A Strong Künneth Theorem for Periodic Topological Cyclic Homology

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Shanks Workshop on Homotopy Theory

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March 25, 2017
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Topological periodic cyclic homology ($TP$) is the analogue of periodic cyclic homology ($HP$) using $THH$ in place of $HH$. If $k$ is a finite field, then smooth and proper d.g. categories over $k$ satisfy a strong Künneth theorem:

$$TP(X) \wedge_{TP(k)}^L TP(Y) \to TP(X \otimes_k Y)$$

is an isomorphism in the derived category of $TP(k)$-modules.

- Joint work with Andrew Blumberg
- Preprint Soon
Overview

Topological periodic cyclic homology \((TP)\) is the analogue of periodic cyclic homology \((HP)\) using \(THH\) in place of \(HH\). If \(k\) is a finite field, then smooth and proper varieties over \(k\) satisfy a strong Künneth theorem:

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1. Introduction to $TP$
Topological periodic cyclic homology ($TP$) is the analogue of periodic cyclic homology ($HP$) using $THH$ in place of $HH$. If $k$ is a finite field, then smooth and proper d.g. algebras over $k$ satisfy a strong Künneth theorem:

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1. Introduction to $TP$
2. Structure and properties of $TP$
Topological periodic cyclic homology (TP) is the analogue of periodic cyclic homology (HP) using THH in place of HH. If \( k \) is a finite field, then smooth and proper d.g. algebras over \( k \) satisfy a strong Künneth theorem:

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Outline
1. Introduction to \( TP \)
2. Structure and properties of \( TP \)
3. The Künneth theorem
Hochschild Homology

Cyclic bar construction

\[ N^c_y R = \underbrace{R \otimes \cdots \otimes R \otimes R}_{q \text{ factors}} \]

Chain complex

Cyclic structure \( \longrightarrow \) Connes’ \( B \) operator

\[ B : N^c_y R \rightarrow N^c_y R[-1] \]
Hochschild Homology

Cyclic bar construction

\[ N^c_R = R \otimes \cdots \otimes R \otimes R \]

\( q \) factors

Chain complex

Cyclic structure \( \implies \) Connes' \( B \) operator

\[ B : N^c_R \rightarrow N^c_R[-1] \]
Hochschild Homology

Cyclic bar construction

\[ N_q^{cy} R = \underbrace{R \otimes \cdots \otimes R \otimes R}_{q \text{ factors}} \]

Chain complex

Cyclic structure \(\Rightarrow\) Connes’ \(B\) operator

\[ B : N^{cy} R \to N^{cy} R[-1] \]

\[ B^2 = 0 \]
Hochschild Homology and Cyclic Homology

Cyclic bar construction

\[ N_q^{cy} R = R \otimes \cdots \otimes R \otimes R \]

\( q \) factors

\[ R \otimes \cdots \otimes R \]

\[ R \]

Chain complex

Cyclic structure \( \Rightarrow \) Connes’ \( B \) operator

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Hochschild Homology and Cyclic Homology

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Hochschild Homology and Cyclic Homology

Cyclic bar construction

\[ N_q^{\text{cy}} R = R \otimes \cdots \otimes R \otimes R \]

\[ q \text{ factors} \]

\[ R \otimes \cdots \otimes R \]

\[ \otimes \quad \otimes \]

\[ R \]

Chain complex

Cyclic structure \( \implies \) Connes’ \( B \) operator

\[ B : N^{\text{cy}} R \to N^{\text{cy}} R[-1] \]
Hochschild Homology and Cyclic Homology

Cyclic bar construction

\[ N_q^{cy} R = R \otimes \cdots \otimes R \otimes R \]

\[ q \text{ factors} \]

\[ R \otimes \cdots \otimes R \]

\[ R \]

Chain complex

Cyclic structure \( \Longrightarrow \) Connes’ \( B \) operator

\[ B : N^{cy} R \rightarrow N^{cy} R[-1] \]

Construct Double Complex:

\[ \cdots \]

\[ \begin{array}{cccc}
\cdot & \Longrightarrow & \cdot & \Longrightarrow \\
\cdot & \Longrightarrow & \cdot & \Longrightarrow \\
\cdot & \Longrightarrow & \cdot & \Longrightarrow \\
\end{array} \]

\[ \begin{array}{cccc}
\cdot & \Longrightarrow & \cdot & \Longrightarrow \\
\cdot & \Longrightarrow & \cdot & \Longrightarrow \\
\cdot & \Longrightarrow & \cdot & \Longrightarrow \\
\end{array} \]

\[ \cdots \]

\[ HP \]
Cyclic bar construction (Bökstedt)

\[ N_q^{cy} R = \underbrace{R \wedge \cdots \wedge R \wedge R}_{q \text{ factors}} \]

Spectrum

Cyclic structure \(\longrightarrow\) circle group action
Topological Hochschild Homology

Cyclic bar construction (Bökstedt)

\[ N^c_y R = R \wedge \cdots \wedge R \wedge R \]

\[ q \text{ factors} \]

\[ R \wedge \cdots \wedge R \]
\[ \wedge \quad \wedge \]
\[ R \]

Spectrum

Cyclic structure \(\rightarrow\) circle group action

\[ \Sigma X \rightarrow S^1 \wedge S^5 \]
\[ X \sim \Omega X \]
Topological Hochschild Homology

Cyclic bar construction (Bökstedt)

\[ N_q^{cy} R = R \wedge \cdots \wedge R \wedge R \]

\[ R \wedge \cdots \wedge R \]

\[ \wedge \quad \wedge \]

