Algebraic Models for Homotopy Types I
Models for Homotopy Theory

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Outline
- Monday: Models in Homotopy Theory
- Tuesday: Algebraic Models in Rational Homotopy Theory
- Thursday: Algebraic Models in \( p \)-adic Homotopy Theory
- Friday: Algebraic Models for Integral Homotopy Types

Web page with slides (available after the talks), exercises, and reference links: http://mypage.iu.edu/~mmandell/YTM
Homotopy Theory

Definition
Let $\mathcal{C}$ be a category.
Let $\mathcal{W}$ be a subcategory, the weak equivalences.
The homotopy category $\text{Ho} \mathcal{C}$ is the category $\mathcal{C}[\mathcal{W}^{-1}]$ obtained by formally inverting the weak equivalences.

Issues
- Are the Hom sets small?
- Is there a reasonable way of describing the Hom sets?
- What kinds of constructions can you do in $\text{Ho} \mathcal{C}$?

Language of Model Categories
Often in addition to weak equivalences, we have other interesting classes of maps that we can use to build homotopy invariant constructions like homotopy colimits and homotopy limits
- Cofibrations
- Fibrations

Quillen’s theory of model categories axiomatizes properties of these.

Additional terminology
- Acyclic cofibration = cofibration + weak equivalence
- Acyclic fibration = fibration + weak equivalence
- Cofibrant / cofibrant object = initial map is a cofibration
- Fibrant / fibrant object = final map is a fibration
Example: Chain Complexes

Let \( C \) be the category of bounded below chain complexes.
Assume enough projectives.

Cofibrant = complex that is degreewise projective
Cofibration = injection whose coker is cofibrant
Fibration = surjection

**Theorem (The Fundamental Lemma of Homological Algebra)**

Let \( P_* \) be cofibrant, and let \( R_* \to A \) be a resolution.
\( (R_* \to A \) is a weak equivalence and \( R_* \) is in non-negative degrees).

For any map \( P_* \to A \), there exists a map \( P_* \to R_* \) over \( A \) and any two are chain homotopy equivalent.

\[
\begin{array}{ccc}
R_* & \to & P_* \\
\downarrow \cong & & \downarrow \\
P_* & \to & A
\end{array}
\quad
\begin{array}{ccc}
P_* \otimes \partial I & \to & R_* \\
\downarrow \cong & & \downarrow \\
P_* \otimes I & \to & A
\end{array}
\]

Example: Spaces

Cofibration = relative cell complex. Cofibrant = cell complex
Fibration = Serre fibration. Fibrant = everything.

\[
\begin{array}{ccc}
D & \to & X \\
\downarrow & & \downarrow \\
D \times I & \to & Y
\end{array}
\]

**Theorem (Whitehead Theorem)**

If \( X \to Y \) is a weak equivalence and \( B \) is a cell complex, then any map \( B \to Y \) lifts up to homotopy to a map \( B \to X \) and any two lifts are homotopic.

**Corollary**

If \( A \) and \( B \) are cell complexes then the set of maps in the homotopy category from \( A \) to \( B \) is just the set of homotopy classes of maps from \( A \) to \( B \).

\[ \text{Ho} \mathcal{T}(A, B) = \pi_0(B^A) \]
Example: Simplicial Sets (Review of Simplicial Sets)

Geometric $n$-simplex = \{ $t_0v_0 + \cdots + t_nv_n$ | $t_i \geq 0$, $\Sigma t_i = 1$ \}

Vertexes are ordered, general position.

Simplicial set: Formed from simplexes by gluing faces along ordered linear maps.

Data = Sets $\overline{X}_n$ of (non-degenerate) $n$-simplices plus gluing data for each face

If we include “degenerate simplices” gluing data consists of maps $\partial_i : X_n \to X_{n-1}$ for $i = 0, \ldots, n$ (one for each face)

$$X_0 = \overline{X}_0$$
$$X_1 = \overline{X}_1 \amalg \overline{X}_0$$
$$X_2 = \overline{X}_2 \amalg (\overline{X}_1 \amalg \overline{X}_1) \amalg \overline{X}_0$$

But then also need maps $\sigma_i : X_{n-1} \to X_n$ to pick out degenerate simplices.

