Polyhedral products and toric topology

The theme of toric topology/polyhedral products is to study spaces that interpolate between a point and a cartesian product of spaces. Reference: "Toric Topology" by Buchstaber-Panov AMS 2015

Ex: \( (X,*) \) a CW pair.
- \( \{*,**,*** \} \subset X \times X \times X \)
- \( \ast \cdots \ast \subset X \times \cdots \times X \).

- G. Porter '82: \( T_i (X, \cdots, X_n) \subset X \times \cdots \times X_n \)
  consisting of \( n \)-tuples with at least \( i \) coordinates = \(*\).
  \( T_i \) is called a fat wedge
  \( T_{n-1} \) is called a wedge.

- \( K \) = simplicial complex, \( R = \text{comm. ring w/d} \)
  The Stanley-Reisner ring: \( R[K] = R[v_1, \cdots, v_n]/I_K \)
  \( I_K = \text{Stanley-Reisner ideal} = \{ v_{i_1} \cdots v_{i_t} \mid \{i_1, \ldots, i_t\} \neq K \} \)

- Davis-Januszkiewicz '91
  \( DJCK(K) \) with \( H^*(DJCK; \mathbb{Z}) \cong \mathbb{Z}[K] \).
  \( \ast^n \leq DJCK(K) \leq (\mathbb{C}P^\infty)^n \)
**Defn:** moment-angle complex: \((D_1^2, S') = \text{pair of spaces}\)

and \(K = \text{simplicial complex with} \ n \ \text{vertices.}\)

\[2_K(D_1^2, S') = (D_1^2, S')^K = \bigcup_{\sigma \in K} D_{\sigma} = \text{colim}_{\sigma \in K} D_{\sigma} \subseteq (D_1^2)^n.\]

where \(D_{\sigma} = \{ (x_1, \ldots, x_n) \in (D_1^n) \mid x_i \in S' \ \text{if} \ i \notin \sigma \} \)

**Ex:** \[K = \{1, 2, 3\}; \quad D_{43} = D_1^2 \times S', \quad D_{43} = S^1 \times D^2 \]

\[\Rightarrow (D_1^2, S')^K = D_1^2 \times S' \cup S^1 \times D^2 = \partial (D_1^2 \times D^2) = S^3.\]

\[K = \Delta_3, \quad (D_1^2, S')^K = D_1^3 \times D^2 \times S' \cup D_1^2 \times S' \times D^2 \cup S^1 \times D^1 \times D^2 = \partial (D_1^2 \times D^2) \times D^2 = S^7.\]

\[K = \Delta^{m-1}, \quad (D_1^2, S')^K = S^{2m-1}.\]

**Defn:** real moment-angle complex: \((D_1^1, S^0) = \text{pair of spaces,}\)

\(K = \text{s.c. on} \ \{n\} = \{1, \ldots, n\}. \ \text{Then}\)

\[2_K(D_1^1, S^0) = (D_1^1, S^0)^K = \text{colim}_{\sigma \in K} D_{\sigma} \subseteq (D_1^1)^n\]

where \(D_{\sigma} = \{ (x_1, \ldots, x_n) \in (D_1^n) \mid x_i \in S^0 \ \text{if} \ i \notin \sigma \} \)

**Ex:** \[K = \{1, 2\} \Rightarrow (D_1^1, S^0)^K = S^1\]

\[K = \Delta^n \Rightarrow (D_1^1, S^0)^K = S^{n-1}.\]

\[K = \text{g-gon} \Rightarrow (D_1^1, S^0)^K = M_g, \quad g = 1 + (m-4) \cdot 2^{m-3}.\]

Coxeter '58
**Polyhedral Product**

\[(X, A) = \left\{ (X_i, A_i) \right\}_{i=1}^n, \] sequence of pairs of spaces, \(X\) s.c. on \(\Delta^n\).

\[Z_k(X, A)^k = \operatorname{colim} D_{\sigma}, \quad D_{\sigma} = \left\{ (x_1, \ldots, x_n) \in \prod X_i \mid x_i \in A_i \right\}_{\sigma \in K}\]

(Can be thought of as a colim of a diagram of spaces \(D: K \to \text{Top}\).)

**Ex:** Moment-angle complexes \(\mathbb{R}\) & real moment-angle cpx

Applications are polyhedral products:

- \(X \times \cdots \times X = (X, *)^k\), where \(K^0 = 0\)-skeleton of \(\Delta^{n-1}\).
- \(T_i (X_1, \ldots, X_n) = (\prod X_i, *)^{K_{n-1}}, K_{n-1} = (n-i)\)-skeleton of \(\Delta^{n-1}\).
- \(D^i (K) = ET^i \times (D^2, S^1)^n \simeq \oplus (BS^1, *)^k = (CP^\infty, *)^k \subset (CP^\infty)^n\).