R

Spectrum

Cyclic structure \(\rightarrow\) circle group action
Topological Hochschild Homology

Cyclic bar construction (Bökstedt)

\[ N_{q}^{\text{cy}} R = R \wedge \cdots \wedge R \wedge R \]

\[ q \text{ factors} \]

\[ R \wedge \cdots \wedge R \]

\[ \wedge \quad \wedge \]

\[ R \]

Construction

\[ \cdots \]

\[ \downarrow \]

\[ \cdots \]

\[ HH \text{ corresponds to } THH \]

Spectrum

Cyclic structure \(\longrightarrow\) circle group action
Topological Hochschild Homology

Cyclic bar construction (Bökstedt)

\[ N^c_y R = R \wedge \cdots \wedge R \wedge R \]

\[ q \text{ factors} \]

\[ R \wedge \cdots \wedge R \wedge \cdots \wedge R \wedge R \]

Spectrum

Cyclic structure \(\rightarrow\) circle group action

Construction

\[ HH \text{ corresponds to } THH \]

\[ HC \text{ corresponds to } THH_{h\mathbb{T}} \]
Topological Hochschild Homology

Cyclic bar construction (Bökstedt)

\[ N_q^{cy} R = R \wedge \ldots \wedge R \wedge R \]

\[ R \wedge \ldots \wedge R \]

\[ \wedge \ldots \wedge R \]

\[ R \]

Construction

\[ \cdots \leftarrow \bullet \leftarrow \bullet \]

\[ \cdots \leftarrow \bullet \leftarrow \bullet \]

\[ \cdots \leftarrow \bullet \leftarrow \bullet \]

\[ \cdots \leftarrow \bullet \leftarrow \bullet \]

\[ \cdots \leftarrow \bullet \]

Spectrum

Cyclic structure \[\longrightarrow\] circle group action

HH corresponds to \( THH \)

HN corresponds to \( THH^hT \)
Topological Hochschild Homology

Cyclic bar construction (Bökstedt)

\[ N^c_y R = R \wedge \cdots \wedge R \wedge R \]

\[ N^c_y R = \underbrace{R \wedge \cdots \wedge R \wedge R}_q \text{ factors} \]

\[ R \wedge \cdots \wedge R \]

\[ \wedge \quad \wedge \]

\[ R \]

Spectrum

Cyclic structure \( \rightarrow \) circle group action

Construction

\[ \cdots \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \cdots \]

\[ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \cdots \]

\[ \cdots \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \cdots \]

\[ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \]

\[ \cdots \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \cdots \]

\[ \downarrow \]

\[ \cdots \leftarrow \bullet \]

\[ \cdots \]

\[ HH \text{ corresponds to } THH \]

\[ HP \text{ corresponds to } THH^{tT} \]

M.A. Mandell (IU)
Topological Periodic Cyclic Homology

**Definition**
For a ring spectrum $R$, define the Topological Periodic Cyclic Homology of $R$ by $TP(R) = THH(R)^Tt^T$.

Not always periodic e.g. $TP(\mathbb{Z})$

But $TP(\mathbb{Z})$ is reduced

$= W_{k} \mathbb{Z}[v, v^{-1}]$  $|v| = -2$. 

$\Psi$
Topological Periodic Cyclic Homology

Definition

For a ring spectrum $R$, define the Topological Periodic Cyclic Homology of $R$ by $TP(R) = THH(R)^{t\mathbb{T}}$.

Highlights

- Major player in trace method $K$-theory calculations
- Characteristic $p$ replacement for $HP$ (?)
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\[ X \quad \text{smooth over } \mathbb{C} \]
\[ \quad \text{or smooth f.g. } \mathbb{C} \text{-alg} \]

$HP_{\cdot} (X)$ is De Rham Cohomology
Topological Periodic Cyclic Homology

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  - (2014–) Hasse-Weil zeta function: Connes-Consani $\sim$ Hesselholt
  - (2011–) Non-commutative motives: Kontsevich, Marcolli-Tabuada
  non-commutative homological motives $\sim ????$
Topological Periodic Cyclic Homology

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For a ring spectrum $R$, define the Topological Periodic Cyclic Homology of $R$ by $TP(R) = THH(R)^T$. 

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Realization functor / Weil cohomology theory

$$HP_*(X) \otimes_{k[t,t^{-1}]} HP_*(Y) \rightarrow HP_*(X \otimes_k Y)$$
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Realization functor / Weil cohomology theory

$$HP_*(X) \otimes_{k[t,t^{-1}]} HP_*(Y) \rightarrow HP_*(X \otimes_k Y)$$

$|t| \leq 2$
Küneth Theorem

**Theorem**

Lax symmetric monoidal functor

\[
TP(X) \wedge_{TP(R)}^L TP(Y) \to TP(X \wedge_R^L Y)
\]

**Definition**

A \( k \)-algebra \( X \) is smooth when it is compact as an \( X \otimes_k X^{\text{op}} \)-module, i.e., when \( R \text{Hom}_{X \otimes_k X^{\text{op}}} (X, -) \) commutes with direct sums.

**Definition**

A \( k \)-algebra \( X \) is proper when it is compact as a \( k \)-module.
Künneth Theorem

**Theorem**

Let $k$ be finite field. The lax symmetric monoidal functor

$$TP(X) \wedge_{TP(k)}^L TP(Y) \rightarrow TP(X \otimes_k Y)$$

is an isomorphism when $X$ and $Y$ are smooth and proper over $k$.

**Definition**

A $k$-algebra $X$ is smooth when it is compact as an $X \otimes_k X^{\text{op}}$-module, i.e., when $R \text{Hom}_X^{X \otimes_k X^{\text{op}}}(X, -)$ commutes with direct sums.

