Algebraic Models for Homotopy Types I
Models for Homotopy Theory

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Indiana University
Young Topologists Meeting 2013

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Outline

- Monday: Models in Homotopy Theory
- Tuesday: Algebraic Models in Rational Homotopy Theory
- Thursday: Algebraic Models in $p$-adic Homotopy Theory
- Friday: Algebraic Models for Integral Homotopy Types

Web page with slides (available after the talks), exercises, and reference links: [http://mypage.iu.edu/~mmandell/](http://mypage.iu.edu/~mmandell/)
Homotopy Theory

Let $C$ be a category. Let $W$ be a subcategory, the weak equivalences. The homotopy category $Ho_C$ is the category $C[W^{-1}]$ obtained by formally inverting the weak equivalences.

Issues
- Are the Hom sets small?
- Is there a reasonable way of describing the Hom sets?
- What kinds of constructions can you do in $Ho_C$?
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Language of Model Categories

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- Cofibrations
- Fibrations

\[ \text{homotopy pullback} \]
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Additional terminology

- Acyclic cofibration = cofibration + weak equivalence
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Additional terminology

- Acyclic cofibration = cofibration + weak equivalence
- Acyclic fibration = fibration + weak equivalence
- Cofibrant / cofibrant object = initial map is a cofibration
- Fibrant / fibrant object = final map is a fibration
Example: Chain Complexes

Let $\mathcal{C}$ be the category of bounded below chain complexes
Assume enough projectives
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Theorem (The Fundamental Lemma of Homological Algebra)

Let $P_* \text{ be cofibrant, and let } R_* \to A \text{ be a resolution.}$
$(R_* \to A \text{ is a weak equivalence and } R_* \text{ is in non-negative degrees}).$
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Then there exists a map $P_\ast \to R_\ast$ over $A$ and any two are chain homotopy equivalent.
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\[
\begin{array}{ccc}
R_* & \simeq & A \\
\downarrow & & \downarrow \\
\cdots & \uparrow & \\
P_* & \longrightarrow & A
\end{array}
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Example: Spaces

\[ \text{Cofibration} = \text{relative cell complex. Cofibrant} = \text{cell complex} \]
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\[
\begin{array}{ccc}
D & \rightarrow & X \\
\downarrow & & \downarrow \\
D \times I & \rightarrow & Y
\end{array}
\]

Theorem (Whitehead Theorem)
If $X \rightarrow Y$ is a weak equivalence and $B$ is a cell complex, then any map $B \rightarrow X$ lifts up to homotopy to a map $B \rightarrow X$ and any two lifts are homotopic.

Corollary
If $A$ and $B$ are cell complexes then maps in the set of maps in the homotopy category from $A$ to $B$ is just the set of homotopy classes of maps from $A$ to $B$.

$\text{Ho} \mathcal{T}(A, B) = \pi_0(B^A)$
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\text{weak equivalence} = \text{iso at the level of } \pi_0
\]

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Corollary

If \(A\) and \(B\) are cell complexes then the set of maps in the homotopy category from \(A\) to \(B\) is just the set of homotopy classes of maps from \(A\) to \(B\). \(\text{Ho} \mathcal{T}(A, B) = \pi_0(B^A)\)
Example: Simplicial Sets

Geometric $n$-simplex = \{ $t_0 v_0 + \cdots + t_n v_n$ | $t_i \geq 0$, $\sum t_i = 1$ \}

Vertexes are ordered, general position.

Simplicial set: Formed from simplexes by gluing faces along ordered linear maps.

Data = Sets $X_n$ of (non-degenerate) $n$-simplices plus gluing data for each face.

If we include “degenerate simplices” gluing data consists of maps $\partial_i: X_n \to X_{n-1}$ for $i = 0, \ldots, n$ (one for each face).