Proved by Buchstaber-Panov.

**Davis-Januszkiewicz** Every quasi-toric manifold can be realized as the quotient of a moment-angle cpx. by the free action of a real torus.

\[(M_p(X) = T^n \times P/\nu, \quad (t, p) \sim (t, u_1 q) \text{ if } p = q \quad \text{and} \quad u_1 \in \text{im}(\chi: T^n \to T^n)\]

\((K)\) dual of \(\mathbb{Z} P\) free action of \(\mathbb{Z}^{m-n} \oplus \text{ker}(\chi)\).

**Goresky-MacPherson** '88 Complement of of subspace arrangements:

\[U(K) = \mathbb{C}^n \setminus \bigcup_{i \notin K} \{ z \in \mathbb{C}^n \mid z_i = 0 \text{ if } i \in I \}\]
G-M studied the cohomology of $U(K)$.

Buchstaber-Panov showed $U(K) \cong (C, C^*)^K \cong (D^2, S^1)^K$ and

$$H^*\left(\mathbb{Z}_k(D^2, S^1); R\right) = \text{Tor}_R \left( \bigotimes_{v_1, \ldots, v_n} R[K], R \right)$$

$H_*$ of free $M$-resolution of $R$, $\otimes_M$ with $R[k]$.

- **Total Rank Conjecture:** $T^k \oplus X$ almost freely, $X$ fin dim $\mathbb{C}$.
  - Then $\dim H^*(X; \mathbb{Q}) \geq 2^k$.
  - Y. Ustinovsky $\Rightarrow$ TRC holds for moment-angle complexes.

- **Bahri-Bendersky-Cohen-Gitter 09:**
  $$\sum (X, A)^K \cong \sum_{\mathcal{I} \subseteq \{n\}} \left( X_{\mathcal{I}}, A_{\mathcal{I}} \right)^K$$

  Far-reaching generalization of $\sum X \times X = \sum X \times X \vee \sum X \times 2X$.

- **Grbic-Panov-Theriault-Wu:** $K = \text{flag complex}$. TFAE:
  1. $\Gamma = K^2$ is a chordal graph
  2. $(D^2, S^1)^K$ is a wedge of spheres
  3. $\text{lk} [K]$ is a Golod ring
  4. Multiplication in $H^*((D^2, S^1)^K; \mathbb{Q})$ is trivial.
### A variety of contexts

- \((X, A)\)
- \((D^2, S^1)\)
- \((D^1, S^0)\)
- \((S^1, S^1)\)
- \((C, C^*)\)
- \((EG, G)\)
- \((BG, *)\)
- \((PX, DX)\)
- \((X, A)^K\)

**Toric geometry & topology**
- Surfaces, number theory
- Robotics
- Complements of hyperplane arrangements
- Free groups
- Monodromy, combinatorics
- Homotopy theory

#### Monodromy representation:

- **G = top group.** Then

\[
\mathbb{Z}_k (EG, G) \rightarrow \mathbb{Z}_k (BG, *)
\]

\[
\downarrow
\]

\[
(BG)^n
\]

**Study the case:**
- when \(G\) is finite.
- (abelian / non-abelian).
Group Actions

Dehham-Suciu: \( G_1, \ldots, G_n \) are topological groups, \( \text{BG}_i, \text{EG}_i \) class. sp. of free \( G_i \)-action.

\[
\mathbb{Z}_k(\text{EG}_i, G_i) \to \prod_{i=1}^n \text{EG}_i \times \prod_{i=1}^n \mathbb{Z}_k(\text{EG}_i, G_i) \to \prod_{i=1}^n \text{BG}_i \quad \text{bundle}
\]

\[
\mathbb{Z}_k(\text{EG}_i, G_i) \to \mathbb{Z}_k(\text{BG}_i) \to \prod_{i=1}^n \text{BG}_i
\]

What happens when \( G_i \) is finite discrete:

\[
\mathbb{Z}_k(\text{EG}_i, G_i) \to \mathbb{Z}_k(\text{BG}_i) \to \prod \text{BG}_i
\]

Thm: \( \mathbb{Z}_k(\text{BG}_i, \ast) \cong K(n_1) \) if \( K \) is a flag complex. (N. Davis \( \iff \))

