Integrals of Equivariant Forms and a Gauss-Bonnet
Theorem for Constructible Sheaves

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Abstract

The Berline-Vergne integral localization formula for equivariantly closed forms ([BV], Theorem 7.11 in [BGV]) is well-known and requires the acting Lie group to be compact. It is restated here as Theorem 2. In this article we extend this result to real reductive Lie groups $G_R$. The main result is Theorem 20.

As an application of this generalization, we prove an analogue of the Gauss-Bonnet theorem for constructible sheaves (Theorem 43). If $\mathcal{F}$ is a $G_R$-equivariant sheaf on a complex projective manifold $M$, then the Euler characteristic of $M$ with respect to $\mathcal{F}$

$$
\chi(M, \mathcal{F}) = \frac{1}{(2\pi)^{\dim C} M} \int_{Ch(\mathcal{F})} \tilde{\chi}_{gC}
$$

as distributions on $g_R$, where $Ch(\mathcal{F})$ is the characteristic cycle of $\mathcal{F}$ and $\tilde{\chi}_{gC}$ is the Euler form of $M$ extended to the cotangent space $T^*M$ (independently of $\mathcal{F}$). We also consider an analogue of Duistermaat-Heckman measures for real reductive Lie groups acting on symplectic manifolds.

In [L3] I apply the results of this article to obtain a Riemann-Roch-Hirzebruch type integral formula for characters of representations of reductive groups.

Keywords: equivariant forms, Berline-Vergne integral localization formula, characteristic cycles of sheaves, integral character formula.

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1 Introduction

Equivariant forms were introduced in 1950 by Henri Cartan. There are many good texts on this subject including [BGV] and [GS].

Let $G_\mathbb{R}$ be a compact Lie group acting on a compact manifold $M$, let $g_\mathbb{R}$ be the Lie algebra of $G_\mathbb{R}$, and let $\alpha(X)$ be an equivariantly closed form on $M$ depending on $X \in g_\mathbb{R}$. For $X \in g_\mathbb{R}$, we denote by $M_0(X)$ the set of zeroes of the vector field on $M$ induced by the infinitesimal action of $X$. We assume that $M_0(X)$ is discrete. Then Theorem 7.11 in [BGV] (which we restate here as Theorem 2) says that the integral of $\alpha(X)$ can be expressed as a sum over the set of zeroes $M_0(X)$ of certain local quantities of $M$ and $\alpha$:

$$\int_M \alpha(X) = \sum_{p \in M_0(X)} \text{local invariant of } M \text{ and } \alpha \text{ at } p.$$ 

This is the essence of the Berline-Vergne integral localization formula for equivariantly closed differential forms which originally appeared in [BV].

In this article we extend this result to reductive groups. So let $G_\mathbb{R}$ be a real reductive Lie group which may not be compact. To avoid pathologies we require the action of $G_\mathbb{R}$ to be complex algebraic. On the other hand, for the purpose of interesting applications we would like to allow integration over homology cycles with non-compact support. Then one encounters the following two problems. First of all, the cycle being infinite, the integral may no longer converge in the usual sense. We resolve this problem by defining a new (more relaxed) notion of integral over the cycle in the sense of distribution on the Lie algebra $g_\mathbb{R}$. Secondly, some cycles simply may not contain enough points fixed by the group action for an integral localization formula to make sense. This is similar to the failure of the Lefschetz fixed point formula for non-compact manifolds – some fixed points may run off to infinity. For this reason we restrict ourselves to the following setting. Let $G_\mathbb{R}$ act algebraically on a complex projective manifold $M$, this action extends naturally to the cotangent space $T^*M$. Let $\Lambda$ be a conic $G_\mathbb{R}$-invariant Lagrangian cycle $\Lambda$ in $T^*M$. We describe a class of differential forms $\tilde{\alpha}(X)$ on $T^*M$ depending on $X \in g_\mathbb{R}$ and define $\int_\Lambda \tilde{\alpha}(X)$ as a distribution on $g_\mathbb{R}$. Let $\{x_1, \ldots, x_d\}$ be the set of zeroes of the vector field on $M$ induced by the infinitesimal action of $X$. The main result (Theorem 20) says that this distribution is given by integration against a function $F$ on $g_\mathbb{R}$ and

$$F(X) = \sum_{k=1}^d m_k(X) \cdot \left( \text{the contribution of } x_k \text{ to the Berline-Vergne localization formula} \right),$$

where $m_k(X)$ is a certain integer multiplicity which is exactly the local contribution of $x_k$ to the Lefschetz fixed point formula, as generalized to sheaf cohomology by M. Goresky and R. MacPherson [GM]. These multiplicities will be determined in terms of local cohomology of $F$, where $F$ is any sheaf with characteristic cycle $Ch(F) = \Lambda$. Existence of such a localization formula was conjectured by W. Schmid in [Sch].

The idea is to observe that the integrand is a closed form (Lemma 16), to pick a sufficiently regular element $X \in g_\mathbb{R}$ and to deform $\Lambda$ into a simple-looking cycle of the following kind:

$$m_1(X)T^*_{x_1}M + \cdots + m_d(X)T^*_{x_d}M,$$

where $m_1(X), \ldots, m_d(X)$ are the integer multiplicities from (1) and each cotangent space $T^*_{x_k}M$ is given a certain orientation. The cycles in question have infinite support, which means one
must deform $\Lambda$ very carefully to ensure that the integral stays unchanged. The precise result is stated in Proposition 31.

This kind of argument fits very well into the cobordism theory of spaces equipped with abstract moment maps as described by V. Guillemin, V. Ginzburg and Y. Karshon in [GGK]. They would probably call Proposition 31 “the linearization theorem for characteristic cycles.” Then Theorem 20 essentially becomes “linearization commutes with integration.” Of course, since we work with cycles with possibly singular support we no longer require that the chains realizing cobordisms have smooth support.

Then, using this generalized localization formula, we prove an analogue of the Gauss-Bonnet theorem for constructible sheaves (Theorem 43). If $\mathcal{F}$ is a $G_\mathbb{R}$-equivariant sheaf on a complex projective manifold $M$, then the Euler characteristic of $M$ with respect to $\mathcal{F}$

$$\chi(M, \mathcal{F}) = \frac{1}{(2\pi)^{\dim \mathbb{C} M}} \int_{Ch(\mathcal{F})} \widetilde{\chi}_{\mathbb{R}}$$

as distributions on $\mathfrak{g}_\mathbb{R}$, where $Ch(\mathcal{F})$ is the characteristic cycle of $\mathcal{F}$ and $\widetilde{\chi}_{\mathbb{R}}$ is the Euler form of $M$ extended to the cotangent space $T^*M$ (independently of $\mathcal{F}$).

In the last section we describe an analogue of Duistermaat-Heckman measures for real reductive Lie groups acting on symplectic manifolds and give a formula for the Fourier transforms of these measures similar to the exact stationary phase approximation formula (Proposition 45).

In [L3] I apply the results of this article to obtain a Riemann-Roch-Hirzebruch type integral formula for characters of representations of reductive groups.

The proof given here is a significant modification of the localization argument which appeared in my Ph.D. thesis [L1]. This thesis provides a geometric proof of an analogue of Kirillov’s character formula for reductive Lie groups. Article [L2] gives a very accessible introduction to [L1] and explains key ideas used there by way of examples.

