Outline

Theorem

*Finite type simply connected space \( X \) and \( Y \) are weakly equivalent if and only if their integral cochain \( E_{\infty} \) algebras \( C^* X \) and \( C^* Y \) are quasi-isomorphic.*

Mandell, Michael A. Cochains and homotopy type, 2006.

1. Spatial realization
2. Mapping spaces and the arithmetic square
3. \( E_{\infty} \) André–Quillen cohomology
4. Proof of the theorem
5. Some open problems and related questions
Spacial Realization

In the last two lectures, the key to the proof was always the unit of the derived spacial realization adjunction (especially for $K(\pi, n)$’s)

$$Y \rightarrow RU_FC^*(Y; F)$$

For $F = \bar{\mathbb{F}}_p$, we get $Y \rightarrow Y_p^\wedge$  
For $F = \mathbb{F}_p$, we get $Y \rightarrow \Lambda Y_p^\wedge$

For $F = \mathbb{Q}$, we get $Y \rightarrow Y_Q$

What do we get for $F = \mathbb{Z}$?

**Theorem**

For $F = \mathbb{Z}$, we get $Y \rightarrow \Lambda_f Y$, the homotopy pullback

$$\Lambda_f Y \rightarrow \Lambda Y^\wedge$$

$$\downarrow \quad \downarrow$$

$$Y \quad \rightarrow \quad Y^\wedge$$

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How to study the unit for $\mathbb{Z}$?

For $F = \mathbb{F}_p, \bar{\mathbb{F}}_p, \mathbb{Q}$, we looked at $Y = K(\pi, n)$

Cofibrant approximation was free or close to it (cohomology).

For $F = \mathbb{Z}$, looking at

$$\Lambda_f Y \rightarrow \Lambda Y^\wedge$$

$$\downarrow \quad \downarrow$$

$$Y \quad \rightarrow \quad Y^\wedge$$

$\Lambda_f K(\mathbb{Z}, n) \simeq K(\mathbb{Z}, n) \times K(\mathbb{Z}^\wedge, n - 1)$

Integral cohomology of $K(\mathbb{Z}, n)$ complicated in terms of free algebras (which we also don’t understand very well)

A different approach: Look at $RU_FC^*(Y; F)$ as a functor of $F$
\( RU_F(C^*(Y; F)) \) as a functor of \( F \)

Let \( A \to C^*(Y; \mathbb{Z}) \) be a cofibrant approximation in \( E_\infty \mathbb{Z} \)-algebras. Then \( A \otimes F \to C^*(Y; F) \) is a cofibrant approximation in \( E_\infty F \)-algebras and

\[
RU_F(C^*(Y; F)) = \mathcal{E}_F(A \otimes F, C^*(\Delta^\bullet; F)) = \mathcal{E}_\mathbb{Z}(A, C^*(\Delta^\bullet; F))
\]

Extension Lemma and Poincaré Lemma for \( C^*(\Delta^\bullet; F) \) means

\[
C^*(\Delta^n; F) \xrightarrow{\partial} \lim_{\partial} C^*(\Delta^{n-1}; F) \text{ and } F = C^*(\Delta^0; F) \xrightarrow{\sim} C^*(\Delta^n; F)
\]

In other words, \( C^*(\Delta^\bullet; F) \) is a simplicial resolution of \( F \)

\( RU_F(C^*(Y; F)) \) is the mapping space \( L\mathcal{E}_\mathbb{Z}(C^*Y; F) \).

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Mapping Spaces and Resolutions

Dwyer–Kan 1980’s: Homotopy categories come enriched over the homotopy category of spaces.

Given a category \( \mathcal{C} \) and a subcategory \( \mathcal{W} \) of weak equivalences, Dwyer and Kan produce a simplicially enriched category \( L_{\mathcal{W}}\mathcal{C} \) such that

\[
\pi_0 L_{\mathcal{W}}\mathcal{C} \cong \mathcal{C}[\mathcal{W}^{-1}]
\]

and which agrees with intrinsic (derived) mapping space in great generality (all reasonable known examples, e.g. simplicial model categories).

In a model category \( L\mathcal{C}(X, Y) \) is equivalent to:

- \( \mathcal{C}(X_0, Y_\bullet) \) where \( X_0 \to X \) is a cofibrant approximation and \( Y \to Y_\bullet \) is a simplicial resolution
- \( \mathcal{C}(X^\bullet, Y^0) \) where \( X^\bullet \to X \) is a cosimplicial resolution and \( Y \to Y^0 \) is a fibrant approximation.