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**Künneth Theorem**

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is an isomorphism when $X$ and $Y$ are smooth and proper over $k$.

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*Definition*  
A $k$-algebra $X$ is proper when it is compact as a $k$-module.
Review of Tate Construction

\[ E \mathbb{T}_+ \to S^0 \to \widetilde{E} \mathbb{T} \]

Smash with \( Z^{E \mathbb{T}} \) and take fixed points

\[
(Z^{E \mathbb{T}} \wedge E \mathbb{T}_+)^T \to (Z^{E \mathbb{T}})^T \to (Z^{E \mathbb{T}} \wedge \widetilde{E} \mathbb{T})^T
\]

\[
(X^{E \mathbb{T}} \wedge E \mathbb{T}_+)^T \cong \Sigma(X^{E \mathbb{T}})_{h \mathbb{T}} \cong \Sigma X h \mathbb{T} \quad \text{(Adams Isomorphism)}
\]

**Definition**

For \( Z \) a \( \mathbb{T} \)-equivariant spectrum \( Z^{t \mathbb{T}} = (Z^{E \mathbb{T}} \wedge \widetilde{E} \mathbb{T})^T \).

(Composite of derived functors.)

\[
\Sigma Z_{h \mathbb{T}} \to Z^{h \mathbb{T}} \to Z^{t \mathbb{T}} \to \Sigma^2 Z_{h \mathbb{T}}
\]

\[
TP(X) = THH(X)^{t \mathbb{T}}
\]
Review of Tate Construction

\[ E^T \xrightarrow{E^T_+} S^0 \rightarrow \widetilde{E^T} \]

Smash with \( Z^{E^T} \) and take fixed points

\[
(Z^{E^T} \wedge E^T_+)^T \rightarrow (Z^{E^T})^T \rightarrow (Z^{E^T} \wedge \widetilde{E^T})^T
\]

\[
(X^{E^T} \wedge E^T_+)^T \simeq \Sigma (X^{E^T})_{ht} \simeq \Sigma X hT \quad \text{(Adams Isomorphism)}
\]

Definition

For \( Z \) a \( T \)-equivariant spectrum \( Z^{tT} = (Z^{E^T} \wedge \widetilde{E^T})^T \).

(Composite of derived functors.)

\[
\Sigma Z_{ht} \rightarrow Z^{ht} \rightarrow Z^{tT} \rightarrow \Sigma^2 Z_{ht}
\]

\[
TP(X) = THH(X)^{tT}
\]
Review of Tate Construction

Smash with $\mathbb{Z}^E_T$ and take fixed points

$$(Z^E_T \wedge E_{T+})^T \rightarrow (Z^E_T)^T \rightarrow (Z^E_T \wedge \widetilde{E}_T)^T$$

$$(X^E_T \wedge E_{T+})^T \simeq \Sigma(X^E_T)_{hT} \simeq \Sigma X h_T \quad \text{(Adams Isomorphism)}$$

Definition

For $Z$ a $T$-equivariant spectrum $Z^{tT} = (Z^E_T \wedge \widetilde{E}_T)^T$.

(Composite of derived functors.)

$$\Sigma Z_{hT} \rightarrow Z^{hT} \rightarrow Z^{tT} \rightarrow \Sigma^2 Z_{hT}$$

$$TP(X) = THH(X)^{tT}$$
Review of Tate Construction

\[ ET \xrightarrow{E_{\mathbb{T}+}} S^0 \xrightarrow{} ET \]

Smash with \( Z^{ET} \) and take fixed points

\[
(Z^{ET} \wedge E_{\mathbb{T}+})^T \rightarrow (Z^{ET})^T \rightarrow (Z^{ET} \wedge \widehat{ET})^T
\]

\[
(X^{ET} \wedge E_{\mathbb{T}+})^T \simeq \Sigma(X^{ET})_{h\mathbb{T}} \simeq \Sigma X h\mathbb{T} \quad \text{(Adams Isomorphism)}
\]

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For \( Z \) a \( \mathbb{T} \)-equivariant spectrum \( Z^{t\mathbb{T}} = (Z^{ET} \wedge \widehat{ET})^T \).

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\[
\Sigma Z_{h\mathbb{T}} \rightarrow Z^{h\mathbb{T}} \rightarrow Z^{t\mathbb{T}} \rightarrow \Sigma^2 Z_{h\mathbb{T}}
\]

\[
TP(X) = THH(X)^{t\mathbb{T}}
\]
Review of Tate Construction

\[ E_T \xrightarrow{E_T +} S^0 \rightarrow \tilde{E}_T \]

Smash with \( Z^{E_T} \) and take fixed points

\[ (Z^{E_T} \wedge E_{T+})^T \rightarrow (Z^{E_T})^T \rightarrow (Z^{E_T} \wedge \tilde{E}_T)^T \]

\[ (X^{E_T} \wedge E_{T+})^T \simeq \Sigma(X^{E_T})_{hT} \simeq \Sigma X h_T \text{ (Adams Isomorphism)} \]

**Definition**

For \( Z \) a \( T \)-equivariant spectrum \( Z^{tT} = (Z^{E_T} \wedge \tilde{E}_T)^T \).

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Smash with \( Z^{E_T} \) and take fixed points

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(Z^{E_T} \wedge E_{T+})^T \rightarrow (Z^{E_T})^T \rightarrow (Z^{E_T} \wedge \widetilde{E}_T)^T
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For \( Z \) a \( \mathbb{T} \)-equivariant spectrum \( Z^{t \mathbb{T}} = (Z^{E \mathbb{T}} \wedge \widetilde{E} \mathbb{T})^T \).

