A Strong Küneth Theorem for Topological Periodic Cyclic Homology

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Workshop on $K$-Theory and Related Fields
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Topological periodic cyclic homology ($TP$) is the analogue of periodic cyclic homology ($HP$) using $THH$ in place of $HH$. If $k$ is a finite field, then smooth and proper d.g. categories over $k$ satisfy a strong Künneth theorem:

$$TP(X) \wedge_{TP(k)}^{L} TP(Y) \to TP(X \otimes_k Y)$$

is an isomorphism in the derived category of $TP(k)$-modules.

- Joint work with Andrew Blumberg

Outline

1. Non-commutative derived algebraic geometry
2. Introduction to $TP$
3. The Künneth theorem
Overview

Topological periodic cyclic homology (\(TP\)) is the analogue of periodic cyclic homology (\(HP\)) using \(THH\) in place of \(HH\). If \(k\) is a finite field, then smooth and proper d.g. categories over \(k\) satisfy a strong Künneth theorem:

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- Preprint arXiv:1706.06846

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1. Non-commutative derived algebraic geometry
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Non-commutative derived algebraic geometry

**Basic objects:** [small] differential graded (or spectral) categories

**Equivalences:** Morita equivalences

**Example:** \(A \sim \rightarrow \text{Mod}_A \sim \rightarrow \text{Mod}_{\text{Mod}_A}\)

**Example:** Tilting \(A M_B, B N_A\) with \(M \otimes^L_N N \simeq A, N \otimes^L_B M \simeq B\)

\[A \sim \rightarrow \text{Mod}_A \sim \rightarrow \text{Mod}_B \leftarrow \sim B\]

**Goal:** Construct/study invariants

**Example:** \(K\) theory of subcategory of compact objects

**Example:** algebraic variety \(X \leftrightarrow\) d.g. cat of perfect complexes \(\mathcal{D}^{\text{dg perf}}(X)\)

\[K(X) = K(\mathcal{D}^{\text{dg perf}}(X))\]
Non-commutative derived algebraic geometry

**Basic objects:** [small] differential graded (or spectral) categories

**EQUIVALENCES:** Morita equivalences

Example: $A \sim \text{Mod}_A \sim \text{Mod}_\text{Mod}_A$

Example: Tilting $\text{Mod}_A$, $\text{Mod}_B$ with $M \otimes^L_N N \sim A$, $N \otimes^L_B M \sim B$

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**Basic objects:** [small] differential graded (or spectral) categories

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**Example:** $\tilde{A} \sim A \rightarrow \text{Mod}_A \sim \text{Mod}_{\text{Mod}_A}$

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\[
\begin{array}{c}
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N \otimes^L_B \text{Mod}_B
\end{array}
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Non-commutative derived algebraic geometry

How is this algebraic geometry?

Use \( \mathcal{D} = \mathcal{D}^{\text{dg}}_{\text{perf}}(X) \) for \( X \)

For reasonable \( X \), [some] properties of \( X \) equivalent to properties of \( \mathcal{D} \)

Example: \( X \) is proper over \( \text{spec} \ k \) if and only if \( \mathcal{D}(a, b) \) is a compact d.g. \( k \)-module for all \( a, b \in \mathcal{D} \). \( \left( \sum \dim H^n(\mathcal{D}(a, b)) < \infty \right) \)

Example: \( X \) is smooth over \( \text{spec} \ k \) if and only if \( \mathcal{D} \) is a compact \( \mathcal{D}^{\text{op}} \otimes_k \mathcal{D} \)-module. \( (\text{RHom}_{\mathcal{D}^{\text{op}} \otimes_k \mathcal{D}}(\mathcal{D}, -) \) commutes with \( \bigoplus \) \)

Definition

Let \( A \) be a d.g. (or spectral) \( R \)-algebra. Then \( A \) is:

- **proper** if it is compact as an \( R \)-module
- **smooth** if it is compact as an \( A^{\text{op}} \otimes_R^L A \)-module (or \( A^{\text{op}} \wedge_R^L A \)-module)
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Use \( D = \mathcal{D}_{\text{perf}}(X) \) for \( X \)

For reasonable \( X \), [some] properties of \( X \) equivalent to properties of \( D \)

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Take away

For theorems in non-commutative derived algebraic geometry:

- Statements are in terms of d.g. (or spectral) categories
- Results are about algebraic varieties (and generalizations)
- Proofs often just need the case of d.g. algebras (or ring spectra)

\[ \mathcal{X} = \text{P} \bigoplus_{r=0}^{m} \text{O}(-r) \sim \rightarrow \text{D}_{\text{dg perf}}(\text{P}^n) \]

\[ H^{-n}(\text{End}(\mathcal{X})) \text{ is a matrix of Ext}^n \text{ groups, Ext}^n(\text{O}(\mathcal{X}), \text{O}(\mathcal{Y})) \]
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Example:

\[ X = \mathbb{P}^m \]

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\text{End} \left( \bigoplus_{r=0}^{m} \mathcal{O}(-r) \right) \cong \mathcal{D}^{dg}_{\text{perf}}(\mathbb{P}^n)
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Hochschild Homology