$X_0 = X_0 \quad X_1 = X_1 \amalg X_0 \quad X_2 = X_2 \amalg (X_1 \amalg X_1) \amalg X_0 \quad \cdots$

But then also need maps $s_i: X_{n-1} \to X_n$ to pick out degenerate simplices.
Example: Simplicial Sets (Review of Simplicial Sets)

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$$\cdots$$
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\end{align*}
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But then also need maps $s_i : X_{n-1} \rightarrow X_n$ to pick out degenerate simplices.
Let $\Delta(m, n)$ be the set of ordered maps from $\{0, \ldots, m\}$ to $\{0, \ldots, n\}$. 
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**Definition**

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**Example**

\[ \Delta^n = \Delta(\bullet, n) \]
\[ \Delta^m = \Delta(m, n) = \text{set of linear ordered maps from a geometric } m\text{-simplex to a geometric } n\text{-simplex.} \]
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$S_n X = T(\Delta^n, X)$
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$S_nX = T(\Delta^n, X)$

Adjunction $S(Z, S\bullet X) \cong T(|Z|, X)$.
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Cofibration = injection. Cofibrant = everything.
Fibration = Kan fibration. Fibrant = Kan complex.

\[
\begin{array}{ccc}
\Lambda_i^n & \rightarrow & X \\
\downarrow & & \downarrow \\
\Delta^n & \rightarrow & Y
\end{array}
\]

(Example. \(S \cdot X\) is a Kan complex; if \(X \rightarrow Y\) is a Serre fibration, then \(S \cdot X \rightarrow S \cdot Y\) is a Kan fibration.)

Definition
A map of simplicial sets is a weak equivalence if it is a weak equivalence after geometric realization.

Theorem
A map of simplicial sets is a weak equivalence if and only if it induces a bijection on homotopy classes of maps into every Kan complex.
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Theorem
A map of simplicial sets is a weak equivalence if and only if it induces a bijection on homotopy classes of maps into every Kan complex.
A closed model category is a category $\mathcal{C}$ with subcategories of weak equivalences, cofibrations, and fibrations such that:

- $\mathcal{C}$ has all limits and colimits.
- The weak equivalences satisfy the 2-out-of-3 property.
- The weak equivalences, cofibrations, and fibrations are closed under retracts.
- The cofibrations satisfy the left lifting property with respect to the acyclic cofibrations and the fibrations satisfy the right lifting property with respect to the acyclic cofibrations.
- Every map factors as a cofibration followed by an acyclic fibration and as an acyclic cofibration followed by a fibration.
Closed Model Category

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\[
\begin{array}{ccc}
X & \rightarrow & Y \\
\downarrow & & \downarrow \\
Z & \rightarrow & W
\end{array}
\]
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- The weak equivalences, cofibrations, and fibrations are closed under retracts

\[ \begin{array}{ccc}
X & \rightarrow & A \\
\downarrow & & \downarrow \\
Y & \rightarrow & B \\
\end{array} \]

\[ \begin{array}{ccc}
& & X \\
\circlearrowleft & & \\
& & \end{array} \]
A closed model category is a category \( \mathcal{C} \) with subcategories of weak equivalences, cofibrations, and fibrations such that:

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Examples: All the examples above.
Maps $f, g : X \to Y$ are (left) \textit{homotopic} means that there exists a diagram:

\[
\begin{array}{ccc}
X \sqcup X & \xrightarrow{f+g} & Y \\
\downarrow \sigma & & \uparrow \partial_0 + \partial_1 \\
X & \xleftarrow{\cong} & IX
\end{array}
\]
Maps $f, g: X \rightarrow Y$ are (left) homotopic means that there exists a diagram

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\downarrow \pmb{\nabla} & & \uparrow \partial_0 + \partial_1 \\
X & \xleftarrow{\sigma} & IX
\end{array}
\]

Can assume without loss of generality that $\partial_0 + \partial_1: X \amalg X \rightarrow IX$ is a cofibration, and then

\[
\begin{array}{ccc}
X \amalg X & \xrightarrow{\partial_0 + \partial_1} & IX \\
\sigma & \cong & \leftarrow
\end{array}
\]

is called a cylinder object.
Homotopy Theory in Model Categories

Definition

Maps \( f, g : X \to Y \) are (left) homotopic means that there exists a diagram

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\[
\begin{array}{ccc}
X \amalg X & \xrightarrow{\partial_0 + \partial_1} & IX \\
\downarrow & & \downarrow \\
X & \xleftarrow{\simeq} & X
\end{array}
\]

is called a cylinder object. If \( \sigma \) is also a fibration, then it is called a special cylinder object.
Fix your favorite special cylinder object.

\[ X \amalg X \xrightarrow{\partial_0 + \partial_1} IX \xrightarrow{\sigma} X \]
Fix your favorite special cylinder object.

\[
\begin{array}{c}
X \amalg X \\
\xrightarrow{\partial_0 + \partial_1} \\
IX \xrightarrow{\sim} X
\end{array}
\]

Assume that \( Y \) is fibrant. Then maps \( f, g : X \to Y \) are homotopic if and only if they are homotopic for your favorite special cylinder object.
Homotopy Theory in Model Categories (cont.)

