family \( \{f_t\} \) on \( U \), we can write \( f_t(z) = P(z)/Q(z) = (a_d(t)z^d + \cdots + a_0(t))/(b_d(t)z^d + \cdots + b_0(t)) \) and denote by \( t \) the minimum among the orders of the \( a_i(t) \) and \( b_j(t) \), at the origin, \( i, j = 1, \cdots, d \). Let

\[ C = \max_{0 \leq i, j \leq d} \{1, \lim_{t \to 0} \frac{|a_i(t)|}{|t|_K}, \lim_{t \to 0} \frac{|b_j(t)|}{|t|_K}\} \]
Let $x \in K$ be an element such that $|x|_K = C$. For $t$ sufficiently small, by considering $g_t(z) = (P_t(z)/x^t)/(Q_t(z)/x^t)$ if necessary, we can assume $\{f_t\} \subset O_K(z)$ and that $f_t$ has at least one coefficient with absolute value 1. Therefore, throughout this paper, for a rational map $\phi(z) \in K(z)$, we assume $\phi(z) \in O_K(z)$ and at least one coefficient has absolute value 1.

For an analytic family

$$\left\{ f_t(z) = \frac{a_d(t)z^d + \cdots + a_0(t)}{b_d(t)z^d + \cdots + b_0(t)} \right\} \subset K(z)$$

of degree $d$ rational maps, let $a_d, \cdots, a_0, b_d, \cdots, b_0$ be the power series expressions of the coefficients $a_d(t), \cdots, a_0(t), b_d(t), \cdots, b_0(t)$, respectively. Then the degree $d$ rational map $f : \mathbb{P}^1(L) \to \mathbb{P}^1(L)$ given by

$$f(z) = \frac{a_dz^d + \cdots + a_0}{b_dz^d + \cdots + b_0}$$

is called, following Kiwi, the rational map associated to $\{f_t\}$. The rational map $f$ induces a map from $\mathbb{P}^1_{Ber}(L)$ to itself. We use the same notation $f$ for the induced map.

2.4. Rescaling limits for an analytic family. A non-Archimedean field is locally compact if and only if it is discretely valued and has finite residue field $\mathbb{F}_q$. Then $K$ is not locally compact, hence neither is $\mathbb{P}^1(K)$. Thus, we define the rescaling limits for an analytic family of rational maps over $K$ in the following sense:

**Definition 2.4.** Let $\{f_t\}$ be an analytic family of rational maps of nontrivial degree at least 2. A moving frame $\{M_t\}$ is called a rescaling for $\{f_t\}$ if there exist an integer $q \geq 1$, a rational map $g : \mathbb{P}^1(K) \to \mathbb{P}^1(K)$ of nontrivial degree $d' \geq 2$ and a finite subset $S$ of $\mathbb{P}^1(K)$ such that, as $t \to 0$,

$$M_t^{-1} \circ f_t^q \circ M_t(z) \to g(z)$$

pointwise on $\mathbb{P}^1(K) \setminus S$. We say $g$ is a rescaling limit for $\{f_t\}$ in $\mathbb{P}^1(K) \setminus S$. The minimal $q \geq 1$ such that the above holds is called the period of the rescaling $\{M_t\}$.

Following Kiwi [18], we define the following equivalence relations on the set of all rescalings.

**Definition 2.5.** Two moving frames $\{M_t\}$ and $\{L_t\}$ are equivalent if there exists $M \in \text{Rat}_1(K)$ such that $M_t^{-1} \circ L_t \to M$ as $t \to 0$.

**Definition 2.6.** Two rescalings $\{M_t\}$ and $\{L_t\}$ for an analytic family $\{f_t\}$ are dynamically dependent if there exist an integer $l \geq 0$ and a nonconstant rational map $g$ such that $L_t^{-1} \circ f_t^l \circ M_t \to g$, as $t \to 0$, pointwise outside some finite set.

If $\{M_t\}$ and $\{L_t\}$ are two equivalent rescalings for an analytic family $\{f_t\}$, then they are dynamically dependent. The converse is not true in general.

3. Reductions

Recall $K$ is any arbitrary complete algebraically closed non-Archimedean field. Let $g \in O_K(z)$ be a rational map. Then reducing the coefficients of $g$ modulo $M_K$ and canceling common factors, we get a rational map $\tilde{g}$ over the residue field $k$, which is called the reduction of $g$. Now we can define a map

$$\rho_K : \text{Rat}_d(K) \to \text{Rat}_d(k),$$
where $\mathrm{Rat}_{d}(k)$ is the space of degree at most $d$ rational maps over $k$, sending $g$ to its reduction $\tilde{g}$. We call $\rho_K$ the reduction map for rational maps over $K$.

We first state an easy proposition and omit the proof.

**Proposition 3.1.** Let $\phi(z), \psi(z) \in K(z)$ be rational maps, and let $\rho(\phi)$ and $\rho(\psi)$ be their reductions, respectively. Then

1. $\rho(\phi \cdot \psi) = \rho(\phi) \cdot \rho(\psi)$,
2. $\rho(\phi + \psi) = \rho(\phi) + \rho(\psi)$,
3. If $\deg(\psi) \geq 1$, then $\rho(\phi \circ \psi) = \rho(\phi) \circ \rho(\psi)$.