The following convention will be in force throughout these notes: whenever $A$ is a subset of $B$, we will denote the inclusion map $A \hookrightarrow B$ by $j_{A \hookrightarrow B}$.

## 2 The Berline-Vergne Localization Formula

In this article we use the same notations as in [BGV].

Let $M$ be a $C^\infty$-manifold of dimension $m$ with an action of a (possibly non-compact) Lie group $G_\mathbb{R}$, and let $\mathfrak{g}_\mathbb{R}$ be the Lie algebra of $G_\mathbb{R}$. The group $G_\mathbb{R}$ acts on $C^\infty(M)$ by the formula $(g \cdot \varphi)(x) = \varphi(g^{-1}x)$. For $X \in \mathfrak{g}_\mathbb{R}$, we denote by $X_M$ the vector field on $M$ given by (notice the minus sign)

$$(X_M \cdot \varphi)(x) = \frac{d}{d\varepsilon} \varphi(\exp(-\varepsilon X)x) \bigg|_{\varepsilon = 0}.$$

This way

$$[X, Y]_M = [X_M, Y_M], \quad \text{for all } X, Y \in \mathfrak{g}_\mathbb{R},$$

which would not be true without this choice of signs.

Let $\Omega^*(M)$ denote the (graded) algebra of smooth complex-valued differential forms on $M$, and let $C^\infty(\mathfrak{g}_\mathbb{R}) \hat{\otimes} \Omega^*(M)$ denote the algebra of all smooth $\Omega^*(M)$-valued functions on $\mathfrak{g}_\mathbb{R}$. The group $G_\mathbb{R}$ acts on an element $\alpha \in C^\infty(\mathfrak{g}_\mathbb{R}) \hat{\otimes} \Omega^*(M)$ by the formula

$$(g \cdot \alpha)(X) = g \cdot (\alpha(g^{-1} \cdot X)) \quad \text{for all } g \in G \text{ and } X \in \mathfrak{g}_\mathbb{R}.$$
Let $\Omega^*_G(M) = (C^\infty(\mathfrak{g}_R) \hat{\otimes} \Omega^*(M))^{G_\mathbb{R}}$ be the subalgebra of $G_\mathbb{R}$-invariant elements. An element $\alpha$ of $\Omega^*_G(M)$ satisfies the relation $\alpha(g \cdot X) = g \cdot \alpha(X)$ and is called an equivariant differential form.

We define the equivariant exterior differential $d_{\mathfrak{g}_R}$ on $C^\infty(\mathfrak{g}_R) \hat{\otimes} \Omega^*(M)$ by the formula

$$(d_{\mathfrak{g}_R}\alpha)(X) = d(\alpha(X)) - \iota(X_M)\alpha(X),$$

where $d$ denotes the ordinary de Rham differential and $\iota(X_M)$ denotes contraction by the vector field $X_M$. This differential $d_{\mathfrak{g}_R}$ preserves $\Omega^*_G(M)$, and $(d_{\mathfrak{g}_R})^2 \alpha = 0$ for all $\alpha \in \Omega^*_G(M)$. The elements of $\Omega^*_G(M)$ such that $d_{\mathfrak{g}_R}\alpha = 0$ are called equivariantly closed forms.

**Example 1** Let $T^*M$ be the cotangent bundle of $M$, and let $\pi : T^*M \to M$ denote the projection map. Let $\sigma : \pi^{-1}(p) \to \mathfrak{g}_R/\mathfrak{g}_R^p$ be the projection map. Let $\sigma$ denote the canonical symplectic form on $T^*M$. It is defined, for example, in [KSch], Appendix A2. The action of the Lie group $G_\mathbb{R}$ on $M$ naturally extends to $T^*M$. Then we always have a canonical equivariantly closed form on $T^*M$, namely, $\mu_\mathbb{R} + \sigma_\mathbb{R}$. Here $\mu_\mathbb{R} : \mathfrak{g}_R \to C^\infty(T^*M)$ is the moment map defined by:

$$\mu_\mathbb{R}(X) : \zeta \mapsto -\langle \zeta, X_M \rangle, \quad X \in \mathfrak{g}_R, \; \zeta \in T^*M. \quad \Box$$

If $\alpha$ is a non-homogeneous equivariant differential form, $\alpha_{[k]}$ denotes the homogeneous component of degree $k$. If $M$ is a compact oriented manifold, we can integrate equivariant differential forms over $M$, obtaining a map

$$\int_M : \Omega^*_G(M) \to C^\infty(\mathfrak{g}_R)^{G_\mathbb{R}},$$

by the formula $(\int_M \alpha)(X) = \int_M \alpha(X)[\dim M]$.

Notice that if $\alpha \in \Omega^*_G(M)$ has top homogeneous component $\alpha_{[k]}$, then $(d_{\mathfrak{g}_R}\alpha)(X)[k+1]$ is exact; and if $p \in M$ is a zero of the vector field $X_M$ (i.e. $X_M(p) = 0$), then $(d_{\mathfrak{g}_R}\alpha)[0](p) = 0$. Hence the map $\alpha \mapsto \alpha(X)[0](p)$ descends to $\Omega^*_G(M)/\text{Im}(d_{\mathfrak{g}_R})$. Similarly, if $M$ is compact, then the map $\alpha \mapsto \int_M \alpha(X)$ also descends to $\Omega^*_G(M)/\text{Im}(d_{\mathfrak{g}_R})$.

Also notice that if $\alpha$ is an equivariantly closed form whose top homogeneous component has degree $k$, then $\alpha(X)[k]$ is closed with respect to the ordinary exterior differential.

We recall some more notations from [BGV]. Let $M_0(X)$ be the set of zeroes of the vector field $X_M$. We state the localization formula in the important special case where $X_M$ has isolated zeroes. Here, at each point $p \in M_0(X)$, the Lie action $\mathcal{L}(X_M)$ acts on the vector fields $\mathcal{X}$ of $M$ gives rise to a linear transformation $L_p$ on $T_pM$.

If the Lie group $G_\mathbb{R}$ is compact, then the transformation $L_p$ is invertible and has only imaginary eigenvalues. Thus the dimension of $M$ is even and there exists an oriented basis $\{e_1, \ldots, e_m\}$ of $T_pM$ such that for $1 \leq i \leq n = m/2$,

$$L_pe_{2i-1} = \lambda_{p,i}e_{2i}, \quad L_pe_{2i} = -\lambda_{p,i}e_{2i-1}.$$ 

We have $\det(L_p) = \lambda_{p,1}^2 \lambda_{p,2}^2 \ldots \lambda_{p,n}^2$, and it is natural to take the following square root (dependent only on the orientation of the manifold):

$$\det^{1/2}(L_p) = \lambda_{p,1} \ldots \lambda_{p,n}.$$

For convenience, we restate Theorem 7.11 from [BGV].
Theorem 2 Let \( G_\mathbb{R} \) be a compact Lie group with Lie algebra \( g_\mathbb{R} \) acting on a compact oriented manifold \( M \), and let \( \alpha \) be an equivariantly closed differential form on \( M \). Let \( X \in g_\mathbb{R} \) be such that the vector field \( X_M \) has only isolated zeroes. Then

\[
\int_M \alpha(X) = (-2\pi)^n \sum_{p \in M_0(X)} \frac{\alpha(X)_0(p)}{\det^{1/2}(L_p)},
\]

where \( n = \text{dim}(M)/2 \), and by \( \alpha(X)_0(p) \), we mean the value of the function \( \alpha(X)_0 \) at the point \( p \in M \).