Review of Simplicial Sets (cont.)

Let $\Delta(m, n)$ be the set of ordered maps from $\{0, \ldots, m\}$ to $\{0, \ldots, n\}$.

Definition

A simplicial set is a functor from $\Delta^{op}$ to sets.

Example

$\Delta^n_\bullet = \Delta(\bullet, n)$

$\Delta^n_m = \Delta(m, n) =$ set of linear ordered maps from a geometric $m$-simplex to a geometric $n$-simplex.

$\text{Hom}(\Delta^m, \Delta^n) = \Delta(m, n)$

Example

$S_nX = T(\Delta^n, X)$  Adjunction $S(Z, S_\bullet X) = T(|Z|, X)$. 

Example: Simplicial Sets

Cofibration = injection. Cofibrant = everything.
Fibration = Kan fibration. Fibrant = Kan complex.

(Example. \( S_n X \) is a Kan complex; if \( X \rightarrow Y \) is a Serre fibration, then \( S_n X \rightarrow S_n Y \) is a Kan fibration.)

Definition

A map of simplicial sets is a weak equivalence if it is a weak equivalence after geometric realization

Theorem

A map of simplicial sets is a weak equivalence if and only if it induces a bijection on homotopy classes of maps into every Kan complex.

Closed Model Category

A closed model category is a category \( C \) with subcategories of weak equivalences, cofibrations, and fibrations such that:

- \( C \) has all limits and colimits
- The weak equivalences satisfy the 2-out-of-3 property
- The weak equivalences, cofibrations, and fibrations are closed under retracts
- The cofibrations satisfy the left lifting property with respect to the acyclic fibrations and the fibrations satisfy the right lifting property with respect to the acyclic cofibrations
- Every map factors as a cofibration followed by an acyclic fibration and as an acyclic cofibration followed by a fibration.

Examples: All the examples above.
Homotopy Theory in Model Categories

Definition
Maps \( f, g : X \to Y \) are (left) homotopic means that there exists a diagram

\[
\begin{array}{ccc}
X \boxtimes X & \xrightarrow{f + g} & Y \\
\downarrow & & \downarrow \\
X & \xleftarrow{\sigma} & IX
\end{array}
\]

Can assume without loss of generality that \( \partial_0 + \partial_1 : X \boxtimes X \to IX \) is a cofibration, and then

\[
X \boxtimes X \xrightarrow{\partial_0 + \partial_1} IX \xrightarrow{\sigma} X
\]

is called a cylinder object. If \( \sigma \) is also a fibration, then it is called a special cylinder object.

Homotopy Theory in Model Categories (cont.)

Fix your favorite special cylinder object.

\[
X \boxtimes X \xrightarrow{\partial_0 + \partial_1} IX \xrightarrow{\sigma} X
\]

Assume that \( Y \) is fibrant. Then maps \( f, g : X \to Y \) are homotopic if and only if they are homotopic for your favorite special cylinder object.

Theorem
Assume that \( X \) is cofibrant and \( Y \) is fibrant. Then the set of maps in the homotopy category from \( X \) to \( Y \) is the set of homotopy classes of maps from \( X \) to \( Y \). \( \text{Ho}\mathcal{C}(X, Y) \cong \mathcal{C}(X, Y)/\text{homotopy} \)

In general: \( \text{Ho}\mathcal{C}(X, Y) \cong \text{Ho}(QX, RY) \) where \( QX \xrightarrow{\sim} X \) and \( Y \xrightarrow{\sim} RY \).
Derived Functors

Basic Problem

\[ \begin{array}{ccc}
\mathcal{C} & \xrightarrow{F} & \mathcal{D} \\
\gamma \downarrow & & \downarrow \\
\text{Ho} \mathcal{C} & \xrightarrow{??} & \mathcal{D}
\end{array} \]

If \( F \) does not factor through \( \text{Ho} \mathcal{C} \), can we find the "closest" one that does?