In Proposition 3.3, if $\deg(\psi) = 0$, the situation is complicated. For example, let $K$ be the completion of the formal Puiseux series over $\mathbb{C}$ and define rational maps $\phi(z) = z^2/t^2$ and $\psi(z) = tz^2$ over $K$. Then $\rho(\phi \circ \psi)(z) = z^4$ but $(\rho(\phi) \circ \rho(\psi))(z) = \infty$ since $\rho(\phi) = \infty$.

Recall that $L$ is the field of Hahn series over $K$. Since $L$ is an algebraically closed and complete non-Archimedean field, we can consider the reduction map $\rho_L : \mathrm{Rat}_d(L) \to \mathrm{Rat}_{d}(K)$ of rational maps over $L$.

**Definition 3.2.** Let $\{f_t\}$ be an analytic family of degree $d \geq 1$ rational maps. We say $\{f_t\}$ has good reduction if the associated rational map $f$ has good reduction, that is, $\deg(\rho_L(f)) = d$. Otherwise, we say $\{f_t\}$ has bad reduction. If there is a moving frame $\{M_t\} \subset \mathbb{P}^3(K)$ such that $\{M_t^{-1} \circ f_t \circ M_t\}$ has good reduction, we say that $\{f_t\}$ has potentially good reduction.

Given $f = [f_1 : f_2] \in \mathbb{P}^{2d+1}(K)$, we can write

$$f = [f_1 : f_2] = [H_f f_1 : H_f f_2] = H_f [\hat{f}_1 : \hat{f}_2] = H_f \hat{f},$$

where $H_f = \gcd(f_1, f_2)$ is a homogeneous polynomial and $\hat{f} = [\hat{f}_1 : \hat{f}_2]$ is a rational map of degree at most $d$.

**Proposition 3.3.** Suppose $\{f_t\}$ is an analytic family of degree $d \geq 1$ rational maps such that $f_t \to H_f \hat{f}$, as $t \to 0$, in $\mathbb{P}^{2d+1}(K)$. Then, as $t \to 0$, $f_t$ converges to $\hat{f}$ pointwise on $\mathbb{P}^1(K) \setminus \{H_f = 0\}$.

**Proof.** Write $f_t = [P_t : Q_t]$ and $\hat{f} = [P : Q]$. Since $f_t$ converges to $H_f \hat{f}$ in $\mathbb{P}^{2d+1}(K)$, there is a $\lambda \in K \setminus \{0\}$ such that for any $[X : Y] \in \mathbb{P}^1(K)$, as $t \to 0$, $P_t(X, Y)$ converges to $\lambda H(X, Y) P(X, Y)$ and $Q_t(X, Y)$ converges to $\lambda H(X, Y) Q(X, Y)$. So if $[X : Y] \notin \{H_f = 0\}$, we have $f_t([X : Y])$ converges to $[P(X, Y) : Q(X, Y)]$. Hence $f_t$ converges to $\hat{f}$ pointwise on $\mathbb{P}^1(K) \setminus \{H_f = 0\}$. □

**Corollary 3.4.** Let $\{f_t\}$ be an analytic family of degree $d \geq 2$ rational maps. If $\deg(\rho_L(f)) \geq 2$, then $\{M_t = z\}$ is a rescaling for $\{f_t\}$ with corresponding rescaling limit $\rho_L(f)$.

**Proof.** Note as $t \to 0$ there is a homogeneous polynomial $H \in K[X, Y]$ such that $f_t$ converges to $H \rho_L(f)$ in $\mathbb{P}^{2d+1}(K)$. The conclusion then follows Proposition 3.3 □

The converse of Proposition 3.3 is also true.

**Proposition 3.5.** Let $\{f_t\}$ be an analytic family of degree $d \geq 1$ rational map and let $S \subset \mathbb{P}^1(K)$ be a finite subset. Suppose $f_t$ converges to $\hat{f}$ pointwise, as $t \to 0$, on $\mathbb{P}^1(K) \setminus S$. Then there exists a homogeneous polynomial $H$ of degree $d - \deg \hat{f}$ with zeros in $S$ such that $f_t \to H \hat{f}$, as $t \to 0$, in $\mathbb{P}^{2d+1}(K)$. □
Proof. Let $f$ be the associated rational map of $\{f_t\}$. Then there exists homogeneous polynomial $H$ such that $f_t(z)$ converges to $H_{\rho_L}(f)$, as $t \to 0$, in $\mathbb{P}^{2d+1}(K)$. Thus, by Proposition 3.3, $\hat{f} = \rho_L(f)$. It is easy to check $H$ satisfies the required conditions.

Corollary 3.6. Suppose $K$ has positive characteristic $p > 0$. Let $\{f_t(z)\} \subset K(z)$ be an analytic family of rational maps of nontrivial degree at least 2. Let $f$ be the associated rational map of $\{f_t\}$. If $f$ is inseparable, then all the rescaling limits of $\{f_t\}$ are inseparable.