Cyclic bar construction

\[ N_q^{cy} A = \underbrace{A \otimes \cdots \otimes A \otimes A}_{q \text{ factors}} \]

Chain complex

Cyclic structure \[\Rightarrow\] Connes’ \( B \) operator

\[ B : N^{cy} A \to N^{cy} A[-1] \]
Hochschild Homology

Cyclic bar construction

\[ N^\text{cy}_q A = \underbrace{A \otimes \cdots \otimes A}_{q \text{ factors}} \]

Chain complex

Cyclic structure $\mapsto$ Connes’ $B$ operator

\[ B: N^\text{cy} A \to N^\text{cy} A[-1] \]
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\[ N^c_y A = A \otimes \cdots \otimes A \otimes A \]

\( q \) factors

\[ A \otimes \cdots \otimes A \]

\[ A \]

Chain complex

Cyclic structure \( \Rightarrow \) Connes’ \( B \) operator

\[ B : N^c_y A \rightarrow N^c_y A[−1] \]

Morita Invariance

Dennis-Waldhausen Argument

Tilting situation \( A M_B, B N_A \)

\[ A \otimes \cdots \otimes A \]

\[ N \]

\[ M \]

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Dennis-Waldhausen Argument
**Hochschild Homology**

Cyclic bar construction

\[ N_q^{cy} A = \underbrace{A \otimes \cdots \otimes A}^{q \text{ factors}} \otimes A \]

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\( q \) factors

\[ A \otimes \cdots \otimes A \]

\[ \otimes \quad \otimes \]

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Chain complex

Cyclic structure \( \longrightarrow \) Connes’ \( B \) operator

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Hochschild Homology and Cyclic Homology

Cyclic bar construction

\[ N^\text{cy}_q A = \underbrace{A \otimes \cdots \otimes A} \quad q \text{ factors} \]

Chain complex

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Hochschild Homology and Cyclic Homology

Cyclic bar construction

\[ N_q^{cy} A = A \otimes \cdots \otimes A \otimes A \]

\( q \) factors

Construct Double Complex:

\[ \cdots \]

\[ \downarrow \]

\[ \cdots \]

\[ A \otimes \cdots \otimes A \]

\[ \otimes \quad \otimes \]

\[ A \]

Chain complex

Cyclic structure \( \Longrightarrow \) Connes’ \( B \) operator

\[ B : N_q^{cy} A \rightarrow N_q^{cy} A[-1] \]
Hochschild Homology and Cyclic Homology

Cyclic bar construction

\[ N_q^{cy} A = A \otimes \cdots \otimes A \otimes A \]

\( q \) factors

\[ A \otimes \cdots \otimes A \]

\[ \otimes \quad \otimes \]

\[ A \]

Chain complex

Construct Double Complex:

\[ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \]

\[ \bullet \quad \bullet \quad \bullet \quad \bullet \quad \cdots \]

\[ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \]

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\[ \downarrow \]

\[ HC \]

Cyclic structure \[\implies\] Connes’ \( B \) operator

\[ B : N^{cy} A \to N^{cy} A[-1] \]
Hochschild Homology and Cyclic Homology

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\[ N_{q}^{cy} A = A \otimes \cdots \otimes A \otimes A \]

\[ q \text{ factors} \]

Chain complex

Cyclic structure \(\implies\) Connes’ \(B\) operator

\[ B : N^{cy} A \to N^{cy} A[-1] \]
Hochschild Homology and Cyclic Homology

Cyclic bar construction

\[ N^c_y A = \underbrace{A \otimes \cdots \otimes A \otimes A}_{q \text{ factors}} \]

\[ A \otimes \cdots \otimes A \]

\[ \otimes \quad \otimes \]

\[ A \]

Chain complex

Cyclic structure \( \Longrightarrow \) Connes’ \( B \) operator

\[ B : N^c_y A \rightarrow N^c_y A[-1] \]
Cyclic bar construction (Bökstedt)

\[ N_q^{cy} A = \underbrace{A \wedge \cdots \wedge A}^{q \text{ factors}} \wedge A \]

Spectrum

Cyclic structure \(\implies\) circle group action
Topological Hochschild Homology

Cyclic bar construction (Bökstedt)

\[ \mathcal{N}_q^{cy} A = \underbrace{A \wedge \cdots \wedge A}_q \wedge A \]

A \wedge \cdots \wedge A

\wedge \wedge \wedge

A

Spectrum

Cyclic structure \(\Rightarrow\) circle group action
Topological Hochschild Homology

Cyclic bar construction (Bökstedt)

\[ N^c_y A = A \wedge \cdots \wedge A \wedge A \]

\[ q \text{ factors} \]

A \wedge \cdots \wedge A

\wedge \quad \wedge \quad \wedge

A

Spectrum

Cyclic structure \rightarrow circle group action
Topological Hochschild Homology

Cyclic bar construction (Bökstedt)

\[ \mathcal{N}^{cy}_q A = A \wedge \cdots \wedge A \wedge A \]

\( q \) factors

\[ A \wedge \cdots \wedge A \]

\[ \wedge \quad \wedge \]

\[ A \]