We looked at \( \mathcal{E}(A, C^*(\Delta^\bullet; F)) \) \( L\mathcal{E}(\cdot, F) \) takes hty colims to hty lims
Equivalent to look at \( \mathcal{E}(A^\bullet, F) \) \( L\mathcal{E}(A, \cdot) \) preserves hty lims
The Arithmetic Square

Observation

The square is a homotopy pullback square in $\mathcal{E}$

\[
\begin{array}{ccc}
Z & \rightarrow & Z^\wedge \\
\downarrow & & \downarrow \\
Q & \rightarrow & Q^\wedge = Z^\wedge \otimes Q
\end{array}
\]

\[0 \rightarrow Z \rightarrow Q \times Z^\wedge \rightarrow Q^\wedge \rightarrow 0\]

Consequence

The square is a homotopy pullback square

\[
RU(C^* Y) \cong L\mathcal{E}(C^* Y, Z) \rightarrow L\mathcal{E}(C^* Y, Z^\wedge) \\
\downarrow \\
L\mathcal{E}(C^* Y, Q) \rightarrow L\mathcal{E}(C^* Y, Q^\wedge)
\]

Theorem

$L\mathcal{E}(C^* Y, Z^\wedge) \rightarrow \prod L\mathcal{E}(C^* Y, \mathbb{F}_p)$ is a weak equivalence
Understanding $L\mathcal{E}(C^*Y, \mathbb{Z}^\wedge)$

\[ \mathbb{Z}^\wedge = \prod \mathbb{Z}_p^\wedge \implies L\mathcal{E}(C^*Y, \mathbb{Z}^\wedge) = \prod L\mathcal{E}(C^*Y, \mathbb{Z}_p^\wedge) \]

\[ \mathbb{Z}_p^\wedge = \text{lim } \mathbb{Z}/p^k \implies L\mathcal{E}(C^*Y, \mathbb{Z}_p^\wedge) = \text{holim } L\mathcal{E}(C^*Y, \mathbb{Z}/p^k) \]

$\mathbb{Z}/p^{k+1}$ is a homotopy pullback:

\[
\begin{array}{ccc}
\mathbb{Z}/p^{k+1} & \longrightarrow & \mathbb{Z}/p \\
\downarrow & & \downarrow \\
\mathbb{Z}/p^k & \longrightarrow & (\mathbb{Z}/p \times \mathbb{Z}/p)
\end{array}
\]

So $L\mathcal{E}(C^*Y, \mathbb{Z}/p^{k+1})$ is a homotopy pullback:

\[
\begin{array}{ccc}
L\mathcal{E}(C^*Y, \mathbb{Z}/p^{k+1}) & \longrightarrow & L\mathcal{E}(C^*Y, \mathbb{Z}/p) \\
\downarrow & & \downarrow \\
L\mathcal{E}(C^*Y, \mathbb{Z}/p^k) & \longrightarrow & L\mathcal{E}(C^*Y, \mathbb{Z}/p \oplus \mathbb{Z}/p[-1])
\end{array}
\]

The Space $L\mathcal{E}(C^*Y, \mathbb{Z} \oplus \mathbb{Z}/p[-1])$

We need to show $L\mathcal{E}(C^*Y, \mathbb{Z}/p) \simeq L\mathcal{E}(C^*Y, \mathbb{Z}/p \oplus \mathbb{Z}/p[-1])$

or equivalently, $L\mathcal{E}_{fp}(C^*(Y; F_p), F_p) \simeq L\mathcal{E}_{fp}(C^*(Y; F_p), F_p \oplus F_p[-1])$.

So it suffices to show

\[ L\mathcal{E}_{fp}(C^*(Y; F_p), F_p \oplus F_p[-1]) \simeq L\mathcal{E}_{fp}(C^*(Y; F_p), F_p) \]

or, since $L\mathcal{E}_{fp}(C^*(Y; F_p), F_p)$ is connected, equivalently,

\[ \text{Fib}(L\mathcal{E}_{fp}(C^*(Y; F_p), F_p \oplus F_p[-1]) \rightarrow L\mathcal{E}_{fp}(C^*(Y; F_p), F_p)) \simeq * \]

You can view this fiber as $L(E_{fp}/F_p)(C^*(Y; F_p), F_p \oplus F_p[-1])$.

The spaces $L(E_{fp}/F_p)(C^*(Y; F_p), F_p \oplus F_p[-n])$ form an $\Omega$-spectrum whose homotopy groups are called the $E_\infty$ André–Quillen cohomology of $C^*(Y; F_p)$.
**Andre–Quillen Homology and Cohomology**

Let $M$ be a differential graded $F$-module. Then $M \oplus F$ becomes a commutative differential graded $F$-algebra by giving $F$ the square zero multiplication.

$$(F \oplus M) \otimes (F \oplus M) \cong (F \otimes F) \oplus (F \otimes M) \oplus (M \otimes F) \oplus (M \otimes M) \rightarrow F \oplus M$$

We can regard this as a functor $Ch(F\text{-Mod}) \rightarrow E_F/F$.

This has a left adjoint indecomposables functor $Q$:

$$QA = A/\langle E \Sigma_n \otimes I^n, n \geq 2 \rangle$$

where $I = \ker(A \to F)$.