Proof. Let $g$ be a rescaling limit for $\{f_t\}$. Then by Definition 2.4 and Proposition 3.5, there exist a rescaling $\{M_t\}$, an integer $q \geq 1$ and homogeneous polynomial $H$ such that $M_{t^{-1}} \circ f_t^q \circ M_t \to Hg$. Note the associated rational map $M_{t^{-1}} \circ f_t^q \circ M_t$ is inseparable since $f$ is inseparable. Considering the coefficients of $g$, we have the map $g$ is inseparable. □

4. BERKOVICH DYNAMICS

In this section, we first summarize the properties of the dynamics on a Berkovich space, see [3, 4, 14, 20]. Then we restate the results in [18], which are proven for a holomorphic family of rational maps over $\mathbb{C}$. These results are still true for an analytic family $\{f_t(z)\}$ of rational maps over a field $K$ with characteristic zero.

Recall that the Berkovich Julia set $J_{\text{Ber}}(\phi)$ of a rational map $\phi : \mathbb{P}^1_{\text{Ber}}(\mathbb{L}) \to \mathbb{P}^1_{\text{Ber}}(\mathbb{L})$ is the set consisting of all points $\xi \in \mathbb{P}^1_{\text{Ber}}(\mathbb{L})$ such that $\bigcup_{n \geq 0} \phi^n(U)$ omits finitely many points of $\mathbb{P}^1_{\text{Ber}}(\mathbb{L})$ for any neighborhood $U$ of $\xi$. The classical Julia set $J_f(\phi)$ is $J_{\text{Ber}}(\phi) \cap \mathbb{P}^1(\mathbb{L})$. Let $\xi \in \mathbb{P}^1_{\text{Ber}}(\mathbb{L}) \setminus \mathbb{P}^1(\mathbb{L})$ be a periodic point of $\phi$ of period $n \geq 1$. The multiplier $\lambda$ of $\xi$ is defined by the local degree of $\phi^n$ at $\xi$, that is, $\lambda := \deg_{\xi}(\phi^n)$. If $\lambda \geq 2$, we say $\xi$ is repelling. If a periodic point $\xi \in \mathbb{P}^1_{\text{Ber}}(\mathbb{L}) \setminus \mathbb{P}^1(\mathbb{L})$ is repelling, then $\xi$ is a type II point. Let $\mathcal{O} \subset \mathbb{P}^1_{\text{Ber}}(\mathbb{L})$ be a $n$-cycle of $\phi$. The basin of $\mathcal{O}$ is the interior of the set of points $\xi \in \mathbb{P}^1_{\text{Ber}}(\mathbb{L})$ such that, for all neighborhoods $U$ of $\mathcal{O}$, the orbit of $\xi$ is eventually contained in $U$.

Recall that the tangent space $T_{\xi}\mathbb{P}^1_{\text{Ber}}(\mathbb{L})$ is the set of all directions at $\xi \in \mathbb{P}^1_{\text{Ber}}(\mathbb{L})$. Let $\phi : \mathbb{P}^1_{\text{Ber}}(\mathbb{L}) \to \mathbb{P}^1_{\text{Ber}}(\mathbb{L})$ be a rational map. Then for any $\vec{w} \in T_{\xi}\mathbb{P}^1_{\text{Ber}}(\mathbb{L})$, there is a unique $\vec{u} \in T_{\phi(\xi)}\mathbb{P}^1_{\text{Ber}}(\mathbb{L})$ such that for any $\xi'$ sufficiently near $\xi$, $\phi(\xi') \in B_{\phi(\xi)}(\vec{w})$. Thus the rational map $\phi$ induces a map

$$\phi_* : T_{\xi}\mathbb{P}^1_{\text{Ber}}(\mathbb{L}) \to T_{\phi(\xi)}\mathbb{P}^1_{\text{Ber}}(\mathbb{L})$$

sending the direction $\vec{u}$ to the corresponding direction $\vec{w}$.

Proposition 4.1. [3] Corollary 9.25, Theorem 9.26, Corollary 9.27, Proposition 9.41 Let $\phi : \mathbb{P}^1_{\text{Ber}}(\mathbb{L}) \to \mathbb{P}^1_{\text{Ber}}(\mathbb{L})$ be a rational map of degree at least 1. Then $\phi(\xi_G) = \xi_G$ if and only if $\deg_{\xi_G}(\phi) \geq 1$. Moreover, (1) Assume $\phi(\xi_G) = \xi_G$. Identifying $T_{\xi_G}\mathbb{P}^1_{\text{Ber}}(\mathbb{L})$ to $\mathbb{P}^1(K)$, the following hold:

(a) $\deg_{\xi_G}(\phi) = \deg_{\xi_G}(\phi)$,

(b) at the Gauss point $\xi_G$, $\phi_* = \rho_L(\phi)$ on $\mathbb{P}^1(K)$.