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\[ E_T \quad E_T^+ \to S^0 \to \tilde{E}_T \]

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(Z^{E_T} \wedge E_T^+)^T \to (Z^{E_T})^T \to (Z^{E_T} \wedge \tilde{E}_T)^T
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(X^{E_T} \wedge E_T^+ )^T \simeq \Sigma (X^{E_T})_h^T \simeq \Sigma X h_T \quad \text{(Adams Isomorphism)}
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Definition

For \( Z \) a \( \mathbb{T} \)-equivariant spectrum \( Z^{t_T} = (Z^{E_T} \wedge \tilde{E}_T)^T \).

(Composite of derived functors.)

\[
\Sigma Z_{h_T} \to Z^{h_T} \to (Z^{t_T}) \to \Sigma^2 Z_{h_T}
\]

\[ TP(X) = \text{THH}(X)^{t_T} \]
The Multiplication

\[ TP(X) \land TP(Y) \rightarrow TP(X \land Y) \]

\[ TP(X) = (THH(X)_{ET} \land \tilde{ET})^T \]

- \( \tilde{ET} \land \tilde{ET} \simeq \tilde{ET} \)
- Use diagonal map \( ET \rightarrow ET \times ET \)
- \( THH(X) \land THH(Y) \cong THH(X \land Y) \)
The Multiplication

\[ TP(X) \wedge TP(Y) \rightarrow TP(X \wedge Y) \]

\[ TP(X) = (THH(X)^{ET} \wedge \tilde{ET})^T \]

- \( \tilde{ET} \wedge \tilde{ET} \simeq \tilde{ET} \)
- Use diagonal map \( ET \rightarrow ET \times ET \)
- \( THH(X) \wedge THH(Y) \cong THH(X \wedge Y) \)
The Multiplication

\[ TP(X) \wedge TP(Y) \to TP(X \wedge Y) \]

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\[ \text{TP}(X) \wedge_{\text{TP}(R)} \text{TP}(Y) \to \text{TP}(X \wedge_R Y) \]
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\[ TP(X) \wedge TP(Y) \rightarrow TP(X \wedge Y) \]

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- \( \widetilde{E_T} \land \widetilde{E_T} \cong \widetilde{E_T} \)
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- \( THH(X) \land THH(Y) \cong THH(X \land Y) \)

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\[ TP(X) \wedge TP(Y) \rightarrow TP(X \wedge Y) \]

\[ TP(X) = (THH(X)^{ET} \wedge \widetilde{ET})^T \]

- \( \widetilde{ET} \wedge \widetilde{ET} \simeq \widetilde{ET} \) ← This can be made coherent!
- Use diagonal map \( ET \rightarrow ET \times ET \) ← This is coherent
- \( THH(X) \wedge THH(Y) \cong THH(X \wedge Y) \)

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\[ TP(X) \wedge TP(R) \wedge TP(Y) \rightarrow TP(X \wedge R \wedge Y) \]
Filtration on $TP(X)$ with associated graded

$$F^i / F^{i-1} \simeq \Sigma^{2i} \text{THH}(X)$$

$TP(X) = (\text{THH}(X)^{ET} \wedge \tilde{E}_T)^T$

Simplicial filtration on $E_T$

$T_+, \Sigma^2 T_+, \Sigma^4 T_+, \ldots$
The Filtration

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$T_+, \Sigma^2 T_+, \Sigma^4 T_+, \ldots$

$$T_+ \wedge (T/\{1\}) \wedge (\Delta[1]/\partial \Delta[1])$$
The Filtration

Filtration on \( TP(X) \) with associated graded

\[
F^i / F^{i-1} \simeq \Sigma^{2i} THH(X)
\]

\[
TP(X) = (THH(X)E_T \wedge \tilde{E}_T)^T
\]

Simplicial filtration on \( E_T \)

\( \mathbb{T}_+, \Sigma^2 \mathbb{T}_+, \Sigma^4 \mathbb{T}_+, \ldots \)

\[
\mathbb{T}_+ \wedge (\mathbb{T} \times \mathbb{T}/(\mathbb{T} \vee \mathbb{T})) \wedge \Delta[2]/\partial \Delta[2]
\]

\[
\mathbb{T} \times \mathbb{T} \times \mathbb{T}
\]
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Simplicial filtration on $E_T$ / on $\tilde{E}_T$

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Filtration on $TP(X)$:

$$F^i TP(X) = \begin{cases} (THH(X)^{(E_T, E_T_{-i-1})} \wedge S^0)^T & i \leq 0 \\ (THH(X)^{E_T} \wedge \widetilde{E_T}_i)^T & i > 0 \end{cases}$$

$$F^i / F^{i-1} = \begin{cases} (THH(X)^{(\Sigma^{2i} T_+)} )^T & i \leq 0 \\ (THH(X)^{E_T} \wedge \Sigma^{2i-1} T_+)^T & i > 0 \end{cases}$$
The Filtration

Filtration on $TP(X)$ with associated graded

$$F^i / F^{i-1} \simeq \Sigma^{2i} \text{THH}(X)$$

$$TP(X) = (\text{THH}(X)E^{\mathbb{T}} \wedge \tilde{E}^{\mathbb{T}})^{\mathbb{T}}$$

Simplicial filtration on $E^{\mathbb{T}}$ / on $\tilde{E}^{\mathbb{T}}$

$\mathbb{T}^+, \Sigma^2 \mathbb{T}^+, \Sigma^4 \mathbb{T}^+, \ldots$ / $S^0, \Sigma \mathbb{T}^+, \Sigma^3 \mathbb{T}^+, \ldots$