3 A Brief Introduction to Characteristic Cycles of Sheaves

Characteristic cycles were introduced by M. Kashiwara and their definition can be found in [KSch]. A comprehensive treatment of characteristic cycles can be found in [Schü]. On the other hand, W. Schmid and K. Vilonen give a geometric way to understand characteristic cycles in [SchVi] which we follow here. In this section we briefly outline the defining properties of characteristic cycles which are analogous to Eilenberg-Steenrod homology axioms for homology in \([\text{SchV}1]\) which we follow here. In this section we briefly outline the defining properties of characteristic cycles in \([KSch]\). A comprehensive treatment of characteristic cycles can be found in \([\text{Sch¨ u}]\). On the other hand, W. Schmid and K. Vilonen give a geometric way to understand characteristic cycles in \([\text{SchVi}]\).

In this section only we assume that \( M \) is an oriented smooth real semi-algebraic manifold which need not be compact. See, for instance, [DM] for the notion of semi-algebraic sets. (In the next section we will further require \( M \) to be a smooth complex projective variety). Now let \( \mathcal{F} \) be a bounded complex of sheaves on \( M \). We say that \( \mathcal{F} \) has \( \mathbb{R} \)-constructible cohomology if there exists a locally finite covering \( M = \bigcup_{j \in J} M_j \) by semi-algebraic subsets such that for all \( k \in \mathbb{Z} \) and all \( j \in J \), the restricted cohomology sheaves \( H^k(\mathcal{F})|_{M_j} \) are constant of finite rank.

Let \( C^b_{\mathbb{R}-c}(M) \) denote the category of bounded complexes of sheaves on \( M \) with \( \mathbb{R} \)-constructible cohomology, and let \( D^b_{\mathbb{R}-c}(M) \) denote the bounded derived category of sheaves on \( M \) with \( \mathbb{R} \)-constructible cohomology. From now on \( \mathcal{F} \) denotes an element in \( C^b_{\mathbb{R}-c}(M) \) or \( D^b_{\mathbb{R}-c}(M) \). The characteristic cycle \( Ch(\mathcal{F}) \) associated to \( \mathcal{F} \) is a Borel-Moore homology cycle (possibly with infinite support) in the cotangent space \( T^*M \) of dimension \( \text{dim}_\mathbb{R} M \). The cycle \( Ch(\mathcal{F}) \) has the following properties: it is conic, i.e. invariant under the scaling action of positive reals \( \mathbb{R}^{>0} \) on \( T^*M \) (but not necessarily under the action of \( \mathbb{R}^+ \)), and its support \( |Ch(\mathcal{F})| \) is Lagrangian. More precisely, there exists a Whitney stratification \( \mathcal{S} \) of \( M \) by semi-algebraic sets such that the cohomology of \( \mathcal{F} \) is constructible with respect to \( \mathcal{S} \). This means that for all \( k \in \mathbb{Z} \) and all \( S \in \mathcal{S} \), the cohomology sheaves restricted to the strata \( H^k(\mathcal{F})|_S \) are (locally) constant of finite rank. Then the support of \( Ch(\mathcal{F}) \) lies in the union of conormal spaces:

\[
|Ch(\mathcal{F})| \subset \bigcup_{S \in \mathcal{S}} T^*_S M.
\]

Let \( \mathcal{L}^+(M) \) denote the abelian group of Borel-Moore homology cycles (with coefficients in \( \mathbb{Z} \)) in the cotangent space \( T^*M \) of dimension \( \text{dim}_M M \) which are conic (i.e. invariant under the scaling action of \( \mathbb{R}^{>0} \) on \( T^*M \)) and whose support lies in \( \bigcup_{S \in \mathcal{S}} T^*_S M \) for some locally finite semi-algebraic Whitney stratification \( \mathcal{S} \) of \( M \).
Example 3 Let $N \subset M$ be a closed semi-algebraic submanifold, $j : N \rightarrow M$ the inclusion map, and let $\mathbb{C}_N$ be the constant sheaf on $N$ of rank 1, then $Ch(j_! \mathbb{C}_N)$ is the conormal space $T^*_N M$ equipped with a certain orientation.

To specify this orientation, pick any point $p \in N$ and choose a positively oriented system of coordinates $(x_1, \ldots, x_{\dim M})$ on $M$ around $p$ such that $N$ is locally given by the equations $x_{\dim N+1} = \cdots = x_{\dim M} = 0$. Let $(\xi_1, \ldots, \xi_{\dim M})$ be the fiber coordinates dual to the frame $dx_1, \ldots, dx_{\dim M}$. Then near points lying in the cotangent space $T^*_p M$, $T^*_N M$ is given by the equations $x_{\dim N+1} = \cdots = x_{\dim M} = \xi_1 = \cdots = \xi_{\dim N} = 0$ and the functions $(x_1, \ldots, x_{\dim N}, \xi_{\dim N+1}, \ldots, \xi_{\dim M})$ form a coordinate system on $T^*_N M$. Finally, $Ch(j_! \mathbb{C}_N)$ is the conormal space $T^*_N M$ with orientation equal $(-1)^{\dim M - \dim N}$ times the orientation given by coordinates $(x_1, \ldots, x_{\dim N}, \xi_{\dim N+1}, \ldots, \xi_{\dim M})$, and this orientation does not depend on the choices made. \qed

Following W. Schmid and K. Vilonen we introduce the notions of families of cycles and their limits. Let $\tilde{M}$ be an ambient manifold which we later take equal $T^* M$, and let $I = (0, b)$ be an open interval.

Definition 4 A family of $k$-cycles in $\tilde{M}$ parametrized by $I$ is a $(k+1)$-cycle $C_I$ in $I \times \tilde{M}$ with the following property: for each $s \in I$, there exists a Whitney stratification of $|C_I|$, such that the “slice” $|C_I| \cap \{(s) \times \tilde{M}\}$ is a Whitney stratified subset of $|C_I|$ of dimension at most $k$.

For each $s \in I$, we identify $\tilde{M}$ with $\{s\} \times \tilde{M}$ and we have a specialization map $C_I \rightarrow C_s$, where $C_s$ is a $k$-cycle in $\tilde{M} \simeq \{s\} \times \tilde{M}$ with support lying in $|C_I| \cap \{(s) \times \tilde{M}\}$. The precise definition can be found in [SchV1], but we skip it because this notion is quite intuitive and in this article all families of cycles will be defined explicitly through the specializations $C_s$. Note that if the dimension of $|C_I| \cap \{(s) \times \tilde{M}\}$ is strictly less than $k$, then $C_s = 0$.