Construction

\[ \cdots \]

\[ \downarrow \]

\[ \cdots \]

\[ \downarrow \]

\[ \cdots \]

\[ \downarrow \]

\[ \cdots \]

Spectrum

Cyclic structure \( \rightarrow \) circle group action

HH corresponds to THH
Topological Hochschild Homology

Cyclic bar construction (Bökstedt)

\[ N_q^{cy} A = \underbrace{A \wedge \cdots \wedge A \wedge A}_q \text{ factors} \]

\[ A \wedge \cdots \wedge A \wedge \cdots \wedge A \]

Spectrum

Cyclic structure \( \rightarrow \) circle group action

Construction

HH corresponds to \( THH \)

HC corresponds to \( THH^h\mathbb{T} \)
Topological Hochschild Homology

Cyclic bar construction (Bökstedt)

\[ N^c_y A = \underbrace{A \wedge \cdots \wedge A}_q \text{ factors} \wedge A \wedge \cdots \wedge A \]

Spectrum

Cyclic structure \(\longrightarrow\) circle group action

Construction

HH corresponds to \( THH \)
HN corresponds to \( THH^{hT} \)
Topological Hochschild Homology

Cyclic bar construction (Bökstedt)

\[ N^c_y A = \underbrace{A \wedge \cdots \wedge A \wedge A}_q \text{ factors} \]

\[ A \wedge \cdots \wedge A \]

\[ \wedge \quad \wedge \quad \wedge \]

\[ A \]

Spectrum

Cyclic structure \(\rightarrow\) circle group action

Construction

\[ HH \text{ corresponds to } THH \]

\[ HP \text{ corresponds to } THH^{tT} \]
For a ring spectrum $A$, define the Topological Periodic Cyclic Homology of $A$ by $TP(A) = THH(A)^{t\mathbb{T}}$. 

Highlights

- Major player in trace method
- $K$-theory calculations
- Characteristic $p$ replacement for $HP(?)$ (2014–)
- Hasse-Weil zeta function: Connes-Consani $\Rightarrow$ Hesselholt (2011–)
- Non-commutative motives: Kontsevich, Marcolli-Tabuada
- Non-commutative homological motives $\Rightarrow$ ???
- Realization functor / Weil cohomology theory

$\text{HP}^\ast(X) \otimes_k [t, t^{-1}] \text{HP}^\ast(Y) \to \text{HP}^\ast(X \otimes_k Y)$
Topological Periodic Cyclic Homology

Definition
For a ring spectrum \( A \), define the Topological Periodic Cyclic Homology of \( A \) by \( TP(A) = THH(A)^{t\mathbb{T}} \).

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    non-commutative homological motives $\rightsquigarrow$ ????
Introduction to \( TP \)

Topological Periodic Cyclic Homology

**Definition**

For a ring spectrum \( A \), define the Topological Periodic Cyclic Homology of \( A \) by \( TP(A) = THH(A)^{t\mathbb{T}} \).

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- **Characteristic \( p \)** replacement for \( HP(\?) \)
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    non-commutative homological motives \( \rightsquigarrow \) ????
Topological Periodic Cyclic Homology

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2017 Tabuada
**Topological Periodic Cyclic Homology**

**Definition**

For a ring spectrum $A$, define the Topological Periodic Cyclic Homology of $A$ by $TP(A) = THH(A)^{t\mathbb{T}}$.

**Highlights**

- Major player in trace method $K$-theory calculations
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  - (2014–) Hasse-Weil zeta function: Connes-Consani $\rightsquigarrow$ Hesselholt
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Realization functor / Weil cohomology theory

\[ HP_*(X) \otimes_{k[t,t^{-1}]} HP_*(Y) \rightarrow HP_*(X \otimes_k Y) \]
Topological Periodic Cyclic Homology

Definition

For a ring spectrum \( A \), define the Topological Periodic Cyclic Homology of \( A \) by \( TP(A) = THH(A)^{t\mathbb{T}} \).

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Realization functor / Weil cohomology theory

\[
HP_* (X) \otimes_{k[t,t^{-1}]} HP_* (Y) \to HP_* (X \otimes_k Y)
\]
Künneth Theorem

**Theorem**

*Lax symmetric monoidal functor*

\[
\text{TP}(X) \wedge_{\text{TP}(R)}^L \text{TP}(Y) \to \text{TP}(X \wedge_R Y)
\]

**Corollary**

There are short exact sequences of graded $\mathbb{W}_k$-modules

\[
0 \to (\text{TP}_*(X) \otimes_{\text{TP}_*(k)} \text{TP}_*(Y))_n \to \text{TP}_n(X \otimes_k Y) \to \text{Tor}_{1,n-1}^{\text{TP}_*(k)}(\text{TP}_*(X), \text{TP}_*(Y)) \to 0
\]

for all $n$, which split but not naturally.