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$$F^i / F^{i-1} = \begin{cases} (\text{THH}(X)(\Sigma^{2i} \mathbb{T}^+))^\mathbb{T} & i \leq 0 \\ (\text{THH}(X)E^{\mathbb{T}} \wedge \Sigma^{2i-1} \mathbb{T}^+)^{\mathbb{T}} & i > 0 \end{cases}$$
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\end{cases}$$

$$F^i / F^{i-1} = \begin{cases} 
(THH(X)^{(\Sigma^{2i} \mathbb{T}_+})^T & i \leq 0 \\
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\end{cases}$$
The Spectral Sequence

Filtration on $TP(X)$ with associated graded

$$F^i / F^{i-1} \simeq \Sigma^{2i} THH(X)$$

Spectral sequence

$$E^1_{i,j} = \pi_{i+j} \Sigma^{2i} THH(X) = THH_{j-i}(X)$$

Renumber: Double filtration degree

$$E^{2r}_{2i,j} = (E^r_{i,i+j})^{old}, \quad d_{2r} = (d_r)^{old}$$

(1, 1) periodic on $E^1$

Spectral Sequence

Conditionally convergent spectral sequence

$$E^2_{2i,j} = \text{THH}(X) \Longrightarrow TP_{2i+j}(X). \quad (E^r_{2i+1,j} = 0)$$
The Spectral Sequence

Filtration on $TP(X)$ with associated graded

$$F^i / F^{i-1} \cong \Sigma^{2i} \text{THH}(X)$$

Spectral sequence

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Combining the Multiplication and Filtration

Multiplicative spectral sequence:

\[ TP(X) \wedge TP(Y) \rightarrow TP(X \wedge Y) \text{ a filtered map} \]
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Combining the Multiplication and Filtration

Multiplicative spectral sequence:

\[ TP(X) \wedge TP(Y) \to TP(X \wedge Y) \] a filtered map

In homotopy category, easy obstruction theory cellular approximation to diagonal & multiplication.

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\[ \Rightarrow \text{map of spectral sequences} \]

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Multiplicative spectral sequence:

$$TP(X) \wedge TP(Y) \to TP(X \wedge Y)$$

Is a filtered map? Coherent model?

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Coherently homotopy associative cellular approximation to diagonal?

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Approximation of the diagonal for simplicial spaces

Let $X_\bullet$ be a simplicial space, $|X_\bullet|$ its geometric realization. $|X_\bullet^n|$ vs $|X_\bullet|^n$

Problem

Parametrize a contractible spaces of filtered approximations of the diagonal maps $|X_\bullet| \to |X_\bullet|^n$ for all $n$ that compose appropriately.

Find an $A_\infty$ operad $\mathcal{A}$ and a map of operads $\mathcal{A}(n) \to \text{Filt}(|X_\bullet|, |X_\bullet|^n) \subset \mathcal{T}(|X_\bullet|, |X_\bullet|^n)$. 
Approximation of the diagonal for simplicial spaces

Let $X_\bullet$ be a simplicial space, $|X_\bullet|$ its geometric realization. $|X^n_\bullet|$ vs $|X^n_\bullet|$

Problem

Parametrize a contractible spaces of filtered approximations of the diagonal maps $|X_\bullet| \rightarrow |X_\bullet|^n$ for all $n$ that compose appropriately.

Find an $A_\infty$ operad $A$ and a map of operads

$A(n) \rightarrow \text{Filt}(|X_\bullet|, |X_\bullet|^n) \subset \mathcal{T}(|X_\bullet|, |X_\bullet|^n)$. 

\[ \begin{array}{c}
\Delta \\
X
\end{array} \]
Approximation of the diagonal for simplicial spaces

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Approximation of the diagonal for simplicial spaces

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**Barycentric Coordinates and Milnor Coordinates on $\Delta[m]$**

Barycentric $t_0, \ldots, t_m$, $\sum t_i = 1 \leftrightarrow$ Milnor $0 \leq u_0 \leq u_1 \leq \cdots \leq u_{m-1} \leq 1$

An element in $|X_\bullet|$ is specified by $(x \in X_m, 0 \leq u_0 \leq \cdots \leq u_{m-1} \leq 1)$
Approximation of the diagonal for simplicial spaces

Let $X_\bullet$ be a simplicial space, $|X_\bullet|$ its geometric realization. $|X_\bullet|^n$ vs $|X_\bullet|^n$

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Approximation of the diagonal for simplicial spaces (ii)

Problem

Parametrize a contractible spaces of filtered approximations of the diagonal maps $|X_{\bullet}| \to |X_{\bullet}|^n$ for all $n$ that compose appropriately.

Solution
Problem

Parametrize a contractible spaces of filtered approximations of the diagonal maps $|X_\bullet| \to |X_\bullet|^n$ for all $n$ that compose appropriately.

Solution

1. The overlapping little 1-cubes operad $C_{\Xi}^1$
### Approximation of the diagonal for simplicial spaces (ii)

**Problem**

Parametrize a contractible spaces of filtered approximations of the diagonal maps $|X_\bullet| \to |X_\bullet|^n$ for all $n$ that compose appropriately.

**Solution**

1. The overlapping little 1-cubes operad $C_1^\Xi$
2. The map $C_1^\Xi \to \mathcal{T}(|X_\bullet|, |X_\bullet|^n)$. 
Approximation of the diagonal for simplicial spaces (ii)

Problem

Parametrize a contractible spaces of filtered approximations of the diagonal maps $|X_{\bullet}| \to |X_{\bullet}|^n$ for all $n$ that compose appropriately.