Next we define the limit of a family of cycles as the parameter $s \rightarrow 0^+$. Recall that $I$ is an open interval $(0, b)$ and set $J = [0, b)$. We consider a family of $k$-cycles $C_I$ in $\tilde{M}$ subject to an additional condition: the closure $\overline{|C_I|}$ in $J \times \tilde{M}$ admits a Whitney stratification such that $\overline{|C_I|} \cap \{(0) \times \tilde{M}\}$ is a stratified subset of $|C_I|$. Note that $|C_I|$ is a subset of $J \times \tilde{M}$ and the latter is a manifold with boundary, so to make sense out of its Whitney stratification we embed $J \times \tilde{M}$ into $\mathbb{R} \times \tilde{M}$. Then it follows that $\overline{|C_I|} \cap \{(0) \times \tilde{M}\}$ has dimension at most $k$. The $(k+1)$-cycle $C_I$ in $I \times \tilde{M}$ can be regarded as a $(k+1)$-chain in $J \times \tilde{M}$, the boundary of this chain $\partial C_I$ is necessarily supported in $\{0\} \times \tilde{M}$. Since $\{0\} \times \tilde{M} \simeq M$, we regard $\partial C_I$ as a cycle in $\tilde{M}$ and define

$$\lim_{s \rightarrow 0^+} C_s = -\partial C_I.$$ 

The negative sign appears for orientation reasons and ensures that the formal definition of a limit agrees with geometric intuition behind it.

Proposition 5 (Proposition 3.25 in [SchV1]) For all $t \in I$,

$$C_t - \left( \lim_{s \rightarrow 0^+} C_s \right) = \partial C_{(0,t)},$$

where $C_{(0,t)}$ denotes the restriction of $C_I$ to $(0,t) \times \tilde{M}$.

Let $U$ be an open semi-algebraic subset in $M$. We are going to define two pushforward maps of cycles $\mathcal{L}^+(U) \rightarrow \mathcal{L}^+(M)$. By a semi-algebraic function on $M$ we mean a function whose graph is a semi-algebraic subset of $M \times \mathbb{R}$. Then one can find a semi-algebraic function
$f$ on $M$ of class $C^2$ such that $f$ is strictly positive on $U$ and the boundary $\partial U$ is precisely the zero set of $f$ (Proposition 4.22 in [DM]). Let $j$ denote the inclusion map $U \hookrightarrow M$, and let $\Lambda \in \mathcal{L}^+(U)$ be a conic Lagrangian cycle in $T^*U$. For each $s > 0$, we regard $s \frac{df}{f}$ as a section in $T^*U$, it induces two mutually inverse homeomorphisms of $T^*U$ defined fiberwise by:

$$\tau_+ : (\zeta, x) \mapsto (\zeta + s \frac{df}{f}(x), x), \quad \tau_- : (\zeta, x) \mapsto (\zeta - s \frac{df}{f}(x), x).$$

We set

$$\Lambda + s \frac{df}{f} = (\tau_+)_*(\Lambda), \quad \Lambda - s \frac{df}{f} = (\tau_-)_*(\Lambda).$$

**Proposition 6** Under the above hypotheses, the cycles $\Lambda \pm s \frac{df}{f}$ in $T^*U$, regarded as chains in $T^*M$, have no boundary, they form two families of cycles in $T^*M$ parametrized by $(0, \infty)$, and the limits

$$\lim_{s \to 0^+} \left( \Lambda + s \frac{df}{f} \right), \quad \lim_{s \to 0^+} \left( \Lambda - s \frac{df}{f} \right).$$

do not depend on the choice of a semi-algebraic function $f$ on $M$ of class $C^2$ such that $f > 0$ on $U$ and the zero set $\{f = 0\} = \partial U$.

This proposition can be extracted from Section 4 of [SchV1]. The growth of $\frac{df}{f}$ near the boundary of $U$ ensures that $\Lambda \pm s \frac{df}{f}$ are cycles in $T^*M$ as opposed to chains with boundary. The proposition implies that the following two maps are well-defined:

$$j_+ : \mathcal{L}^+(U) \to \mathcal{L}^+(M), \quad \Lambda \mapsto \lim_{s \to 0^+} \left( \Lambda + s \frac{df}{f} \right)$$

and

$$j_1 : \mathcal{L}^+(U) \to \mathcal{L}^+(M), \quad \Lambda \mapsto \lim_{s \to 0^+} \left( \Lambda - s \frac{df}{f} \right).$$

**Example 7** Let $M = \mathbb{R}$ with its standard orientation, and let $U = (0, \infty)$. Take $\Lambda \in \mathcal{L}^+(0, \infty)$ equal the zero section of $T^*U$ oriented the same way $U$ is. This $\Lambda$ is the characteristic cycle of $\mathcal{C}_{(0,\infty)}$ – the constant sheaf on $(0, \infty)$ of rank 1. Note that $\Lambda$, regarded as a chain in $T^*\mathbb{R}$, has non-zero boundary. We can take the defining function of $(0, \infty)$ to be $f(x) = x$, where $x$ is the standard coordinate on $\mathbb{R}$. Then $\Lambda + s \frac{df}{f} = s \frac{dx}{x}$ is a piece of hyperbola $\xi = \frac{x}{2}, x > 0$, oriented so that the direction of increasing $x$ is positive. Evidently, these cycles have no boundary in $T^*\mathbb{R}$. As $s \to 0^+$, these cycles tend to $j_+ (\Lambda)$ which has support $\{\xi > 0\} \cup \{x > 0\}$ and oriented along decreasing $\xi$ and increasing $x$. Similarly, $\Lambda - s \frac{df}{f} = -s \frac{dx}{x}$ is a piece of hyperbola $\xi = -\frac{x}{2}, x > 0$, oriented along increasing $x$. As $s \to 0^+$, these cycles tend to $j_1 (\Lambda)$ which has support $\{\xi < 0\} \cup \{x > 0\}$ and oriented along increasing $\xi$ and increasing $x$.

We will see in a moment that the limit cycles $j_+ (\Lambda)$ and $j_1 (\Lambda)$ are the characteristic cycles of $Rj_+ (\mathcal{C}_{(0,\infty)})$ and $Rj_1 (\mathcal{C}_{(0,\infty)})$ respectively. \qed

**Definition 8** The characteristic cycle is a map

$$Ch : C^b_{\mathbb{R}, c}(M) \to \mathcal{L}^+(M)$$

which is uniquely determined by the following properties:
1. **Normalization**: Let $\mathcal{C}_M$ be the constant sheaf on $M$ of rank 1, then

$$Ch(\mathcal{C}_M) = [M] = \text{zero section of } T^*M \text{ oriented by the fixed orientation of } M.$$  

2. **Additivity**: The map $Ch$ descends to $D^b_{R-c}(M)$ – the bounded derived category of sheaves on $M$ with $R$-constructible cohomology – and is additive on distinguished triangles of $D^b_{R-c}(M)$:

$$Ch(\mathcal{F}) = Ch(\mathcal{F}') + Ch(\mathcal{F}'')$$

whenever there is a distinguished triangle

$$\mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to +1$$

in $D^b_{R-c}(M)$;

3. **Ch Is Local**: For any open semi-algebraic subset $U \subset M$ the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{C}^b_{R-c}(M) & \xrightarrow{Ch} & \mathcal{L}^+(M) \\
\downarrow \quad & & \downarrow \\
\mathcal{C}^b_{R-c}(U) & \xrightarrow{Ch} & \mathcal{L}^+(U),
\end{array}$$

where the left vertical arrow is the restriction map of complexes of sheaves and the right vertical arrow is the restriction map of cycles with infinite support from $T^*M$ to its open subset $T^*U$;

4. **Open Embedding**: For any open semi-algebraic subset $U \subset M$ the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{C}^b_{R-c}(U) & \xrightarrow{Ch} & \mathcal{L}^+(U) \\
R_*j \downarrow & & \downarrow j_* \\
\mathcal{C}^b_{R-c}(M) & \xrightarrow{Ch} & \mathcal{L}^+(M).
\end{array}$$

As was explained in [SchV1], these properties uniquely determine the characteristic cycle map $Ch : \mathcal{C}^b_{R-c}(M) \to \mathcal{L}^+(M)$, however, from this point of view proving its existence becomes a highly non-trivial matter. Below we state more properties of characteristic cycles, starting with a stronger open embedding property.