Left derived functor is the closest from the left: Final natural transformation \( LF \circ \gamma \rightarrow F \)

Right derived functor is the closest from the right: Initial natural transformation \( F \rightarrow RF \circ \gamma \)

Theorem

If \( F \) sends weak equivalences between cofibrant objects to isomorphisms, then the left derived functor exists and \( LF(X) = F(QX) \) for \( QX \xrightarrow{\sim} X \) a cofibrant approximation.

Example: Derived functors of an additive functor

Let \( \Phi: \mathcal{A} \rightarrow \mathcal{B} \) be an additive functor between abelian categories. Assume \( \mathcal{A} \) has enough projectives.

Consider \( F: Ch^+ (\mathcal{A}) \xrightarrow{\Phi} Ch^+ (\mathcal{B}) \rightarrow \text{Ho}(Ch^+ (\mathcal{B})) \).

Cofibrant objects = complexes of projectives.

A weak equivalence between cofibrant objects is a chain homotopy equivalence. Thus, \( F \) sends weak equivalences between cofibrant objects to isomorphisms.

For \( A \) in \( \mathcal{A} \), \( LF(A) = \Phi(P_*) \) where \( P_* \rightarrow A \) is a projective resolution.

Then \( H_n(LF(A)) = \text{L}^n \Phi(A) \).
Quillen Adjunctions and Quillen Equivalences

**Definition**

Let $\mathcal{C}$ and $\mathcal{D}$ be model categories and $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ an adjunction $\mathcal{D}(FX, Y) \cong \mathcal{C}(X, GY)$. The adjunction $F, G$ is a Quillen adjunction if one of the following equivalent conditions hold:

- $F$ preserves cofibrations and $G$ preserves fibrations
- $F$ preserves cofibrations and acyclic cofibrations
- $G$ preserves fibrations and acyclic fibrations.

It is a Quillen equivalence if for any cofibrant $X$ in $\mathcal{C}$ and any fibrant $Y$ in $\mathcal{D}$, a map $FX \to Y$ is a weak equivalence if and only if the adjoint map $X \to GY$ is a weak equivalence.

**Theorem**

If $F, G$ is a Quillen adjunction, then the derived functors $LF$ and $RG$ exist and are adjoint functors $Ho\mathcal{D}(X, RGY) \cong Ho\mathcal{C}(LFX, Y)$. The Quillen adjunction is a Quillen equivalence if and only if the derived adjunction on homotopy categories is an equivalence.

**Example: Tor and Ext**

Let $A$ be a commutative ring.
Let $\mathcal{C}_A = Ch(A\text{-Mod})$.
Let $M$ be a differential graded $A$-module.
Consider $\quad M \otimes_A (\_ : \mathcal{C}_A \rightleftarrows \mathcal{C}_A : \text{Hom}_A(M, \_))$

On the left, give $\mathcal{C}_A$ the model structure with fibrations the surjections; cofibrations are injections with projective cokernel (+ filt. hyp).

On the right give $\mathcal{C}_A$ the model structure with cofibrations the injections; fibrations are surjective with injective kernel (+ filt. hyp.).

This is then a Quillen adjunction and we get an adjunction on the homotopy category $\quad Ho\mathcal{C}_A(M \otimes_A X, Y) \cong Ho\mathcal{C}_A(X, R\text{Hom}_A(M, Y))$. 
Example: Simplicial Sets and Spaces

Adjunction

\[ \cdot : S \rightleftharpoons T : \cdot \]

\[ \cdot \] takes injections to inclusions of subcomplexes.
\[ S : \cdot \] takes Serre fibrations to Kan fibrations. \[ \implies \text{Quillen Adjunction} \]

Simplicial Approximation Theorem implies:

- Every cell complex is homotopy equivalent to the geometric realization of a simplicial set, so every space is weakly equivalent to the geometric realization of a simplicial set.
- The map \( X_\bullet \to S_\cdot |X_\bullet| \) is a weak equivalence for all \( X \) (and so \( |S_\cdot Y| \to Y \) is also a weak equivalence for all \( Y \)).

Then \( |X_\bullet| \to Y \) is a weak equivalence if and only if \( X_\bullet \to S_\cdot Y \) is a weak equivalence. \[ \implies \text{Quillen Equivalence} \]

\[ \implies \text{equivalence } \text{Ho } S \simeq \text{Ho } T \]