**Corollary**

\[
\text{TP}_*(X)[1/p] \otimes_{\text{TP}_*(k)[1/p]} \text{TP}_*(Y)[1/p] \to \text{TP}_*(X \otimes_k Y)[1/p] \text{ is an isomorphism.}
\]
Künneth Theorem

**Theorem**

Let $k$ be finite field. The lax symmetric monoidal functor

$$TP(X) \wedge_{TP(k)}^L TP(Y) \to TP(X \otimes_k Y)$$

is an isomorphism when $X$ and $Y$ are smooth and proper over $k$.

**Corollary**

There are short exact sequences of graded $\mathbb{W}_k$-modules

$$0 \to (TP_*(X) \otimes_{TP_*(k)} TP_*(Y))_n \to TP_n(X \otimes_k Y) \to \text{Tor}_{1,n-1}^{TP_*(k)}(TP_*(X), TP_*(Y)) \to 0$$

for all $n$, which split but not naturally.

**Corollary**

$$TP_*(X)[1/p] \otimes_{TP_*(k)[1/p]} TP_*(Y)[1/p] \to TP_*(X \otimes_k Y)[1/p]$$

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**Corollary**

$$TP_*(X)[1/p] \otimes_{TP_*(k)[1/p]} TP_*(Y)[1/p] \to TP_*(X \otimes_k Y)[1/p]$$

is an isomorphism.
Künnett Theorem

**Theorem**

Let $k$ be finite field. The lax symmetric monoidal functor

$$TP(X) \wedge^L_{TP(k)} TP(Y) \to TP(X \otimes_k Y)$$

is an isomorphism when $X$ and $Y$ are smooth and proper over $k$.

**Corollary**

There are short exact sequences of graded $W_k$-modules

$$0 \to (TP_* (X) \otimes_{TP_* (k)} TP_* (Y))_n \to TP_n (X \otimes_k Y) \to \text{Tor}_{1,n-1}^{TP_* (k)} (TP_* (X), TP_* (Y)) \to 0$$

for all $n$, which split but not naturally.

**Corollary**

$$TP_* (X)[1/p] \otimes_{TP_* (k)[1/p]} TP_* (Y)[1/p] \to TP_* (X \otimes_k Y)[1/p]$$ is an isomorphism.
Künneth Theorem

**Theorem**

Let $k$ be finite field. The lax symmetric monoidal functor

$$TP(X) \wedge_{TP(k)}^L TP(Y) \to TP(X \otimes_k Y)$$

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**Corollary**

There are short exact sequences of graded $\mathbb{W}k$-modules

$$0 \to (TP_*(X) \otimes_{TP_*(k)} TP_*(Y))_n \to TP_n(X \otimes_k Y) \to \text{Tor}_{1,n-1}^{TP_*(k)}(TP_*(X), TP_*(Y)) \to 0$$

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**Corollary**

$$TP_*(X)[1/p] \otimes_{TP_*(k)[1/p]} TP_*(Y)[1/p] \to TP_*(X \otimes_k Y)[1/p]$$

is an isomorphism.
**Review of Tate Construction**

\[ \mathcal{E}_T \xrightarrow{E_{T+}} S^0 \xrightarrow{\sim} \mathcal{E}_T \]

Smash with \( \mathcal{Z}^{\mathcal{E}_T} \) and take fixed points

\[
(Z^{\mathcal{E}_T} \wedge E_{T+})^T \rightarrow (Z^{\mathcal{E}_T})^T \rightarrow (Z^{\mathcal{E}_T} \wedge \mathcal{E}_T)^T
\]

\[
(X^{\mathcal{E}_T} \wedge E_{T+})^T \simeq \Sigma (X^{\mathcal{E}_T})_{hT} \simeq \Sigma X_{hT} \quad \text{(Adams Isomorphism)}
\]

**Definition**

For \( Z \) a \( T \)-equivariant spectrum \( Z^{tT} = (Z^{\mathcal{E}_T} \wedge \mathcal{E}_T)^T \).

(Composite of derived functors.)

\[
\Sigma Z_{hT} \rightarrow Z^{hT} \rightarrow Z^{tT} \rightarrow \Sigma^2 Z_{hT}
\]

**Formula**

\[ TP(X) = THH(X)^{tT} \]
Review of Tate Construction

\[ E_T \quad E_{T^+} \to S^0 \to \widetilde{E}_T \]

Smash with \( Z^{E_T} \) and take fixed points

\[(Z^{E_T} \wedge E_{T^+})^T \to (Z^{E_T})^T \to (Z^{E_T} \wedge \widetilde{E}_T)^T \]

\[(X^{E_T} \wedge E_{T^+})^T \simeq \Sigma (X^{E_T})_{h_T} \simeq \Sigma X_{h_T} \quad \text{(Adams Isomorphism)} \]

Definition

For \( Z \) a \( T \)-equivariant spectrum \( Z^{tT} = (Z^{E_T} \wedge \widetilde{E}_T)^T \).