(2) For $\xi \in \mathbb{P}^1_{\text{Ber}}(\mathbb{L})$ and $\vec{u} \in T_{\xi}\mathbb{P}^1_{\text{Ber}}(\mathbb{L})$, the image $\phi(B_{\xi}(\vec{v}))$ always contains $B_{\phi(\xi)}(\phi_* \vec{v})$, and either $\phi(B_{\xi}(\vec{v})) = B_{\phi(\xi)}(\phi_* \vec{v})$ or $\phi(B_{\xi}(\vec{v})) = \mathbb{P}^1_{\text{Ber}}(\mathbb{L})$. There exists an integer $m \geq 1$ such that

(a) if $\phi(B_{\xi}(\vec{v})) = B_{\phi(\xi)}(\phi_* \vec{v})$, then each $\zeta \in B_{\phi(\xi)}(\phi_* \vec{v})$ has $m$ preimages in $B_{\xi}(\vec{v})$, counting multiplicities;
Proposition 4.5. Suppose we have referred to Proposition 3.4, Lemma 3.6, Lemma 3.7. Let $K$ be a type II point and $\xi \in \mathcal{B}_\phi(\mathcal{B}(\phi))^\sim$ has $N$ preimages in $\mathcal{B}_\phi(\mathcal{B}(\phi))^\sim$ and each $\zeta \in \mathcal{B}_\phi(\mathcal{B}(\phi)) \setminus \mathcal{B}_\phi(\mathcal{B}(\phi))^\sim$ has $N-m$ preimages in $\mathcal{B}_\phi(\mathcal{B}(\phi))^\sim$, counting multiplicities.

Based on Proposition 3.3 and Proposition 1.1, we have

**Proposition 4.2.** Let $\{f_t(z)\} \subset K(z)$ be an analytic family of rational maps of nontrivial degree at least 2, and let $\{M_t\}$ and $\{L_t\}$ be moving frames. Denote by $f$, $M$ and $L$ the associated rational maps. Then

1. For all $l \geq 1$, the following are equivalent:
   a. There exists a rational map $g : \mathbb{P}^1(K) \to \mathbb{P}^1(K)$ of degree at least $d \geq 1$
      such that $M_t^{-1} \circ f_t^l \circ M_t$ converges to $g$ pointwise, as $t \to 0$, on $\mathbb{P}^1(K)$
      off a finite subset.
   b. $f_t(\xi) = \xi$, where $\xi = M(\xi_G)$ and $\deg f_t^l = d$.
      In the case in which (a) and (b) hold, the map $(f_t^l)_* : T_\xi \mathcal{B}(\mathcal{B}(\phi)) \to T_\xi \mathcal{B}(\mathcal{B}(\phi))$
      is conjugate via a $M \in \text{Rat}_d(K)$ to $g : \mathbb{P}^1(K) \to \mathbb{P}^1(K)$.

2. Moving frames $\{M_t\}$ and $\{L_t\}$ are equivalent if and only if $M(\xi_G) = L(\xi_G)$.

3. The following are equivalent:
   a. $f \circ M(\xi_G) = L(\xi_G)$.
   b. As $t \to 0$, $L_t^{-1} \circ f_t \circ M_t$ converges to some nonconstant rational map
      $g : \mathbb{P}^1(K) \to \mathbb{P}^1(K)$ pointwise outside some finite subset.

**Corollary 4.3.** Let $\{f_t\} \subset K(z)$ be an analytic family of degree at least 2 rational maps. Suppose $\{f_t\}$ has (potentially) good reduction. Then there is at most one rescaling, up to equivalence, for $\{f_t\}$, and this rescaling is of period 1.

**Proof.** Let $f$ be the associated rational map of $\{f_t\}$. Then $f$ has (potentially) good reduction. Then the classical Julia set $J_f(f) = \emptyset$ and the Berkovich Julia set $J_{\mathcal{B}(\phi)}(f)$ is a singleton set [3, Lemma 10.53]. Thus $\phi$ has no repelling periodic points of type I and has only one repelling periodic point [3, Theorems 10.81,10.82]. By Proposition 4.2 all the rescalings of $\{f_t\}$ are equivalent and they are of period 1.

To relate the critical points of $f$ and the rescaling limits of $\{f_t\}$, we first state the following non-Archimedean Rolle’s theorem:

**Lemma 4.4.** Suppose $K$ has characteristic zero and residue characteristic zero. Let $\phi \in K(z)$ be a rational map of degree at least 1. If $\phi$ has two distinct zeros in the closed disk $\overline{D}(a,r)$, then it has a critical point in $\overline{D}(a,r)$.

We should mention here Lemma 4.4 is not true in general. If $K$ has characteristic zero and residue characteristic $p > 0$, then under some assumptions, $\phi$ is only guaranteed to have a critical point in $\overline{D}(a,r|p|^{-1/(p-1)})$ which is strictly larger than $\overline{D}(a,r)$. If $K$ has characteristic $p > 0$, consider the field $\mathbb{L}$ and $\phi(z) = z^p - z \in \mathbb{L}(z)$. Then $\phi$ has only one critical point, which is $\infty \in \mathbb{P}^1(\mathbb{L})$. However, $\phi$ has $p$ zeros in $\overline{D}(0,1) \subset \mathbb{L}$. For more details about rational maps with one critical point, we refer to [12].