**Definition (Andre–Quillen Cohomology)**

$$D^n(A; M) = \text{Ho}(E_{F/F})(A, F \oplus M[-n])$$

**Theorem**

$$D^n(A; M) \cong H^n(\text{Hom}_F(Q(\bar{A}), M)) \text{ where } \bar{A} \to A \text{ is a cofibrant approximation.}$$

### Andre–Quillen Cohomology of $C^*(Y; F)$

First consider the case $F = \mathbb{Q}$, $M = \mathbb{Q}$.

$$D^{-n}(A; \mathbb{Q}) = \text{Ho}(C_{\mathbb{Q}/\mathbb{Q}})(A, \mathbb{Q} \oplus \mathbb{Q}[n])$$

But for $n > 0$, $\mathbb{Q} \oplus \mathbb{Q}[n] \cong C^*(S^n; \mathbb{Q})$, so

$$D^{-n}(C^*(Y; \mathbb{Q}), \mathbb{Q}) \cong \pi_n Y \otimes \mathbb{Q}.$$

For $F = \mathbb{F}_p$, $M = \mathbb{F}_p$, and $A$ cofibrant

$$D^{-n}(A; \mathbb{Q}) \cong H^n(\text{Hom}_F(QA, M))$$

For $B_n = \mathbb{E}\langle x, y | dy = (1 - P^0)x \rangle \cong C^*(K(\mathbb{Z}/p, n); \mathbb{F}_p)$,

$$QB_n = \langle x, y | dy = x \rangle \cong 0$$

In general, $LQ(C^*(Y; \mathbb{F}_p)) \cong 0$.

This computes

$$L(E_{\mathbb{F}_p/\mathbb{F}_p})(C^*(Y; \mathbb{F}_p), \mathbb{F}_p \oplus \mathbb{F}_p[-n]) \cong *$$
Finishing the Proof of the Theorem

This shows \( RU(C^* Y) \simeq \Lambda_f Y \)

\[
\Lambda_f Y \rightarrow \Lambda Y^\wedge \\
\downarrow \downarrow \\
Y \rightarrow Y^\wedge
\]

So unit map \( Y \rightarrow RU(C^* Y) \) is split by \( \Lambda_f Y \rightarrow Y \)

and composite

\[ Ho S(X, Y) \xrightarrow{C^*} Ho E(C^* Y, C^* X) \cong Ho S(X, RU(C^* Y) \rightarrow Ho S(X, Y) \]

is the identity: A retraction of \( C^* : Ho S(X, Y) \rightarrow Ho E(C^* Y, C^* X) \).

Check that for any map in \( Ho E(C^* Y, C^* X) \), the map you get in \( Ho S(X, Y) \)
induces the same map on cohomology.

Consequence:

**Theorem**

*For finite type simply connected spaces \( X \) and \( Y \), if \( C^* X \) and \( C^* Y \) are quasi-isomorphic \( E_\infty \) algebras, then \( X \) and \( Y \) are weakly equivalent.*

Some Open Problems and Other Questions

We do not get an equivalence of the homotopy category with a homotopy category of \( E_\infty \) algebras like we did in the rational and \( p \)-adic case.

**Problem**

Find an algebraic model that captures all the homotopy theory of finite type simply connected spaces

The argument in the integral context was to reduce to the \( p \)-adic and rational cases.

**Problem**

Do the integral context directly: Specifically, understand the cohomology of free \( E_\infty \) algebras, of the integral Eilenberg–MacLane spectra, and the algebra of doing these cofibrant approximations.

These two are probably very hard problems
Some Open Problems and Other Questions II

In the rational and $p$-adic cases, we can identify which commutative or $E_\infty$ differential graded algebras are equivalent to the DeRham forms or cochains of a finite type simply connected space.

**Problem**

Identify which $E_\infty \mathbb{Z}$-algebras are quasi-isomorphic to the cochains of a finite type simply connected space.

The equivalence in the rational and $p$-adic cases extends to finite type nilpotent spaces. In the rational context, we know which commutative or $E_\infty$ differential graded algebras come from finite type nilpotent spaces.

**Problem**

Identify which $E_\infty \mathbb{F}_p$-algebras (or $E_\infty \overline{\mathbb{F}}_p$-algebras) come from finite type nilpotent spaces.

Some Open Problems and Other Questions III

A question Dennis Sullivan has been asking for years.

**Problem**

Combine the theory here with the total surgery obstruction to have a purely algebraic invariant classifying manifolds up to homeomorphism (or other -eomorphism).

This will probably necessitate developing surgery categories for an $E_\infty$ algebra $A$ that reduce to (or are canonically equivalent to) the surgery categories on a simplicial complex $Y$ when $A = C^* Y$. 
Problem

Find a good non-calculation reason why $B_n \simeq C^*(K(\mathbb{Z}/p,n); \mathbb{F}_p)$, or just why $H^0 B_n = \mathbb{F}_p$ and $H^q B_n = 0$ for $q < 0$.

A closely related problem: Find a good non-calculation reason why $L \mathcal{E}_{\mathbb{F}_p}(C^*(K(\mathbb{Z}/p,n); \mathbb{F}_p), \mathbb{F}_p)$ is connected and simply connected for all $n > 1$ (or, equivalently, connected for all $n \geq 1$).