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\[ \Sigma Z_{h_T} \to Z^{h_T} \to Z^{tT} \to \Sigma^2 Z_{h_T} \]

\[ TP(X) = THH(X)^{tT} \]
Review of Tate Construction

\[ E_T \xrightarrow{E_{T+}} S^0 \rightarrow \tilde{E}_T \]

Smash with \( Z^{E_T} \) and take fixed points

\[ (Z^{E_T} \wedge E_{T+})^T \rightarrow (Z^{E_T})^T \rightarrow (Z^{E_T} \wedge \tilde{E}_T)^T \]

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\[ TP(X) = THH(X)^{t_T} \]
Review of Tate Construction

\[ E_T \rightarrow E_{T+} \rightarrow S^0 \rightarrow \widetilde{E}_T \]

Smash with \( Z^{E_T} \) and take fixed points:

\[
\begin{align*}
(Z^{E_T} \wedge E_{T+})^T & \rightarrow (Z^{E_T})^T \\
& \rightarrow (Z^{E_T} \wedge \widetilde{E}_T)^T
\end{align*}
\]

\[
(X^{E_T} \wedge E_{T+})^T \simeq \Sigma(X^{E_T})_{hT} \simeq \Sigma X_{hT} \quad \text{(Adams Isomorphism)}
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\[ E_T \xrightarrow{E_T^+} S^0 \xrightarrow{\sim} \tilde{E}_T \]

Smash with \( Z^{E_T} \) and take fixed points

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(Z^{E_T} \wedge E_{T^+})^T \to (Z^{E_T})^T \to (Z^{E_T} \wedge \tilde{E}_T)^T
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\]

Definition

For \( Z \) a \( T \)-equivariant spectrum \( Z^{t_T} = (Z^{E_T} \wedge \tilde{E}_T)^T \).

(Composite of derived functors.)

\[
\Sigma Z_{h_T} \to Z^{h_T} \to Z^{t_T} \to \Sigma^2 Z_{h_T}
\]

\[ TP(X) = THH(X)^{t_T} \]
Review of Tate Construction

\[ ET_\mathbb{T} \quad ET_\mathbb{T}^+ \to S^0 \to \widetilde{ET}_\mathbb{T} \]

Smash with \( Z^{ET}_\mathbb{T} \) and take fixed points

\[
(Z^{ET}_\mathbb{T} \wedge ET_\mathbb{T}^+)^\mathbb{T} \to (Z^{ET}_\mathbb{T})^\mathbb{T} \to (Z^{ET}_\mathbb{T} \wedge \widetilde{ET}_\mathbb{T})^\mathbb{T}
\]

\[
(X^{ET}_\mathbb{T} \wedge ET_\mathbb{T}^+)^\mathbb{T} \cong \Sigma(X^{ET}_\mathbb{T})h_\mathbb{T} \cong \Sigma X_{h_\mathbb{T}} \quad \text{(Adams Isomorphism)}
\]

Definition

For \( Z \) a \( \mathbb{T} \)-equivariant spectrum \( Z^{t_\mathbb{T}} = (Z^{ET}_\mathbb{T} \wedge \widetilde{ET}_\mathbb{T})^\mathbb{T} \).

(Composite of derived functors.)

\[
\Sigma Z_{h_\mathbb{T}} \to Z^{h_\mathbb{T}} \to Z^{t_\mathbb{T}} \to \Sigma^2 Z_{h_\mathbb{T}}
\]

\[ TP(X) = \text{THH}(X)^{t_\mathbb{T}} \]
Künneth Theorem

Review of Tate Construction

$E_T \quad E_{T+} \to S^0 \to \widetilde{E_T}$

Smash with $Z^{E_T}$ and take fixed points

$$(Z^{E_T} \wedge E_{T+})^T \to (Z^{E_T})^T \to (Z^{E_T} \wedge \widetilde{E_T})^T$$

$$(X^{E_T} \wedge E_{T+})^T \simeq \Sigma(X^{E_T})_{hT} \simeq \Sigma X_{hT} \quad \text{(Adams Isomorphism)}$$

Definition

For $Z$ a $\mathbb{T}$-equivariant spectrum $Z^{tT} = (Z^{E_T} \wedge \widetilde{E_T})^T$. (Composite of derived functors.)

$$\Sigma Z_{hT} \to Z^{hT} \to Z^{tT} \to \Sigma^2 Z_{hT}$$

$$TP(X) = THH(X)^{tT}$$
Künneth Theorem

Review of Tate Construction

\[ E_T \rightarrow E_{T+} \rightarrow S^0 \rightarrow \widetilde{E}_T \]

Smash with \( Z^{E_T} \) and take fixed points

\[ (Z^{E_T} \wedge E_{T+})^T \rightarrow (Z^{E_T})^T \rightarrow (Z^{E_T} \wedge \widetilde{E}_T)^T \]

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Definition

For \( Z \) a \( T \)-equivariant spectrum \( Z^{t_T} = (Z^{E_T} \wedge \widetilde{E}_T)^T \).