In general: \( \prod \mathbb{Z}_k(\text{BG}_i) = \prod G_i = \text{graph product of } G_i \).

\[
\mathbb{Z}(\text{EG}_i, G_i) \to \mathbb{Z}(\text{BG}_i) \to \prod \text{BG}_i \to \mathbb{Z}
\]

\[
\mathbb{Z}(\text{EG}_i, G_i) \to \mathbb{Z}(\text{BG}_i) \to \prod \text{BG}_i \to \mathbb{Z}
\]

\[
K=\text{flag}: \quad 1 \to \mathbb{Z}(\mathbb{Z}_k(\text{EG}_i, G_i)) \to \prod G_i \to \prod G_i \to \mathbb{Z}
\]

\[
\mathbb{Z}(\text{EG}_i, G_i) \to \mathbb{Z}(\text{BG}_i) \to \prod \text{BG}_i \to \mathbb{Z}
\]

\[
K=\text{flag} = K_0: \quad 1 \to \mathbb{Z}(\mathbb{Z}_k(\text{EG}_i, G_i)) \to \prod G_i \to \prod G_i \to \mathbb{Z}
\]

Prep: Rank of free sp is \( (n-1) \sum_{i=1}^n \text{rank } G_i - \sum_{i=1}^n \left( \prod_{j \neq i} \text{rank } G_j \right) + 1 \)

An early result of Nielsen using algebra. Here we use topology.

When is \( \mathbb{Z}_5(\text{fib}) \) free? \( \text{Ans. } \iff K_1 \) is a chordal graph (i.e. triangulated)

What is the monodromy rep?

\[
K_0 \text{ w/ 2 vertices, } G_1 = \mathbb{Z}/2, \ G_2 = \mathbb{Z}/3 = \{1, y, y^2\}
\]

\[
\begin{align*}
\langle x \rangle & = \langle y \rangle \\
\{1, x \} & = \langle y \rangle
\end{align*}
\]

\[
1 \to F_2 \to \mathbb{Z}_2 \times \mathbb{Z}_3 \to \mathbb{Z}_2 \times \mathbb{Z}_3 \to 1
\]
$F_2 = \{ w_1, w_2 \}$, $w_1 = [x,y]$, $w_2 = [xy^2]$. $xw_1 x^{-1} = w_1^{-1}$, $xw_2 x^{-1} = w_2^{-1}$.

$\mathbb{Z}_2 \times \mathbb{Z}_3 \hookrightarrow \text{Aut}(F_2)$
$\mathbb{Z}_2 \times \mathbb{Z}_3 \hookrightarrow \text{SL}_2(\mathbb{Z})$.

For any finite cyclic $\mathbb{Z}_n$, $\mathbb{Z}_m$

$\mathbb{Z}_n \times \mathbb{Z}_m \hookrightarrow \text{Aut}(F_n)$
$\mathbb{Z}_n \times \mathbb{Z}_m \hookrightarrow \text{SL}_n(\mathbb{Z})$.

(*) Note that none of these aut. are in $\text{IA}_N$.

If we choose non-abelian groups, then the reps don't necessarily land in $\text{SL}_N(\mathbb{Z})$. e.g. $G_1 = \mathbb{Z}/2$, $G_2 = \Sigma_3$.

For a finite # of $G_1, \ldots, G_\ell$:

The fibre has fundamental group $w$ with a generally set of iterated commutators

$[g_i, g_j], [g_i, [g_i, g_j]], \ldots, [g_1, \ldots, [g_k], [g_i, g_j]] \ldots$, where $g_i \in G_i$, $w$, $k_1 < \ldots < k_\ell < j \quad \& \quad j > i \neq k_r \quad \forall r$.

E.g. $G_1 = \{1, x\}$, $G_2 = \{1, y\}$, $G_3 = \{1, x, x^2\}$

$S = \{ (x^i, x), (x^j, x), (x^i, y), (x^j, y), (y, x), \ldots, (x, (x^i, y)), \ldots \}$

$|S| = q$.

Connections

w/ Feit-Thompson Theorem.

For the monodromy rep., see Magma codes.

Theorem: Let $G_1, \ldots, G_\ell$ be finite abelian gps. Then the faithful monodromy rep induces a faithful representation $G_1 \times \ldots \times G_\ell \rightarrow \text{SL}(\mathbb{Z}_n, \mathbb{Z})$.

Theorem: $\prod_{K^i} G_i \hookrightarrow \text{Aut}(F_{p_1})$ $K^i = \text{chordal}$

Right-angled Artin/Coxeter groups.