Fix your favorite special cylinder object.

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Assume that \( Y \) is fibrant. Then maps \( f, g : X \rightarrow Y \) are homotopic if and only if they are homotopic for your favorite special cylinder object.

**Theorem**

Assume that \( X \) is cofibrant and \( Y \) is fibrant. Then the set of maps in the homotopy category from \( X \) to \( Y \) is the set of homotopy classes of maps from \( X \) to \( Y \).

\[ \text{Ho} \mathcal{C}(X, Y) \cong \mathcal{C}(X, Y) / \text{homotopy} \]
Homotopy Theory in Model Categories (cont.)

Fix your favorite special cylinder object.

\[ X \sqcup X \xrightarrow{\partial_0 + \partial_1} I X \xrightarrow{\sigma} X \]

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\[ \text{Ho} \mathcal{C}(X, Y) \cong \mathcal{C}(X, Y)/\text{homotopy} \]

In general: \( \text{Ho} \mathcal{C}(X, Y) \cong \text{Ho}(QX, RY) \) where \( QX \xrightarrow{\sim} X \) and \( Y \xrightarrow{\sim} RY \).
Derived Functors

Basic Problem

If $F$ does not factor through $\text{Ho} \, C$, can we find the “closest” one that does?

$C \xrightarrow{F} D$
**Basic Problem**

If $F$ does not factor through $\text{Ho } C$, can we find the “closest” one that does?

- **Left derived functor** is the closest from the left: Final natural transformation $L F \circ \gamma \rightarrow F$

- **Right derived functor** is the closest from the right: Initial natural transformation $F \rightarrow R F \circ \gamma$
Derived Functors

Basic Problem

\[ C \xrightarrow{F} D \]
\[ \gamma \downarrow \quad \gamma \downarrow \]
\[ Ho C \quad Ho C \]

If \( F \) does not factor through \( Ho C \), can we find the “closest” one that does?

Left derived functor is the closest from the left: Final natural transformation \( LF \circ \gamma \rightarrow F \)

Right derived functor is the closest from the right: Initial natural transformation \( F \rightarrow RF \circ \gamma \)

Theorem

If \( F \) sends weak equivalences between cofibrant objects to isomorphisms, then the left derived functor exists and \( LF(X) = F(QX) \) for \( QX \xrightarrow{\sim} X \) a cofibrant approximation.
Example: Derived functors of an additive functor

Let $\Phi: \mathcal{A} \rightarrow \mathcal{B}$ be an additive functor between abelian categories. Assume $\mathcal{A}$ has enough projectives.

Consider $F: \text{Ch}^+ (\mathcal{A}) \xrightarrow{\Phi^{-}} \text{Ch}^+ (\mathcal{B}) \rightarrow \text{Ho} (\text{Ch}^+ (\mathcal{B}))$.

Cofibrant objects = complexes of projectives. A weak equivalence between cofibrant objects is a chain homotopy equivalence. Thus, $F$ sends weak equivalences between cofibrant objects to isomorphisms.

For $A \in \mathcal{A}$, $LF(A) = \Phi (P^*)$ where $P^* \to A$ is a projective resolution.

Then $H^n (LF(A)) = L^n \Phi (A)$. 

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Consider $F: \text{Ch}^+(\mathcal{A}) \xrightarrow{\Phi} \text{Ch}^+(\mathcal{B}) \rightarrow \text{Ho}(\text{Ch}^+(\mathcal{B}))$.
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\[ LF(A) = \Phi(P^\ast) \] where $P^\ast \rightarrow A$ is a projective resolution.

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M.A. Mandell (IU)  
Models for Homotopy Theory  
July 2013
Quillen Adjunctions and Quillen Equivalences

Definition

Let $\mathcal{C}$ and $\mathcal{D}$ be model categories and $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ an adjunction $\mathcal{D}(FX, Y) \cong \mathcal{C}(X, GY)$. The adjunction $F, G$ is a Quillen adjunction if one of the following equivalent conditions hold:

- $F$ preserves cofibrations and $G$ preserves fibrations
- $F$ preserves cofibrations and acyclic cofibrations
- $G$ preserves fibrations and acyclic fibrations.