**Theorem 9 (Open Embedding Theorem 4.2 in [SchV1])** Let $U$ be an open semi-algebraic subset in $M$, and let $f$ be semi-algebraic function on $M$ of class $C^2$ such that $f > 0$ on $U$ and the boundary $\partial U$ is precisely the zero set of $f$. Let $\mathcal{F} \in \mathcal{C}^b_{R-c}(U)$ be a bounded complex of sheaves on $U$ with $R$-constructible cohomology. Then

$$Ch(Rj_*\mathcal{F}) = \lim_{s \to 0^+} \left( Ch(\mathcal{F}) + s \frac{df}{f} \right) = j_*(Ch(\mathcal{F})),$$

$$Ch(Rj!\mathcal{F}) = \lim_{s \to 0^+} \left( Ch(\mathcal{F}) - s \frac{df}{f} \right) = j!(Ch(\mathcal{F})).$$
The Open Embedding Theorem not only provides a means of computing the characteristic cycles of \( R_j^a \mathcal{F} \) and \( R_j^i \mathcal{F} \), but also a way of deforming them. The following observation will play a crucial role in Section 5. It immediately follows from the Open Embedding Theorem and Proposition 5.

**Corollary 10** The following pairs of cycles are homologous:

\[
\text{Ch}(R_j^a \mathcal{F}) \sim \text{Ch}(\mathcal{F}) + \frac{df}{f} \quad \text{and} \quad \text{Ch}(R_j^i \mathcal{F}) \sim \text{Ch}(\mathcal{F}) - \frac{df}{f}.
\]

Moreover, the chains realizing these homology relations can be chosen to lie inside the sets

\[
\bigcup_{0 \leq s \leq 1} \left( \text{Ch}(\mathcal{F}) + s \frac{df}{f} \right) \quad \text{and} \quad \bigcup_{0 \leq s \leq 1} \left( |\text{Ch}(\mathcal{F})| - s \frac{df}{f} \right)
\]

respectively.

Let \( K(D^b_{\mathbb{R}_{\mathbb{C}}}(M)) \) denote the Grothendieck group of \( D^b_{\mathbb{R}_{\mathbb{C}}}(M) \), i.e. the abelian group generated by the objects of \( D^b_{\mathbb{R}_{\mathbb{C}}}(M) \) with one relation \( \mathcal{F} = \mathcal{F}' + \mathcal{F}'' \) for each distinguished triangle \( \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow +1 \) in \( D^b_{\mathbb{R}_{\mathbb{C}}}(M) \). The additivity property of characteristic cycles implies that \( \text{Ch} \) descends to a homomorphism

\[
\text{Ch} : K(D^b_{\mathbb{R}_{\mathbb{C}}}(M)) \rightarrow \mathcal{L}^+(M).
\]

M. Kashiwara and P. Schapira (Theorem 9.7.10 in [KSch]) show that this homomorphism is in fact an isomorphism of abelian groups. In particular, every conic Lagrangian cycle in \( T^*M \) can be realized as the characteristic cycle of some \( \mathcal{F} \in C^b_{\mathbb{R}_{\mathbb{C}}}(M) \). If a group \( G_\mathbb{R} \) acts on \( M \) semi-algebraically and \( \mathcal{F} \in C^b_{\mathbb{R}_{\mathbb{C}}}(M) \) is \( G_\mathbb{R} \)-equivariant (see [KSchm] for the definition), then \( \text{Ch}(\mathcal{F}) \) is \( G_\mathbb{R} \)-invariant. Furthermore,

\[
\langle \mu_\mathbb{R}(\zeta), X \rangle = -\langle \zeta, X_M \rangle = 0 \quad \text{for all } \zeta \in |\text{Ch}(\mathcal{F})|, \ X \in g_\mathbb{R},
\]

where the vector field \( X_M \) and the real moment map \( \mu_\mathbb{R} \) are defined in Section 2. Conversely, every \( G_\mathbb{R} \)-invariant cycle \( \Lambda \in \mathcal{L}^+(M) \) can be realized as the characteristic cycle of some \( G_\mathbb{R} \)-equivariant \( \mathcal{F} \in C^b_{\mathbb{R}_{\mathbb{C}}}(M) \).

**Theorem 11** (Hopf Index Theorem (Corollary 9.5.2 in [KSch])) Suppose that a complex of sheaves \( \mathcal{F} \in C^b_{\mathbb{R}_{\mathbb{C}}}(M) \) has compact support, then the Euler characteristic of \( M \) with respect to \( \mathcal{F} \)

\[
\chi(M, \mathcal{F}) = \#([M] \cap \text{Ch}(\mathcal{F})),
\]

where the right hand side denotes the intersection number between the cycles \([M]\) and \( \text{Ch}(\mathcal{F}) \).

For \( k \in \mathbb{Z} \), let \( \mathcal{F}[k] \) denote the complex \( \mathcal{F} \) with degrees shifted by \( k \), then \( \text{Ch}(\mathcal{F}[k]) = (-1)^k \text{Ch}(\mathcal{F}) \). Let \( \mathbb{D}_M : D^b_{\mathbb{R}_{\mathbb{C}}}(M) \rightarrow D^b_{\mathbb{R}_{\mathbb{C}}}(M) \) denote the Verdier duality operator, then \( \text{Ch}(\mathbb{D}_M(\mathcal{F})) = \text{Ch}(\mathcal{F})^a \), where \( a : T^*M \rightarrow T^*M \) is the antipodal map \( \zeta \mapsto -\zeta \) and \( \text{Ch}(\mathcal{F})^a \) denotes the image of \( \text{Ch}(\mathcal{F}) \) under this map.

If \( f : M \rightarrow N \) is a proper map of real semi-algebraic manifolds, there is an explicit description of the effect on characteristic cycles by the pushforward map \( Rf_* : D^b_{\mathbb{R}_{\mathbb{C}}}(M) \rightarrow D^b_{\mathbb{R}_{\mathbb{C}}}(N) \). Similarly, if \( f : M \rightarrow N \) is a map of real semi-algebraic manifolds, \( \mathcal{G} \in D^b_{\mathbb{R}_{\mathbb{C}}}(N) \) and \( f \) is “normally non-singular with respect to \( \mathcal{G} \)” (a transversality condition on the induced map \( df : T^*N \rightarrow T^*M \) and the stratification \( \mathcal{S} \) of \( M \) making the cohomology of \( \mathcal{G} \) constant), there is an explicit description of \( \text{Ch}(Rf^*(\mathcal{G})) \). (See, for instance, [KSch], [SchV1]).
4 Statement of the Main Result

Let $G_C$ be a connected complex algebraic reductive group which is defined over $\mathbb{R}$, and let $G_\mathbb{R}$ be a subgroup of $G_C$ lying between the group of real points $G_C(\mathbb{R})$ and the identity component $G_C(\mathbb{R})^0$. We regard $G_\mathbb{R}$ as a real reductive Lie group. Let $\mathfrak{g}_C$ and $\mathfrak{g}_\mathbb{R}$ be their respective Lie algebras. We pick another subgroup $U_\mathbb{R}$ of $G_C$ such that, letting $\mathfrak{u}_\mathbb{R}$ be the Lie algebra of $U_\mathbb{R}$, we have an isomorphism $\mathfrak{u}_\mathbb{R} \otimes_\mathbb{R} \mathbb{C} \simeq \mathfrak{g}_C$. In the applications we have in mind we will choose $U_\mathbb{R}$ to be a compact real form of $G_C$ (i.e. a maximal compact subgroup of $G_C$), but we do not require $U_\mathbb{R}$ to be compact for now; for instance, $U_\mathbb{R}$ may equal $G_\mathbb{R}$. Let $M$ be a smooth complex projective variety of dimension $n$ with a complex algebraic $G_C$-action on it. We denote by $\Omega^{(p,q)}(M)$ the space of complex-valued differential forms of type $(p,q)$ on $M$.