(Composite of derived functors.)

\[ \Sigma Z_{h_T} \rightarrow Z^{h_T} \rightarrow Z^{t_T} \rightarrow \Sigma^2 Z_{h_T} \]

\[ TP(X) = THH(X)^{t_T} \]
Künneth Theorem

Review of Tate Construction

\[ E_{\mathbb{T}} \rightarrow E_{\mathbb{T}+} \rightarrow S^0 \rightarrow \widetilde{E_{\mathbb{T}}} \]

Smash with \( Z^{E_{\mathbb{T}}} \) and take fixed points

\[
(Z^{E_{\mathbb{T}}} \wedge E_{\mathbb{T}+})^\mathbb{T} \rightarrow (Z^{E_{\mathbb{T}}})^\mathbb{T} \rightarrow (Z^{E_{\mathbb{T}}} \wedge \widetilde{E_{\mathbb{T}}})^\mathbb{T}
\]

\[
(X^{E_{\mathbb{T}}} \wedge E_{\mathbb{T}+})^\mathbb{T} \cong \Sigma(X^{E_{\mathbb{T}}})_{h\mathbb{T}} \cong \Sigma X_{h\mathbb{T}} \quad \text{(Adams Isomorphism)}
\]

**Definition**

For \( Z \) a \( \mathbb{T} \)-equivariant spectrum \( Z^{t\mathbb{T}} = (Z^{E_{\mathbb{T}}} \wedge \widetilde{E_{\mathbb{T}}})^\mathbb{T} \).

(Composite of derived functors.)

\[
\Sigma(Z_{h\mathbb{T}}) \rightarrow Z^{h\mathbb{T}} \rightarrow Z^{t\mathbb{T}} \rightarrow \Sigma^2 Z_{h\mathbb{T}}
\]

\[ TP(X) = THH(X)^{t\mathbb{T}} \]
The Multiplication

$$TP(X) \land TP(Y) \to TP(X \land Y)$$

$$TP(X) = (THH(X)^{ET} \land \tilde{ET})^T$$

- $\tilde{ET} \land \tilde{ET} \simeq \tilde{ET}$
- Use diagonal map $ET \to ET \times ET$
- $THH(X) \land THH(Y) \cong THH(X \land Y)$
The Multiplication

\[ TP(X) \wedge TP(Y) \rightarrow TP(X \wedge Y) \]

\[ TP(X) = (THH(X)^{ET} \wedge \widetilde{ET})^T \]

- \( \widetilde{ET} \wedge \widetilde{ET} \simeq \widetilde{ET} \)
- Use diagonal map \( ET \rightarrow ET \times ET \)
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- \( THH(X) \wedge THH(Y) \cong THH(X \wedge Y) \)
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- Use diagonal map \( ET \to ET \times ET \)
- \( THH(X) \wedge THH(Y) \cong THH(X \wedge Y) \)
The Multiplication

\[ TP(X) \wedge TP(Y) \rightarrow TP(X \wedge Y) \]

\[ TP(X) = (\text{THH}(X)^{E_T} \wedge \tilde{E}_T)^T \]

- \( \tilde{E}_T \wedge \tilde{E}_T \cong \tilde{E}_T \)
- Use diagonal map \( E_T \rightarrow E_T \times E_T \)
- \( \text{THH}(X) \wedge \text{THH}(Y) \cong \text{THH}(X \wedge Y) \)
The Multiplication

\[ TP(X) \land TP(Y) \rightarrow TP(X \land Y) \]

\[ TP(X) = (THH(X)^{ET} \land \widetilde{ET})^T \]

- \( \widetilde{ET} \land \widetilde{ET} \simeq \widetilde{ET} \)
- Use diagonal map \( ET \rightarrow ET \times ET \)
- \( THH(X) \land THH(Y) \cong THH(X \land Y) \)
The Multiplication

\[ TP(X) \wedge TP(Y) \rightarrow TP(X \wedge Y) \]

\[ TP(X) = (\text{THH}(X)^{ET} \wedge \widetilde{ET})^T \]

- \( \widetilde{ET} \wedge \widetilde{ET} \simeq \widetilde{ET} \)

- Use diagonal map \( ET \rightarrow ET \times ET \)

- \( \text{THH}(X) \wedge \text{THH}(Y) \cong \text{THH}(X \wedge Y) \)
The Multiplication

\[ TP(X) \wedge TP(Y) \rightarrow TP(X \wedge Y) \]

\[ TP(X) = (THH(X)^{ET} \wedge \widehat{ET})^T \]

- \( \widehat{ET} \wedge \widehat{ET} \cong \widehat{ET} \)
- Use diagonal map \( ET \rightarrow ET \times ET \)
- \( THH(X) \wedge THH(Y) \cong THH(X \wedge Y) \)
The Multiplication

\[ TP(X) \wedge TP(Y) \rightarrow TP(X \wedge Y) \]

\[ TP(X) = (THH(X)^{\mathbb{E}_T} \wedge \mathbb{E}_T)^{\mathbb{T}} \]

- \( \mathbb{E}_T \wedge \mathbb{E}_T \approx \mathbb{E}_T \)
- Use diagonal map \( \mathbb{E}_T \rightarrow \mathbb{E}_T \times \mathbb{E}_T \)
- \( THH(X) \wedge THH(Y) \cong THH(X \wedge Y) \)
The Multiplication

\[ TP(X) \wedge TP(Y) \rightarrow TP(X \wedge Y) \]

\[ TP(X) = \left( THH(X)^{E_T} \wedge \sim E_T \right)^T \]