Applying the non-Archimedean Rolle’s theorem and using the same proof in [18], we have

**Proposition 4.5.** Suppose $K$ has characteristic zero. Consider a rational map $\phi : \mathbb{P}^1(\mathbb{L}) \to \mathbb{P}^1(\mathbb{L})$ of degree at least 2. Let $\xi \in \mathcal{B}(\mathcal{B}(\phi))$ be a type II point and
let \( \vec{v} \in T_{\xi}^{1} \mathbb{P}^{1}_{\text{Ber}}(L) \). If \( \phi \) is not injective on \( B_{\xi}(\vec{v})^{-} \), then there is a critical point of \( \phi \) in \( B_{\xi}(\vec{v})^{-} \) such that the corresponding critical value \( \phi(c) \in B_{\phi(\xi)}(\phi_{*}(\vec{v}))^{-} \).

**Proposition 4.6.** Suppose \( K \) has characteristic zero. Consider a rational map \( \phi : \mathbb{P}^{1}_{\text{Ber}}(L) \to \mathbb{P}^{1}_{\text{Ber}}(L) \) of degree at least 2. Let \( \mathcal{O} \) be a type II periodic orbit of period \( q \geq 1 \) of \( \phi \). Assume the basin of \( \mathcal{O} \) is free of critical points of \( \phi \). Then, for all \( \xi \in \mathcal{O} \), every \( \vec{v} \in T_{\xi}^{1} \mathbb{P}^{1}_{\text{Ber}}(L) \) with \( \phi^{q}(B_{\xi}(\vec{v})^{-}) = \mathbb{P}^{1}_{\text{Ber}}(L) \) has a finite forward orbit under \( (\phi^{q})^{*} \). Moreover, if \( \deg(\phi^{q}) \geq 2 \), then \( (\phi^{q})^{*} \) is postcritically finite.

In the next section, we establish analogs of these two propositions in positive characteristic, and from this deduce our main result.

5. **RATIONAL MAPS OVER FIELDS WITH POSITIVE CHARACTERISTIC**

Assume that the field \( K \) has positive characteristic \( p > 0 \). A nonconstant rational map \( \phi \in K(z) \) can then be written as \( \phi(z) = \phi_{1}(z^{p^{j}}) \) for some integer \( j \geq 0 \), where \( \phi_{1} \) is separable. Recall the ramification locus \( \mathcal{R}_{\phi} = \{ \xi \in \mathbb{P}^{1}_{\text{Ber}}(K) : \deg_{\xi} \phi \geq 2 \} \) and \( \phi \) is tame if \( \mathcal{R}_{\phi} \) contains finitely many points with valence at least 3. We say a rational map \( \phi \in K(z) \) is tamely ramified if the characteristic of \( K \) does not divide the multiplicity \( m_{\phi}(z) \) for any \( z \in \mathbb{P}^{1}(K) \). The space \( \mathbb{P}^{1}_{\text{Ber}}(K) \setminus \mathbb{P}^{1}(K) \) carries a natural metrizable topology, the strong topology, see [3,10]. With respect to this metric, there exists \( r > 0 \) such that the ramification locus \( \mathcal{R}_{\phi} \) is in an \( r \)-neighborhood of the connected hull \( \text{Hull}(\text{Crit}(\phi)) \) of critical set if and only if \( \phi \) is tamely ramified [11, Theorem E]. If \( \phi \) is separable, the extreme case \( \mathcal{R}_{\phi} \subseteq \text{Hull}(\text{Crit}(\phi)) \) is equivalent to \( \phi \) is tame [10, Corollary 7.13].

We can prove the following non-Archimedean Rolle’s theorem for a separable tame rational map over a field \( K \) with positive characteristic.

**Lemma 5.1.** Suppose \( K \) has positive characteristic \( p > 0 \). Let \( \phi \in K(z) \) be a separable tame rational map of degree at least 1. If \( \phi \) has two distinct zeros in the closed disk \( \overline{D(a,r)} \), then it has a critical point in \( \overline{D(a,r)} \).

**Proof.** Suppose there is no critical point in \( \overline{D(a,r)} \). Let \( \xi_{a,r} \in \mathbb{P}^{1}_{\text{Ber}}(K) \) be the point corresponding to the closed disk \( \overline{D(a,r)} \). Then \( \xi_{a,r} \notin \text{Hull}(\text{Crit}(\phi)) \). Let \( \vec{v} \in T_{\xi_{a,r}}^{1} \mathbb{P}^{1}_{\text{Ber}}(K) \) be the direction such that \( \overline{D(a,r)} \subset \mathbb{P}^{1}_{\text{Ber}}(K) \setminus B_{\xi_{a,r}}(\vec{v})^{-} \). Then the set \( \mathbb{P}^{1}_{\text{Ber}}(K) \setminus B_{\xi_{a,r}}(\vec{v})^{-} \) is disjoint with \( \text{Hull}(\text{Crit}(\phi)) \). So

\[
\mathbb{P}^{1}_{\text{Ber}}(K) \setminus B_{\xi_{a,r}}(\vec{v})^{-} \cap \mathcal{R}_{\phi} = \emptyset.
\]