It is a Quillen equivalence if for any cofibrant $X$ in $\mathcal{C}$ and any fibrant $Y$ in $\mathcal{D}$, a map $FX \to Y$ is a weak equivalence if and only if the adjoint map $X \to GY$ is a weak equivalence.

Theorem

If $F, G$ is a Quillen adjunction, then the derived functors $LF$ and $RG$ exist and are adjoint functors $Ho \mathcal{D}(X, GY) \cong Ho \mathcal{C}(LFX, Y)$. The Quillen adjunction is a Quillen equivalence if and only if the derived adjunction on homotopy categories is an equivalence.
**Quillen Adjunctions and Quillen Equivalences**

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Quillen Adjunctions and Quillen Equivalences

Definition

Let $C$ and $D$ be model categories and $F : C \leftrightarrow D : G$ an adjunction $D(FX, Y) \cong C(X, GY)$. The adjunction $F, G$ is a Quillen adjunction if one of the following equivalent conditions hold:

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Theorem

If $F, G$ is a Quillen adjunction, then the derived functors $LF$ and $RG$ exist and are adjoint functors $\text{Ho } D(X, RGY) \cong \text{Ho } C(LFX, Y)$. The Quillen adjunction is a Quillen equivalence if and only if the derived adjunction on homotopy categories is an equivalence.
Example: Tor and Ext

Let $A$ be a commutative ring.
Let $C_A = Ch(A\text{-Mod})$.
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Example: Tor and Ext

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On the left, give $\mathcal{C}_A$ the model structure with fibrations the surjections; cofibrations are injections with projective cokernel (+ filt. hyp).

On the right give $\mathcal{C}_A$ the model structure with cofibrations the injections; fibrations are surjective with injective kernel (+ filt. hyp.).
Example: Tor and Ext

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On the right give \( C_A \) the model structure with cofibrations the injections; fibrations are surjective with injective kernel (+ filt. hyp.).

This is then a Quillen adjunction and we get an adjunction on the homotopy category
\[
\text{Ho} C_A(M \otimes_A^L X, Y) \cong \text{Ho} C_A(X, R\text{Hom}_A(M, Y)).
\]
Example: Simplicial Sets and Spaces

Adjunction

\[ |·| : S \leftrightarrow T : S_\bullet \]

\[ |X_\cdot| \to Y \]

is a weak equivalence if and only if \( X_\cdot \to S_\bullet Y \) is a weak equivalence.
Example: Simplicial Sets and Spaces

Adjunction

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Adjunction

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\(|\cdot|\) takes injections to inclusions of subcomplexes.

\(S\) takes Serre fibrations to Kan fibrations.

The map \(X \rightarrow S\) is a weak equivalence for all \(X\) (and so \(|S\) \(Y| \rightarrow Y\) is also a weak equivalence for all \(Y\)).

Then \(|X| \rightarrow Y\) is a weak equivalence if and only if \(X \rightarrow S\) is a weak equivalence.

Quillen Equivalence
Example: Simplicial Sets and Spaces

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Simplicial Approximation Theorem implies:

- Every cell complex is homotopy equivalent to the geometric realization of a simplicial set, so every space is weakly equivalent to the geometric realization of a simplicial set.
- The map \(X_\bullet \to S_\bullet |X_\bullet|\) is a weak equivalence for all \(X\) (and so \(|S_\bullet Y| \to Y\) is also a weak equivalence for all \(Y\)).
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Then \( |X.| \to Y \) is a weak equivalence if and only if \( X. \to S. Y \) is a weak equivalence. \quad \implies \text{Quillen Equivalence}
Example: Simplicial Sets and Spaces

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Then \( \| X_\bullet \| \to Y \) is a weak equivalence if and only if \( X_\bullet \to S_\bullet Y \) is a weak equivalence.  

\( \implies \) Quillen Equivalence

\( \implies \) equivalence \( \text{Ho} S \simeq \text{Ho} T \)
Web page with slides (available after the talks), exercises, and reference links: http://mypage.iu.edu/~mmandell/