Solution

1. The overlapping little 1-cubes operad $C_1^{\Xi}$
2. The map $C_1^{\Xi} \to T(|X_{\bullet}|, |X_{\bullet}|^n)$.

An element $c \in C^{\Xi}(n)$ specifies $n$ monotonic PL maps $g_i: I \to I$
Approximation of the diagonal for simplicial spaces (ii)

**Problem**

Parametrize a contractible spaces of filtered approximations of the diagonal maps $|X\bullet| \to |X\bullet|^n$ for all $n$ that compose appropriately.

**Solution**

1. The overlapping little 1-cubes operad $C_1^\Xi$
2. The map $C_1^\Xi \to T(|X\bullet|, |X\bullet|^n)$.

An element $c \in C_1^\Xi(n)$ specifies $n$ monotonic PL maps $g_i : I \to I$

$$
(x, 0 \leq u_0 \leq \cdots \leq u_{m-1} \leq 1) \mapsto
((x, 0 \leq g_1(u_0) \leq \cdots \leq g_1(u_{m-1}) \leq 1), \ldots, (x, 0 \leq g_n(u_0) \leq \cdots \leq g_n(u_{m-1}) \leq 1))
$$
Approximation of the diagonal for simplicial spaces (ii)

**Problem**

Parametrize a contractible spaces of filtered approximations of the diagonal maps $|X| \to |X|^n$ for all $n$ that compose appropriately.

**Solution**

1. The overlapping little 1-cubes operad $\mathcal{C}_1^\Xi$
2. The map $\mathcal{C}_1^\Xi \to \mathcal{T}(|X|, |X|^n)$.
3. little 1-cubes operad $\mathcal{C}_1 \subset \mathcal{C}_1^\Xi$.

An element $c \in \mathcal{C}^\Xi(n)$ specifies $n$ monotonic PL maps $g_i : I \to I$

\[(x, 0 \leq u_0 \leq \cdots \leq u_{m-1} \leq 1) \mapsto ((x, 0 \leq g_1(u_0) \leq \cdots \leq g_1(u_{m-1}) \leq 1), \ldots, (x, 0 \leq g_n(u_0) \leq \cdots \leq g_n(u_{m-1}) \leq 1))\]
A filtered approximation of the diagonal gives a map

\[ E_{T_{i+j-1}} \to (E_{T_{i-1}} \times E_T) \cup (E_T \times E_{T_{j-1}}) \subset E_T \times E_T \]

Hence a map \((E_T, E_{T_{i+j-1}}) \to (E_T, E_{T_{i-1}}) \times (E_T, E_{T_{j-1}})\)

Applied to \(TP\)
Filtered Monoidal Structure

A filtered approximation of the diagonal gives a map

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Applied to \(TP\)
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A filtered approximation of the diagonal gives a map

$$E^{T_{i+j-1}} \rightarrow (E^{T_{i-1}} \times E^{T}) \cup (E^{T} \times E^{T_{j-1}}) \subset E^{T} \times E^{T}$$

Hence a map $$(E^{T}, E^{T_{i+j-1}}) \rightarrow (E^{T}, E^{T_{i-1}}) \times (E^{T}, E^{T_{j-1}})$$

Applied to $TP$

$$F^{-i} TP(X) \land F^{-j} TP(X) \rightarrow F^{-i-j} TP(X \land Y)$$
Filtered Monoidal Structure

A filtered approximation of the diagonal gives a map

\[ E^T_{i+j-1} \to (E^T_{i-1} \times E^T) \cup (E^T \times E^T_{j-1}) \subset E^T \times E^T \]

Hence a map \((E^T, E^T_{i+j-1}) \to (E^T, E^T_{i-1}) \times (E^T, E^T_{j-1})\)

Parametrized

\[ \mathcal{C}_1(n) \wedge F^{-i_1} \text{TP}(X_1) \wedge \cdots \wedge F^{-i_n} \text{TP}(X_n) \to F^{-i_1-\cdots-i_n} \text{TP}(X_1 \wedge \cdots \wedge X_n) \]
A filtered approximation of the diagonal gives a map

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Hence a map \((E^T, E^T_{i+j-1}) \to (E^T, E^T_{i-1}) \times (E^T, E^T_{j-1})\)

Parametrized

\[ C_1(n)_+ \wedge F^{-i_1} \text{TP}(X_1) \wedge \cdots \wedge F^{-i_n} \text{TP}(X_n) \to F^{-i_1 - \cdots - i_n} \text{TP}(X_1 \wedge \cdots \wedge X_n) \]

Little 1-cubes: Moore construction (Moore loop space)

Use length parameter to make fully associative

\[ F^{-i} \text{TP}(X) \wedge \mathbb{R}^0_+ \wedge F^{-j} \text{TP}(Y) \wedge \mathbb{R}^0_+ \to F^{-i-j} \text{TP}(X \wedge Y) \wedge \mathbb{R}^0_+ \]
Filtered Monoidal Structure

A filtered approximation of the diagonal gives a map

\[ E^T_{i+j-1} \to (E^T_{i-1} \times E^T) \cup (E^T \times E^T_{j-1}) \subset E^T \times E^T \]

Hence a map \((E^T, E^T_{i+j-1}) \to (E^T, E^T_{i-1}) \times (E^T, E^T_{j-1})\)

Parametrized

\[ C_1(n)_+ \wedge F^{-i_1} TP(X_1) \wedge \cdots \wedge F^{-i_n} TP(X_n) \to F^{-i_{i_1} \cdots i_n} TP(X_1 \wedge \cdots \wedge X_n) \]