Since \( \mathcal{R}_{\phi} \) is closed, there exist \( \xi \in \mathbb{P}^{1}_{\text{Ber}}(K) \) and \( \vec{w} \in T_{\xi}^{1} \mathbb{P}^{1}_{\text{Ber}}(K) \) such that \( \mathbb{P}^{1}_{\text{Ber}}(K) \setminus B_{\xi_{a,r}}(\vec{v})^{-} \subset B_{\xi}(\vec{w})^{-} \) and \( B_{\xi}(\vec{w})^{-} \cap \mathcal{R}_{\phi} = \emptyset \). Hence \( \phi \) is injective on \( B_{\xi}(\vec{w})^{-} \) [10, Corollary 3.8]. So \( \phi \) is injective on \( \mathbb{P}^{1}_{\text{Ber}}(K) \setminus B_{\xi_{a,r}}(\vec{v})^{-} \). Thus \( \phi \) is injective on the closed disk \( \overline{D(a,r)} \). So \( \phi \) has at most one zero in \( \overline{D(a,r)} \). It is a contradiction. \( \square \)

Applying Lemma 5.1 and the argument in [18, Lemma 4.2], we obtain an analog of Proposition 4.5.

**Proposition 5.2.** Suppose \( K \) has positive characteristic \( p > 0 \) and consider a separable tame rational map \( \phi : \mathbb{P}^{1}_{\text{Ber}}(L) \to \mathbb{P}^{1}_{\text{Ber}}(L) \) of degree at least 2. Let \( \xi \in \mathbb{P}^{1}_{\text{Ber}}(L) \) be a type II point and let \( \vec{v} \in T_{\xi}^{1} \mathbb{P}^{1}_{\text{Ber}}(L) \). If \( \phi \) is not injective on \( B_{\xi}(\vec{v})^{-} \), then there is a critical point of \( \phi \) in \( B_{\xi}(\vec{v})^{-} \) such that the corresponding critical value \( \phi(c) \in B_{\phi(\xi)}(\phi_{*}(\vec{v}))^{-} \).

We now prove an analogy of Proposition 4.6.
Proposition 5.3. Suppose $K$ has positive characteristic $p > 0$. Consider a rational map $\phi : \mathbb{P}^1_{\text{Ber}}(L) \to \mathbb{P}^1_{\text{Ber}}(L)$ of nontrivial degree at least 2 and suppose the separable part $\phi_1$ of $\phi$ is tame. Let $\mathcal{O}$ be a type II periodic orbit of period $q \geq 1$ of $\phi$. Assume the basin of $\mathcal{O}$ is free of critical values of $\phi_1$. Then, for all $\xi \in \mathcal{O}$, every $\vec{v} \in T_\xi \mathbb{P}^1_{\text{Ber}}(L)$ with $\phi^n(\mathcal{B}_\xi(\vec{v})^-) = \mathbb{P}^1_{\text{Ber}}(L)$ has a finite forward orbit under $(\phi^n)_*$.  

Proof. Let $\vec{v} \in T_\xi \mathbb{P}^1_{\text{Ber}}(L)$ such that $\phi^n(\mathcal{B}_\xi(\vec{v})^-) = \mathbb{P}^1_{\text{Ber}}(L)$ and $\vec{v}$ has an infinite forward orbit under $(\phi^n)_*$. We will show there exists a critical value of $\phi_1$ in the basin of $\mathcal{O}$. Let $q_0 \geq 1$ be the smallest integer such that

$$
\phi_1(\mathcal{B}_{\phi_{q_0+1}(\xi)}((\phi^{q_0+1})_*(\vec{v})^-)) = \mathbb{P}^1_{\text{Ber}}(L).
$$

Then by Proposition 4.1 and Proposition 5.2, there is a critical point $c \in \text{Crit}(\phi_1)$ such that $\phi_1(c) \in \mathcal{B}_{\phi_{q_0+1}(\xi)}((\phi^{q_0+1})_*(\vec{v})^-)$. Now we show for each $n \geq q_0 + 1$, $\mathcal{B}_{\phi^n(\xi)}((\phi^n)_*(\vec{v}))^-$ contains a point in the forward orbit of a critical value of $\phi_1$. By induction, suppose it holds for $n = k \geq q_0 + 1$. If $\phi_1(\mathcal{B}_{\phi^k(\xi)}((\phi^k)_*(\vec{v})^-)) = \mathcal{B}_{\phi^{k+1}(\xi)}((\phi^{k+1})_*(\vec{v})^-)$, then $\mathcal{B}_{\phi^{k+1}(\xi)}((\phi^{k+1})_*(\vec{v})^-)$ contains a point in the forward orbit of a critical value of $\phi_1$. If $\phi_1(\mathcal{B}_{\phi^k(\xi)}((\phi^k)_*(\vec{v})^-)) \neq \mathcal{B}_{\phi^{k+1}(\xi)}((\phi^{k+1})_*(\vec{v})^-)$, then $\phi_1(\mathcal{B}_{\phi^k(\xi)}((\phi^k)_*(\vec{v})^-)) \neq \mathcal{B}_{\phi^{k+1}(\xi)}((\phi^{k+1})_*(\vec{v})^-)$ contains a critical value of $\phi_1$.