- \[ \sim E_T \wedge \sim E_T \simeq \sim E_T \]
- Use diagonal map \( E_T \rightarrow E_T \times E_T \)
- \( THH(X) \wedge THH(Y) \cong THH(X \wedge Y) \)
The Multiplication

\[ TP(X) \land TP(Y) \to TP(X \land Y) \]

\[ TP(X) = (THH(X)^E_T \land \tilde{E}_T)^T \]

- \[ \tilde{E}_T \land \tilde{E}_T \simeq \tilde{E}_T \]
- Use diagonal map \( E_T \to E_T \times E_T \)
- \( THH(X) \land THH(Y) \cong THH(X \land Y) \)
The Multiplication

\[
TP(X) \wedge TP(Y) \rightarrow TP(X \wedge Y)
\]

\[
TP(X) = (THH(X)^{E_T} \wedge \tilde{E}_T)^T
\]

- \(\tilde{E}_T \wedge \tilde{E}_T \cong \tilde{E}_T\)
- Use diagonal map \(E_T \rightarrow E_T \times E_T\)
- \(THH(X) \wedge THH(Y) \cong THH(X \wedge Y)\)

\[
TP(X) \wedge_{TP(R)} TP(Y) \rightarrow TP(X \wedge_R Y)
\]
The Multiplication

\[ TP(X) \wedge TP(Y) \to TP(X \wedge Y) \]

\[ TP(X) = \left( THH(X)^{E_T} \wedge \tilde{E_T} \right)^T \]

- \( \tilde{E_T} \wedge \tilde{E_T} \cong \tilde{E_T} \)
- Use diagonal map \( E_T \to E_T \times E_T \)
- \( THH(X) \wedge THH(Y) \cong THH(X \wedge Y) \)
The Multiplication

\[ TP(X) \wedge TP(Y) \rightarrow TP(X \wedge Y) \]

\[ TP(X) = (THH(X)^{ET} \wedge \tilde{ET})^T \]

- \( \tilde{ET} \wedge \tilde{ET} \simeq \tilde{ET} \)
- Use diagonal map \( ET \rightarrow ET \times ET \quad \leftarrow \text{This is coherent} \)
- \[ THH(X) \wedge THH(Y) \cong THH(X \wedge Y) \]

\[ TP(X) \wedge_{TP(R)} TP(Y) \rightarrow TP(X \wedge_R Y) \]

\[ TP(X) \wedge TP(R) \wedge TP(Y) \rightarrow TP(X \wedge R \wedge Y) \]
The Multiplication

\[ TP(X) \wedge TP(Y) \to TP(X \wedge Y) \]

\[ TP(X) = (THH(X)^{E_T} \wedge \tilde{E_T})^T \]

- \( \tilde{E_T} \wedge \tilde{E_T} \cong \tilde{E_T} \quad \text{← This can be made coherent} \)
- Use diagonal map \( E_T \to E_T \times E_T \quad \text{← This is coherent} \)
- \( THH(X) \wedge THH(Y) \cong THH(X \wedge Y) \)

\[ TP(X) \wedge_{TP(R)} TP(Y) \to TP(X \wedge_R Y) \]

\[ TP(X) \wedge TP(R) \wedge TP(Y) \to TP(X \wedge R \wedge Y) \]
The Filtration

Filtration on $TP(X)$ with associated graded

$$F^i/F^{i-1} \simeq \Sigma^{2i} \text{THH}(X)$$

$TP(X) = (\text{THH}(X)^{E_T} \land \widetilde{E_T})^T$

Simplicial filtration on $E_T$

$T_+, \Sigma^2 T_+, \Sigma^4 T_+, \ldots$
The Filtration

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$$T_+, \Sigma^2 T_+, \Sigma^4 T_+, \ldots$$

$$T_+ \wedge (T/\{1\}) \wedge \Delta[1]/\partial \Delta[1]$$
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$$T_+ \wedge (T \times T / (T \vee T)) \wedge \Delta[2] / \partial \Delta[2]$$

$$T \times T \times T$$

$$T \times T$$

$$T$$
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$T_+, \Sigma^2 T_+, \Sigma^4 T_+, \ldots$ / $S^0, \Sigma T_+, \Sigma^3 T_+, \ldots$
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$$F^i TP(X) = \begin{cases} (\text{THH}(X)^{E_T, E_T_{-i-1}} \wedge S^0)^T & i \leq 0 \\ (\text{THH}(X)^{E_T} \wedge \tilde{E}_{T_i})^T & i > 0 \end{cases}$$

$$F^i / F^{i-1} = \begin{cases} (\text{THH}(X)^{\Sigma^{2i} T_+})^T & i \leq 0 \\ (\text{THH}(X)^{E_T} \wedge \Sigma^{2i-1} T_+)^T & i > 0 \end{cases}$$
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The Spectral Sequence