Let $T^*M$ be the holomorphic cotangent bundle of $M$, and let $\pi : T^*M \to M$ denote the projection map. Let $\sigma$ denote the canonical complex algebraic holomorphic symplectic form on $T^*M$ defined similarly to the form $\sigma_\mathbb{R}$ from Example 1. The action of the Lie group $G_C$ on $M$ naturally extends to $T^*M$. Then we always have a canonical equivariantly closed form on $T^*M$, namely, $\mu + \sigma$. Here $\mu : \mathfrak{g}_C \to \mathcal{C}^\infty(T^*M)$ is the moment map defined by:

$$\mu(X) : \zeta \mapsto -\langle \zeta, X_M \rangle, \quad X \in \mathfrak{g}_C, \quad \zeta \in T^*M. \tag{2}$$

Remark 12 If $M$ is a complex manifold and $M^\mathbb{R}$ is the underlying real analytic manifold, then there are at least two different but equally natural ways to identify the real cotangent bundle $T^*(M^\mathbb{R})$ with the holomorphic cotangent bundle $T^*M$ of the complex manifold $M$. We use the convention (11.1.2) of [KSch], Chapter XI; the same convention is used in [L1], [L2] and [SchV2]. Under this convention, if $\sigma_\mathbb{R}$ is the canonical real symplectic form on $T^*M^\mathbb{R}$ described in Example 1 and $\sigma$ is the canonical complex symplectic form on $T^*M$, then $\sigma_\mathbb{R}$ gets identified with $2 \text{Re} \sigma$. (And $\mu_\mathbb{R} = 2 \text{Re} \mu$.)

In this article we consider integrals over Borel-Moore homology cycles $\Lambda$ in $T^*M$ (with coefficients in $\mathbb{Z}$) which satisfy the following three properties:

- $\Lambda$ is real-Lagrangian, i.e. $\dim \mathbb{R} \Lambda = \dim \mathbb{R} M$ and there exists a locally finite semi-algebraic Whitney stratification $S$ of $M^\mathbb{R}$ such that, regarding $\Lambda$ as a cycle in $T^*(M^\mathbb{R})$ via the identification with $T^*M$, the support of $\Lambda$ lies in $\bigcup_{S \in S} T_S^*M$;

- $\Lambda$ is conic, i.e. invariant under the scaling action of positive reals $\mathbb{R}_{>0}$ on $T^*M$ (but not necessarily under the actions of $\mathbb{C}^\times$ or $\mathbb{R}^\times$);

- $\Lambda$ is $G_\mathbb{R}$-invariant.

We denote the abelian group of such cycles by $\mathcal{L}_G(\mathbb{R})(M)$. Note that the Lagrangian condition together with $G_\mathbb{R}$-equivariance imply $\text{Re} \sigma|_\Lambda \equiv 0$ and $\mu(\|\Lambda\|) \subset i\mathfrak{g}_\mathbb{R}^\mathbb{R}$. As was mentioned earlier, given any $\Lambda \in \mathcal{L}_G(\mathbb{R})(M)$, there exists a $G_\mathbb{R}$-equivariant complex of sheaves $\mathcal{F} \in \mathcal{C}_G^b(M)$ such that $\Lambda = Ch(\mathcal{F})$. The reason for restricting ourselves to the conic Lagrangian cycles in $T^*M$ was explained in Section 1.

Example 13 Consider $G_\mathbb{R} = GL(l, \mathbb{R}) \subset GL(l, \mathbb{C}) = G_C$ acting naturally on a complex Grassmanian $Gr_C(k, l)$. Let $N$ be the real Grassmanian $Gr_\mathbb{R}(k, l) \subset Gr_C(k, l)$ and $\Lambda = T^*_{Gr_\mathbb{R}(k, l)}Gr_C(k, l)$ equipped with some orientation. \[\square\]

Conditions 14 We consider forms $\alpha : \mathfrak{g}_C \to \Omega^*(M)$ which satisfy the following three conditions:
1. The assignment $X \mapsto \alpha(X) \in \Omega^*(M)$ depends holomorphically on $X \in \mathfrak{g}_\mathbb{C}$;

2. For each $k \in \mathbb{N}$ and each $X \in \mathfrak{g}_\mathbb{C}$,

$$\alpha(X)[2k] \in \bigoplus_{p + q = 2k, p \geq q} \Omega^{(p,q)}(M);$$

(3)

3. The restriction of $\alpha$ to $\mathfrak{u}_\mathbb{R}$ is an equivariantly closed form with respect to $U_\mathbb{R}$.

**Example 15** A $U_\mathbb{R}$-equivariant characteristic form $\alpha : \mathfrak{u}_\mathbb{R} \to \Omega^*(M)$ (defined in Section 7.1 of [BGV]) satisfies the third condition. Since it depends on $X \in \mathfrak{u}_\mathbb{R}$ polynomially, $\alpha$ extends uniquely to a map $\alpha : \mathfrak{g}_\mathbb{C} \to \Omega^*(M)$ so that the first condition is satisfied. Finally, for each $X \in \mathfrak{g}_\mathbb{C}$,

$$\alpha(X) \in \bigoplus_k \Omega^{(k,k)}(M),$$

so that the second condition is satisfied too. This is the most important class of forms satisfying Conditions 14. □

We regard $M$ as a submanifold of $T^*M$ via the zero section inclusion. We consider the form

$$\tilde{\alpha}(X) = e^{\mu(X)+\sigma} \wedge \pi^*(\alpha(X)), \quad X \in \mathfrak{g}_\mathbb{C}.$$

The restriction of $\tilde{\alpha}(X)$ to $M$ is just $\alpha(X)$. We will see later that, in a way, $\tilde{\alpha}$ is the most natural equivariant extension of $\alpha$ to $T^*M$. To avoid cumbersome notations, we denote the image of an element $\beta \in \Omega^*(M)$ under the inclusion $\pi^* : \Omega^*(M) \hookrightarrow \Omega^*(T^*M)$ by $\beta$ as well instead of $\pi^*(\beta)$. Thus

$$\tilde{\alpha}(X) = e^{\mu(X)+\sigma} \wedge \alpha(X).$$

Recall that $n = \dim_{\mathbb{C}} M$, so that the cycle $\Lambda \in \mathcal{L}^+_{G_\mathbb{R}}(M)$ has dimension $2n$.  