Little 1-cubes: Moore construction (Moore loop space)

Use length parameter to make fully associative

\[ F^{-i} TP(X) \wedge \mathbb{R}^{>0}_+ \wedge F^{-j} TP(Y) \wedge \mathbb{R}^{>0}_+ \to F^{-i-j} TP(X \wedge Y) \wedge \mathbb{R}^{>0}_+ \]
Filtered Monoidal Structure

A filtered approximation of the diagonal gives a map

\[ E^T_{i+j-1} \to (E^T_{i-1} \times E^T) \cup (E^T \times E^T_{j-1}) \subset E^T \times E^T \]

Hence a map \((E^T, E^T_{i+j-1}) \to (E^T, E^T_{i-1}) \times (E^T, E^T_{j-1})\)

Parametrized

\[ C_1(n)_+ \wedge F^{-i_1} TP(X_1) \wedge \cdots \wedge F^{-i_n} TP(X_n) \to F^{-i_1-\cdots-i_n} TP(X_1 \wedge \cdots \wedge X_n) \]

Little 1-cubes: Moore construction (Moore loop space)

Use length parameter to make fully associative

\[ F^{-i} TP(X) \wedge \mathbb{R}_+^{>0} \wedge F^{-j} TP(Y) \wedge \mathbb{R}_+^{>0} \to F^{-i-j} TP(X \wedge Y) \wedge \mathbb{R}_+^{>0} \]
Filtered Monoidal Structure

A filtered approximation of the diagonal gives a map
\[ E^T_{i+j-1} \to (E^T_{i-1} \times E^T) \cup (E^T \times E^T_{j-1}) \subseteq E^T \times E^T \]

Hence a map \( (E^T, E^T_{i+j-1}) \to (E^T, E^T_{i-1}) \times (E^T, E^T_{j-1}) \)

Parametrized
\[ C_1(n)_+ \wedge F^{-i_1} TP(X_1) \wedge \cdots \wedge F^{-i_n} TP(X_n) \to F^{-i_1-\cdots-i_n} TP(X_1 \wedge \cdots \wedge X_n) \]

Little 1-cubes: Moore construction (Moore loop space)

Use length parameter to make fully associative
\[ F^{-i} TP(X) \wedge R_+^{\geq 0} \wedge F^{-j} TP(Y) \wedge R_+^{\geq 0} \to F^{-i-j} TP(X \wedge Y) \wedge R_+^{\geq 0} \]
Digression

Filtered Monoidal Structure

A filtered approximation of the diagonal gives a map

\[ E_{i+j-1} \rightarrow (E_{i-1} \times ET) \cup (ET \times E_{j-1}) \subset ET \times ET \]

Hence a map \((ET, E_{i+j-1}) \rightarrow (ET, E_{i-1}) \times (ET, E_{j-1})\)

Parametrized

\[ C_1(n)_+ \land F^{-i_1} TP(X_1) \land \cdots \land F^{-i_n} TP(X_n) \rightarrow F^{-i_1-\cdots-i_n} TP(X_1 \land \cdots \land X_n) \]

Little 1-cubes: Moore construction (Moore loop space)

Use length parameter to make fully associative

\[ F^{-i} TP(X) \land R_+^{>0} \land F^{-j} TP(Y) \land R_+^{>0} \rightarrow F^{-i-j} TP(X \land Y) \land R_+^{>0} \]

Filtered monoidal

\[ TP^M(X) \land TP^M(Y) \rightarrow TP^M(X \land Y) \]
Filtered Monoidal Structure

A filtered approximation of the diagonal gives a map

\[ E_{T_{i+j-1}} \to (E_{T_{i-1}} \times E_T) \cup (E_T \times E_{T_{j-1}}) \subset E_T \times E_T \]

Hence a map \((E_T, E_{T_{i+j-1}}) \to (E_T, E_{T_{i-1}}) \times (E_T, E_{T_{j-1}})\)

Parametrized

\[ C_1(n)_+ \land F^{-i_1}TP(X_1) \land \cdots \land F^{-i_n}TP(X_n) \to F^{-i_1-\cdots-i_n}TP(X_1 \land \cdots \land X_n) \]

Little 1-cubes: Moore construction (Moore loop space)

Use length parameter to make fully associative

\[ F^{-i}TP(X) \land R_+^{>0} \land F^{-j}TP(Y) \land R_+^{>0} \to F^{-i-j}TP(X \land Y) \land R_+^{>0} \]

Filtered monoidal

\[ TP^M(X) \land TP^M(Y) \to TP^M(X \land Y) \]

\[ TP^M(X) \land_{TP\bar{M}(R)} TP^M(Y) \to TP^M(X \land_R Y) \]
Küneth Theorem

Filtered map \( TP(X) \wedge_{TP(R)} TP(Y) \to TP(X \wedge_R Y) \)

\[ \implies \text{map of spectral sequences} \]

Righthand spectral sequence is Tate spectral sequence for

\[ THH(X \wedge_R Y) \cong THH(X) \wedge_{THH(R)} THH(Y) \]

\( E^2 \) periodic with \( \pi_*(THH(X) \wedge_{THH(R)} THH(Y)) \) in each even column

Lefthand spectral sequence has (renumbered) \( E^2 \)-term

\[ \pi_* \text{Gr}(TP(X) \wedge_{TP(R)} TP(Y)) \cong \pi_*(\text{Gr} TP(X) \wedge_{\text{Gr} TP(R)} \text{Gr} TP(Y)) \]

\( E^2 \)-term is \( \pi_* \text{Gr} TP(R) \)-module \( \implies (2, 0) \)-periodic
Künneu Theorem