Filtration on $TP(X)$ with associated graded

$F^i / F^{i-1} \simeq \Sigma^{2i} \text{THH}(X)$

Spectral sequence

$E_{i,j}^1 = \pi_{i+j} \Sigma^{2i} \text{THH}(X) = \text{THH}_{j-i}(X)$

Renumber: Double filtration degree

$E_{2i,j}^2 = (E_{i,i+j}^r)^{\text{old}}, \quad d_{2r} = (d_r)^{\text{old}}$

Greenlees Tate Spectral Sequence

Conditionally convergent spectral sequence

$E_{2i,j}^2 = \text{THH}_j(X) \Longrightarrow TP_{2i+j}(X). \quad (E_{2i+1,j}^r = 0)$
The Spectral Sequence

Filtration on $TP(X)$ with associated graded

$$F^i / F^{i-1} \cong \Sigma^{2i} THH(X)$$

Spectral sequence

$$E^{1}_{i,j} = \pi_{i+j} \Sigma^{2i} THH(X) = THH_{j-i}(X)$$

Renumber: Double filtration degree

$$E^{2r}_{2i,j} = (E^r_{i,j})^{\text{old}}, \quad d_{2r} = (d^r)^{\text{old}}$$

Greenlees Tate Spectral Sequence

Conditionally convergent spectral sequence

$$E^{2}_{2i,j} = THH_j(X) \implies TP_{2i+j}(X). \quad (E^r_{2i+1,j} = 0)$$
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Filtration on $TP(X)$ with associated graded

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$$E_{2i,j}^2 = THH_j(X) \implies TP_{2i+j}(X). \quad (E_{2i+1,j}^r = 0)$$
Multiplicative spectral sequence:

\[ TP(X) \wedge TP(Y) \rightarrow TP(X \wedge Y) \text{ a filtered map} \]
Combining the Multiplication and Filtration

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In homotopy category, easy obstruction theory cellular approximation to diagonal & multiplication.

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Combining the Multiplication and Filtration

Multiplicative spectral sequence:

\[ TP(X) \wedge TP(Y) \to TP(X \wedge Y) \]
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In homotopy category, easy obstruction theory cellular approximation to diagonal & multiplication.

What about \( TP(X) \wedge_{TP(R)} TP(X) \to TP(X \wedge_R Y) \)?

\[ \mapsto \text{map of spectral sequences} \]

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Multiplicative spectral sequence:

\[ TP(X) \wedge TP(Y) \rightarrow TP(X \wedge Y) \] a filtered map? Coherent model?

In homotopy category, easy obstruction theory cellular approximation to diagonal & multiplication.

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Künneth Theorem

Filtered map \( TP(X) \wedge_{TP(R)} TP(Y) \rightarrow TP(X \wedge_R Y) \)

\[ \rightarrow \text{map of spectral sequences} \]

Righthand spectral sequence is Tate spectral sequence for

\[ THH(X \wedge_R Y) \cong THH(X) \wedge_{THH(R)} THH(Y) \]

\( E^2 \) periodic with \( \pi_\ast(THH(X) \wedge_{THH(R)} THH(Y)) \) in each even column

Lefthand spectral sequence has (renumbered) \( E^2 \)-term

\[ \pi_\ast \text{Gr}(TP(X) \wedge_{TP(R)} TP(Y)) \cong \pi_\ast(\text{Gr} TP(X) \wedge_{\text{Gr} TP(R)} \text{Gr} TP(Y)) \]

\( E^2 \)-term is \( \pi_\ast \text{Gr} TP(R) \)-module \( \rightarrow (2, 0) \)-periodic
Künneth Theorem

Filtered map $TP(X) \wedge_{TP(R)} TP(Y) \to TP(X \wedge_R Y)$

$\Rightarrow$ map of spectral sequences

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$$THH(X \wedge_R Y) \cong THH(X) \wedge_{THH(R)} THH(Y)$$

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Küneth Theorem

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Küneth Theorem

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Proposition

Map of spectral sequences is an isomorphism on $E^2$
Outline of Proof of Künneth Theorem

Filtered map $TP(X) \wedge_{TP(R)} TP(Y) \to TP(X \wedge_R Y)$ induces isomorphism of $E^2$-terms of spectral sequences

RHSS: Tate spectral sequence $\Rightarrow$ conditionally convergent.

Theorem

If $R = Hk$, $k$ a perfect field of characteristic $p > 0$, and $X$ and $Y$ are smooth and proper over $k$, then the LHSS is conditionally convergent.
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Where do we use hypotheses?

- $X$ smooth and proper $\Rightarrow$ $THH(X)$ compact $THH(R)$-module.
- $TP_*(k) = \mathbb{W}k[v, v^{-1}]$ finite global dimension.
  $THH(X)$ compact $\Rightarrow$ $TP(X)$ compact.
- Equivariantly, $Hk$ is a compact $THH(Hk)$-module.
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If $R = Hk$, $k$ a perfect field of characteristic $p > 0$, and $X$ and $Y$ are smooth and proper over $k$, then the LHSS is conditionally convergent.

Where do we use hypotheses?

- $X$ smooth and proper $\Rightarrow THH(X)$ compact $THH(R)$-module.
- $TP_*(k) = \mathbb{W}k[v, v^{-1}]$ finite global dimension. $THH(X)$ compact $\Rightarrow TP(X)$ compact.
- Equivariantly, $Hk$ is a compact $THH(Hk)$-module.