**Lemma 16** For each $X \in \mathfrak{g}_\mathbb{C}$, the form $\tilde{\alpha}(X)[2n]$ is closed.  

*Proof.* First we show that $\tilde{\alpha}(X)[2n+2] = 0$. Indeed,

$$\tilde{\alpha}(X)[2n+2] = e^{\mu(X)} \sum_{k=0}^{k=n+1} \frac{1}{(n-k+1)!} \sigma^{n-k+1} \wedge \alpha(X)[2k],$$

so it suffices to show that each term $\sigma^{n-k+1} \wedge \alpha(X)[2k] = 0$. But this follows from (3) and an observation

$$\sigma^{n-k+1} \wedge \Omega^{(p,q)}(M) = 0 \quad \text{for} \quad p \geq k.$$

The restriction of $\tilde{\alpha}$ to $\mathfrak{u}_\mathbb{R}$ is equivariantly closed with respect to the action of $U_\mathbb{R}$ on $T^*M$ for the reason that it is “assembled” from $U_\mathbb{R}$-equivariantly closed forms. Hence $\tilde{\alpha}(X)[2n]$ is closed for all $X \in \mathfrak{u}_\mathbb{R}$. But since $d\tilde{\alpha}(X)$ depends on $X \in \mathfrak{g}_\mathbb{C}$ holomorphically, $d\tilde{\alpha}(X)[2n] = 0$ for all $X \in \mathfrak{g}_\mathbb{C}$. □

If $\varphi$ is a smooth compactly supported differential form on $\mathfrak{g}_\mathbb{R}$ of top degree, then we define its Fourier transform as in [L1], [L2] and [SchV2]:

$$\hat{\varphi}(\xi) = \int_{\mathfrak{g}_\mathbb{R}} e^{\langle X, \xi \rangle} \varphi(X), \quad X \in \mathfrak{g}_\mathbb{R}, \xi \in \mathfrak{g}_\mathbb{C}^*,$$

(4)
without the customary factor of $i = \sqrt{-1}$ in the exponent.

Similarly we define $\widehat{\phi} : \mathfrak{g}_C^* \to \Omega^*(M)$:

$$\widehat{\phi}(\xi) = \int_{\mathfrak{g}_R} e^{\langle X, \xi \rangle} \varphi(X) \wedge \alpha(X), \quad X \in \mathfrak{g}_R, \ \xi \in \mathfrak{g}_C^*,$$

where $\varphi(X) \wedge \alpha(X)$ is a form on $\mathfrak{g}_R \times M$. For each $\xi \in \mathfrak{g}_C^*$, the form $\widehat{\phi}(\xi)$ belongs to $\Omega^*(M)$ and decays rapidly as $\xi \to \infty, \xi \in i\mathfrak{g}_R^*$.

We can regard the moment map (2) as a map $\mu : T^*M \to \mathfrak{g}_C^* \times M$ via

$$\mu(\zeta) : X \mapsto -\langle \zeta, X \rangle, \quad X \in \mathfrak{g}_C, \ \zeta \in T^*M.$$ (5)

Abusing notation we denote by $\mu^*(\widehat{\phi}) \in \Omega^*(T^*M)$ the pullback of $\widehat{\phi} \in \Omega^*(\mathfrak{g}_C^* \times M)$ via the composition of

$$T^*M \hookrightarrow T^*M \times M \quad \text{and} \quad T^*M \times M \to \mathfrak{g}_C^* \times M \\
\zeta \mapsto (\zeta, \pi(\zeta)) \quad \text{and} \quad (\zeta, x) \mapsto (\mu(\zeta), x).$$

Then

$$\mu^*(\widehat{\phi}) = \int_{\mathfrak{g}_R} e^{\langle X, \mu(\zeta) \rangle} \varphi(X) \wedge \alpha(X), \quad \zeta \in T^*M, \ X \in \mathfrak{g}_R.$$ (6)

We will be studying integrals of the kind

$$\int_{\Lambda} \mu^*(\widehat{\phi}) \wedge e^\sigma = \int_{\Lambda} \left( \int_{\mathfrak{g}_R} e^{\langle X, \mu(\zeta) \rangle + \sigma} \wedge \varphi(X) \wedge \alpha(X) \right) = \int_{\Lambda} \left( \int_{\mathfrak{g}_R} \widehat{\phi} \wedge \varphi(X) \right).$$ (6)

Of course, the cycle $\Lambda$ being infinite it is not clear at all whether this integral converges. We denote by

$$\text{supp}(\sigma|_{\Lambda})$$

the closure in $T^*M$ of the set of smooth points of the support $|\Lambda|$ where $\sigma|_{\Lambda} \neq 0$.

**Lemma 17** If the moment map $\mu : T^*M \to \mathfrak{g}_C^*$ is proper on the set $\text{supp}(\sigma|_{\Lambda})$, then the integral (6) converges. In particular, the integral (6) converges if the moment map $\mu$ is proper on the support $|\Lambda|$.

**Proof.** Note that $M$ is compact, so the only unbounded directions of $\Lambda$ are those along the fibers of $T^*M \to M$. We fix any norm $\|\cdot\|_{\mathfrak{g}_C^*} \text{ on } \mathfrak{g}_C^*$. For $R > 0$ we denote by $B_R$ the open ball of radius $R$ in $\mathfrak{g}_C^*$:

$$B_R = \{ \xi \in \mathfrak{g}_C^* : \|\xi\|_{\mathfrak{g}_C^*} < R \}$$ (7)

and by $\overline{B_R}$ its closure in $\mathfrak{g}_C^*$. We already know that

$$\widehat{\phi}(\xi) = \int_{\mathfrak{g}_R} e^{\langle X, \xi \rangle} \varphi(X) \wedge \alpha(X)$$

decays rapidly as $\|\xi\|_{\mathfrak{g}_C^*} \to \infty, \xi \in i\mathfrak{g}_R^*$.

Since the cycle $\Lambda$ is $G_R$-invariant, $\mu(|\Lambda|) \subseteq i\mathfrak{g}_R^*$. On the other hand, $\mu$ being proper on $\text{supp}(\sigma|_{\Lambda})$ implies that the set $\text{supp}(\sigma|_{\Lambda}) \cap \mu^{-1}(\overline{B_R})$ is compact. Since the cycle $\Lambda$ is conic along the fibers of $T^*M \to M$ and the integrand decays rapidly on $\text{supp}(\sigma|_{\Lambda})$ along those fibers as $R \to \infty$, it is clear that the limit

$$\lim_{R \to \infty} \int_{\Lambda \cap \mu^{-1}(B_R)} \mu^*(\widehat{\phi}) \wedge e^\sigma = \lim_{R \to \infty} \int_{\Lambda \cap (M \cup \text{supp}(\sigma|_{\Lambda})) \cap \mu^{-1}(B_R)} \mu^*(\widehat{\phi}) \wedge e^\sigma$$

is finite. □
Example 18 The condition of the lemma is automatically satisfied if the support $|Λ| = M$ (which happens when $Λ = Ch(\mathcal{C}_M)$, where $\mathcal{C}_M$ is the constant sheaf on $M$ of rank 1).