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$\pi_* Gr(TP(X) \wedge_{TP(R)} TP(Y)) \cong \pi_*(Gr TP(X) \wedge_{Gr TP(R)} Gr TP(Y))$

$E^2$-term is $\pi_*$ $Gr TP(R)$-module $\implies$ $(2, 0)$-periodic
Künneth Theorem

Filtered map \( TP(X) \wedge_{TP(R)} TP(Y) \rightarrow TP(X \wedge_R Y) \)

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Künneth Theorem

Filtered map $TP(X) \wedge_{TP(R)} TP(Y) \to TP(X \wedge_R Y)$

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Righthand spectral sequence is Tate spectral sequence for

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Künnett Theorem

Filtered map $TP(X) \wedge_{TP(R)} TP(Y) \rightarrow TP(X \wedge_R Y)$

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Righthand spectral sequence is Tate spectral sequence for

$$THH(X \wedge_R Y) \simeq THH(X) \wedge_{THH(R)} THH(Y)$$

$E^2$ periodic with $\pi_*(THH(X) \wedge_{THH(R)} THH(Y))$ in each even column

Lefthand spectral sequence has (renumbered) $E^2$-term

$$\pi_* \text{Gr}(TP(X) \wedge_{TP(R)} TP(Y)) \simeq \pi_*(\text{Gr} TP(X) \wedge_{\text{Gr} TP(R)} \text{Gr} TP(Y))$$

$E^2$-term is $\pi_* \text{Gr} TP(R)$-module $\implies$ $(2, 0)$-periodic
Künneth Theorem

Filtered map $TP(X) \land_{TP(R)} TP(Y) \to TP(X \land_R Y)$

$\implies$ map of spectral sequences preserving periodicity op. on $E^2$

Righthand spectral sequence is Tate spectral sequence for

$THH(X \land_R Y) \cong THH(X) \land_{THH(R)} THH(Y)$

$E^2$ periodic with $\pi_*(THH(X) \land_{THH(R)} THH(Y))$ in each even column

Lefthand spectral sequence has (renumbered) $E^2$-term

$\pi_* Gr(TP(X) \land_{TP(R)} TP(Y)) \cong \pi_* (Gr TP(X) \land_{Gr TP(R)} Gr TP(Y))$

$E^2$-term is $\pi_* Gr TP(R)$-module $\implies (2, 0)$-periodic
Künneth Theorem

Filtered map \( TP(X) \wedge_{TP(R)} TP(Y) \rightarrow TP(X \wedge_{R} Y) \)

\[ \implies \text{map of spectral sequences preserving periodicity op. on } E^2 \]

Righthand spectral sequence is Tate spectral sequence for

\[ THH(X \wedge_{R} Y) \cong THH(X) \wedge_{THH(R)} THH(Y) \]

\( E^2 \) periodic with \( \pi_*(THH(X) \wedge_{THH(R)} THH(Y)) \) in each even column

Lefthand spectral sequence has (renumbered) \( E^2 \)-term

\[ \pi_*(Gr(TP(X) \wedge_{TP(R)} TP(Y))) \cong \pi_*(Gr TP(X) \wedge_{Gr TP(R)} Gr TP(Y)) \]

\( E^2 \)-term is \( \pi_*\text{Gr} TP(R) \)-module \( \implies (2, 0) \)-periodic

Proposition

Map of spectral sequences is an isomorphism on \( E^2 \)
Outline of Proof of Künneth Theorem

Filtered map $TP(X) \wedge_{TP(R)} TP(Y) \to TP(X \wedge_R Y)$ induces isomorphism of $E^2$-terms of spectral sequences

RHSS: Tate spectral sequence $\Longrightarrow$ conditionally convergent.

Theorem

If $R = Hk$ and $X$ and $Y$ are smooth and proper, then the LHSS is conditionally convergent.
Outline of Proof of Künneth Theorem

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**Theorem**

If \( R = Hk \) and \( X \) and \( Y \) are smooth and proper, then the LHSS is conditionally convergent.

**Theorem**

If \( R \) is an \( E_\infty \) ring spectrum and \( A \) is a smooth and proper \( R \)-algebra, then \( THH(A) \) is a compact \( THH(R) \)-module.
Outline of Proof of Künneth Theorem

Filtered map $TP(X) \wedge_{TP(R)} TP(Y) \rightarrow TP(X \wedge_R Y)$ induces isomorphism of $E^2$-terms of spectral sequences

RHSS: Tate spectral sequence $\Longrightarrow$ conditionally convergent.

**Theorem**

*If $R = \mathcal{H}k$ and $X$ and $Y$ are smooth and proper, then the LHSS is conditionally convergent. In fact, strongly convergent.*

**Theorem**

*If $R$ is an $E_\infty$ ring spectrum and $A$ is a smooth and proper $R$-algebra, then $THH(A)$ is a compact $THH(R)$-module.*
Outline of Proof of Künneth Theorem

Filtered map $TP(X) \wedge_{TP(R)} TP(Y) \to TP(X \wedge_R Y)$ induces isomorphism of $E^2$-terms of spectral sequences

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$TP_*(k) = \mathbb{W}k[v, v^{-1}]$ is periodic
Outline of Proof of Künneth Theorem

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**Theorem**

*If $R$ is an $E_\infty$ ring spectrum and $A$ is a smooth and proper $R$-algebra, then $\text{THH}(A)$ is a compact $\text{THH}(R)$-module.*

$TP_* (k) = \mathbb{W}k[v, v^{-1}]$ is periodic

$\pi_* (F^0 TP(k)) = \mathbb{W}k[v, pv^{-1}]$ has finite global dimension