Thus, for $n$ large, $\mathcal{B}_{\phi^n(\xi)}((\phi^n)_*(\vec{v}))^-$ contains a point in the forward orbit of a critical value of $\phi_1$. Note for $n$ sufficiently large, say $n \geq n_0$,

$$
\phi(\mathcal{B}_{\phi^{n+1}(\xi)}((\phi^{n+1})_*(\vec{v}))^-) \neq \mathbb{P}^1_{\text{Ber}}(L).
$$

Suppose $\phi(\phi_1(c)) \in \mathcal{B}_{\phi^{n+1}(\xi)}((\phi^{n+1})_*(\vec{v}))^-$ for some $c \in \text{Crit}(\phi_1)$ and $l \geq 0$, then $\phi^{n+1}(\phi_1(c)) \to \xi$, as $n \to \infty$, in the weak topology. Thus $\phi_1(c)$ is in the basin of the periodic cycle $\mathcal{O}$. 

Corollary 5.4. Under the same assumptions in Proposition 5.3, if $\deg_0(\phi^n)_* \geq 2$, then $(\phi^n)_*$ is postcritically finite at the nontrivial critical points.

Proof. Suppose $(\phi^n)_*$ is not postcritically finite at the nontrivial critical points. Let $\vec{v} \in T_\xi \mathbb{P}^1_{\text{Ber}}(L)$ be a nontrivial critical point of $(\phi^n)_*$ with infinite forward orbit, then there exists $j \geq 0$ such that $(\phi_2 \circ \phi^j)^{-1}(\text{Crit}(\phi_1)) \cap \mathcal{B}_\xi(\vec{v})^- \neq \emptyset$. If $\phi^n(\mathcal{B}_\xi(\vec{v})^-) \neq \mathbb{P}^1_{\text{Ber}}(L)$ for all $n \geq 1$, then

$$
\phi_1(\text{Crit}(\phi_1)) \cap \mathcal{B}_{\phi^{j+1}(\xi)}((\phi^{j+1})_*(\vec{v})^-) \neq \emptyset.
$$

Hence for all $n \geq 0$,

$$
\phi^n(\phi_1(\text{Crit}(\phi_1))) \cap \mathcal{B}_{\phi^{n+j+1}(\xi)}((\phi^{n+j+1})_*(\vec{v})^-) \neq \emptyset.
$$

So there exists $c \in \text{Crit}(\phi_1)$ such that $\phi_1(c)$ in the basin of $\mathcal{O}$. If there exists $n_0 \geq 1$ such that $\phi^{n_0}(\mathcal{B}_\xi(\vec{v})^-) = \mathbb{P}^1_{\text{Ber}}(L)$, then $\vec{v}$ has a finite forward orbit by Proposition 5.3. 

Based on Proposition 4.6 and Corollary 5.4, applying the argument in [18], we can prove Theorem 1.1.

Proof of Theorem 1.1. Let $\{M_t^{(1)}\}, \ldots, \{M_t^{(n)}\}$ be pairwise dynamically independent rescalings for $\{f_t\}$ of periods $q_1, \ldots, q_n$ such that the corresponding rescaling limits are not postcritically finite at nontrivial critical points. Let $f$ be the associated rational map of $\{f_t\}$ and $M^{(1)}, \ldots, M^{(n)}$ be the associated rational maps of $\{M_t^{(1)}\}, \ldots, \{M_t^{(n)}\}$. Let $\xi_j = M^{(j)}(\xi_{\ell}) \in \mathbb{P}^1_{\text{Ber}}(L)$ for $j = 1, \ldots, n$. Then by
Proposition 4.2 for all $j = 1, \ldots, n$, $f^{(j)} : T_{\xi_j} \mathcal{P}^1_{\text{Ber}}(\mathbb{L}) \to T_{\xi_j} \mathcal{P}^1_{\text{Ber}}(\mathbb{L})$ is not postcritically finite at the nontrivial critical points, and the points $\xi_1, \ldots, \xi_n$ are in pairwise distinct periodic orbits of $f$. Note the separable part of $f$ has at most $2 \deg_0 f - 2$ critical points, hence it has at most $2 \deg_0 f - 2$ critical values. Then by Proposition 4.6 for the case $\text{char } K = 0$ and Corollary 5.3 for the case $\text{char } K > 0$, we have $n \leq 2 \deg_0 f - 2$.

6. Examples

In this section, we give some examples to illustrate rescaling limits in non-Archimedean dynamics. We refer [18] for more examples. All examples in [18] are holomorphic families over $\mathbb{C}$, which can be considered as analytic families over non-Archimedean fields.

Now let $p > 0$ be a prime number. Denote by $K$ the field $\overline{\mathbb{F}_p}[\![s^q]\!]$ of Hahn series over $\mathbb{F}_p$ with respect to its nontrivial non-Archimedean absolute value. Then $\text{char } K = p > 0$.

Example 6.1. Polynomials with quadratic separable parts.

Given sufficiently small $t \in K \setminus \{0\}$. Consider the map

$$f_t(z) = (z^p - (s + t))(z^p - s) \in K(z).$$

Then associated map of $\{f_t\}$ has separable part $f_1(z) = (z - (s + t))(z - s)$. Note that $f_1$ is tame if and only if $p \geq 3$. In fact, if $p = 2$, $f_1$ has only one critical point at $\infty$.

Note $\{f_t\}$ is an analytic family that has good reduction. Then, up to equivalence, the moving frame $\{M_t(z) = z\}$ is the unique possible rescaling. The corresponding limit is $g(z) = (z^p - s)^2$. If $p = 2$, then $g(z) = z^4 + s^2$ has nontrivial degree 1. If $p \geq 3$, $g(z) = (z^p - s)^2$ has separable part $g_1(z) = (z - s)^2$. Note $\text{Crit}(g_1) = \{s, \infty\}$ and $g_1(s)$ has an infinite forward orbit under $g$. Thus $g$ is a rescaling limit that is not postcritically finite at nontrivial critical points.

Example 6.2. Connected Julia sets.

Let $a \in K$ with $0 < |a|_K < 1$ and $b \in K$ with $|b|_K = |b - 1|_K = 1$. Define

$$\phi_{a,b}(z) = \frac{az^6 + 1}{az^6 + z(z - 1)(z - b)} \in K(z).$$

Then the Berkovich Julia set $J_{\text{Ber}}(\phi_{a,b})$ is connected but not contained in a line segment $\overline{2}$.

For sufficiently small $t \in K \setminus \{0\}$, let $a = t^q$, where $q = 3p^2 + 1$, and fix $b \in K$. Define $f_t(z) = \phi_{t^q,b}(z^p)$. Then $\{f_t\}$ is a degenerated analytic family defined in a neighborhood of $t = 0$. Let $f$ be the associated map of $\{f_t\}$. Then the separable part $f_1$ of $f$ is $f_1(z) = (t^{q}z^6 + 1)/(t^{q}z^6 + z(z - 1)(z - b))$, which has a connected Berkovich Julia set $J_{\text{Ber}}(f_1)$.

First the moving frame $\{M_t(z) = z\}$ is a rescaling of period 1 with rescaling limit

$$g(z) = \frac{1}{z(z - 1)(z - b)} \circ z^p.$$

Note $f_1$ maps the segment $[\xi_0,|t|^{-q/3},\xi_G]$ isometrically onto the segment $[\xi_0,|t|^{-q/2},\xi_G]$. And $f_1$ maps the segment $[\xi_G, \xi_0,|t|^{-q/2},\xi_G]$ bijectively to the segment $[\xi_G, \xi_0,|t|^{-q/2},\xi_G]$ and the segment $[\xi_0,|t|^{-q/2},\xi_0,|t|^{-q/2},\xi_G]$ bijectively to the segment $[\xi_0,|t|^{-q/2},\xi_G]$, stretching
by a factor of 3, respectively. We may expect there exists a point $\xi_{0,r} \in \mathbb{P}^1_{\text{Ber}}(L)$ such that $f^2(\xi_{0,r}) = \xi_{0,r}$. Indeed, we can choose $r = |t|$. Let $L_t(z) = tz$. Then the moving frame $\{L_t(z)\}$ is a rescaling of period 2 leading to rescaling limit

$$h(z) = \begin{cases} \frac{1}{1+p^\infty} \circ z^4, & \text{if } p = 2, \\ -\frac{1}{p^2} \circ z^{2^7}, & \text{if } p = 3, \\ -\frac{1}{p^2} \circ z^{2^\infty}, & \text{if } p \geq 5. \end{cases}$$

Example 6.3. McMullen maps.

This example is an analog of [18, 2.4]. Given sufficiently small $t \in K \setminus \{0\}$, consider the map

$$f_t(z) = z^{2p} + \frac{t^{1+2p}}{2p} \in K(z).$$

Then $\{f_t\}$ is a degenerate analytic family defined in a neighborhood of $t = 0$. The associated map $f$ of $\{f_t\}$ has separable part $f_1(z) = z^2 + t^{1+2p}/z$. By [10] Corollary 6.6, when $p \geq 5$, the map $f_1(z)$ is tame. In fact, when $p = 3$, at every type II point, $\phi$ has separable reduction. Then by [10] Corollary 7.13, $\phi$ is tame. When $p = 2$, $\phi$ has a unique critical point. Thus, $\phi$ is tame if and only if $p \geq 3$.

Obviously, the moving frame $\{M_t(z) = z\}$ is a rescaling of period 1. The corresponding rescaling limit is $g(z) = z^{2p}$, which has a tame separable part $g_1(z) = z^2$. Note the nontrivial critical set $\text{Crit}_0(g) = \{0, \infty\}$. So the rescaling limit $g$ is postcritically finite at the nontrivial critical points.

Moreover, the moving frame $\{L_t(z) = tz\}$ is a rescaling of period 2 for $\{f_t\}$, which leads to the rescaling limit $h(z) = z^{-2p^2}$. The rescaling limit $h(z)$ is also postcritically finite at the nontrivial critical points.

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References