This condition is also satisfied when $M$ is a homogeneous space $G_C/P_C$, where $P_C \subset G_C$ is a parabolic subgroup, and $Λ \in L^+_G(M)$ is any cycle at all. \[
\]

Integrals of this kind generalize the integral character formula due to W. Schmid and K. Vilonen [SchV2] for representations of $G_\mathbb{R}$ constructed from a $G_\mathbb{R}$-equivariant sheaf $\mathcal{F}$. In that character formula the manifold $M$ is the flag variety $\mathcal{B}$ of $g_C$, $Λ = Ch(\mathcal{F})$, and the integrand is the pullback of a naturally defined form on a complex coadjoint orbit to $T^*\mathcal{B}$ via the “twisted moment map” and can be be put into the shape $\tilde{\alpha}$.

Let $T_C$ be a maximal complex torus contained in $G_C$, i.e. $T_C$ is a maximal subgroup of $G_C$ isomorphic to $\mathbb{C}^n$ for representations of $G$ isomorphic to $\mathbb{C}$. Since all the maximal tori of $G_C$ are conjugate, if this assumption holds for one torus $T_C$ then it holds for all maximal tori.

We denote by $g_C^{rs}$ the set of regular semisimple elements in $g_C$. These are elements $X \in g_C$ such that the adjoint action of $ad(X)$ on $g_C$ is diagonalizable and has maximal possible rank. We also denote by $g_\mathbb{R}^{rs} = g_\mathbb{R} \cap g_C^{rs}$ the set of regular semisimple elements of $g_\mathbb{R}$. It is an open and dense subset of $g_\mathbb{R}$.

For a regular semisimple element $X \in g_C^{rs}$ we set $t_C(X) \subset g_C$ to be the unique Cartan subalgebra of $g_C$ containing $X$ and $T_C(X) = \exp(t_C(X))$ to be the corresponding maximal torus. Let $p \in M$ be a point fixed by $T_C(X)$, then the complex Lie action $X \mapsto L(X_M)X = [X_M, X]$ on the holomorphic vector fields $X$ of $M$ gives rise to a linear transformation $L_p^C$ on $T_pM$. We define a function

$$\text{Den}_p(X) = \det(L_p^C)$$

which will appear in the denominator of the contribution of $p \in M_0(X)$ to the localization formula.

We will use the following description of $\text{Den}_p(X)$ which can serve as an alternative definition. The maximal torus $T_C(X)$ acts linearly on $T_pM$. Thus $T_pM$, as a representation of $T_C(X)$, decomposes into a direct sum of one-dimensional representations

$$\mathbb{C}_{\beta_{p,1}} \oplus \cdots \oplus \mathbb{C}_{\beta_{p,n}}, \quad \beta_{p,1}, \ldots, \beta_{p,n} \in t_C(X)^*,$$

where the action of $T_C(X)$ on the one-dimensional complex vector space $\mathbb{C}_{\beta_{p,k}}$ is given by

$$\exp(Y) \cdot v = e^{\beta_{p,k}(Y)}v, \quad Y \in t_C(X), \quad v \in \mathbb{C}_{\beta_{p,k}}.$$

The set of weights $\{\beta_{p,1}, \ldots, \beta_{p,n}\}$ is determined uniquely up to permutation. Then we have

$$\text{Den}_p(X) = \beta_{p,1}(X) \cdots \beta_{p,n}(X).$$

Notice that if the eigenvalues of $ad(X)$ are all purely imaginary (that is $X$ lies in the Lie algebra of a compact subgroup of $G_C$), then we have the following relationship:

$$\text{Den}_p(X) = i^{n} \cdot \det^{1/2}(L_p).$$

We let $\Delta(X)$ denote the set of all weights that occur this way:

$$\Delta(X) = \{\beta_{p,k} \in t_C(X)^*; \beta_{p,k} \text{ appears in the weight decomposition} \}$$

$$\hspace{1cm} T_pM \simeq \mathbb{C}_{\beta_{p,1}} \oplus \cdots \oplus \mathbb{C}_{\beta_{p,k}} \oplus \cdots \oplus \mathbb{C}_{\beta_{p,n}} \text{ for some } p \in M_0(X).$$
It is a finite subset of $t_C(X)^* \setminus \{0\}$.

For instance, if $M$ is the flag variety of $g_C$, then $\Delta(X)$ is the root system of $g_C(X)$ corresponding to the choice of Cartan algebra $t_C(X)$.

Let $g'_R$ denote the set of regular semisimple elements $X \in g'_{R}$ which satisfy the following additional properties. If $t_R(X) \subset g_R$ and $t_C(X) \subset g_C$ are the unique Cartan subalgebras in $g_R$ and $g_C$ respectively containing $X$, then:

1. The set of zeroes $M_0(X)$ is exactly the set of points in $M$ fixed by the complex torus $T_C(X) = \exp(t_C(X)) \subset G_C$;
2. $\beta(X) \neq 0$ for all $\beta \in \Delta(X) \subset t_C(X)^*$;
3. For each $\beta \in \Delta(X)$, we have either
   \[
   \text{Re}(\beta)|_{t_R(X)} \equiv 0 \quad \text{or} \quad \text{Re}(\beta(X)) \neq 0.
   \] (8)

Clearly, $g'_R$ is an open subset of $g_R$; since $M$ is compact and $\Delta(X)$ is finite, the complement of $g'_R$ in $g_R$ has measure zero; and $\text{Den}_p(X) \neq 0$ for all $X \in g'_R$.

The contribution to the integral of each zero $p \in M_0(X)$ will be counted with some multiplicity $m_p \in \mathbb{Z}$ which we describe next. We use the Bialynicki-Birula decomposition as restated in Theorem 2.4.3 in [ChG]. Let $C^\times$ be a subgroup of $G_C$ such that the set of fixed points $M^{C^\times}$ in $M$ is finite. We embed $C^\times$ into $C$ in the most natural way so that $C^\times = C \setminus \{0\}$. For each fixed point $p \in M^{C^\times}$ we define the attracting set

\[
O_p = \{x \in M; \lim_{z \to 0} z^{-1} \cdot x = p\}.
\]

Clearly $p$ is the only point in $O_p$ fixed by $C^\times$. There is also a natural $C^\times$-action on the tangent space $T_pM$. It decomposes into a direct sum

\[
T_pM = T_p^- M \oplus T_p^+ M,
\] (9)

where

\[
T_p^- M = \bigoplus_{k<0, k \in \mathbb{Z}} T_p M(k), \quad T_p^+ M = \bigoplus_{k>0, k \in \mathbb{Z}} T_p M(k),
\]

Then we get the Bialynicki-Birula decomposition of $M$ into attracting sets $O_p$, each isomorphic to $T_p^- M$:

**Theorem 19 (Bialynicki-Birula Decomposition [BB])**

1. The attracting sets form a decomposition

\[
M = \coprod_{p \in M^{C^\times}} O_p
\]

into smooth locally closed algebraic varieties;

2. There are natural isomorphisms of algebraic varieties

\[
O_p \simeq T_p(O_p) \simeq T_p^- M
\] (10)

which commute with the $C^\times$-action.
Now let $X \in g'_R$, and let $t_C(X)$ and $T_C(X) = \exp(t_C(X))$ be the corresponding complex Cartan subalgebra and subgroup respectively. Pick any $X' \in t_R(X) \cap g'_R$ in the same connected component of $t_R(X) \cap g'_R$ as $X$ and such that

$$\text{Re} \beta(X) > 0 \iff \text{Re} \beta(X') > 0 \quad \text{and} \quad \text{Re} \beta(X) < 0 \iff \text{Re} \beta(X') < 0$$

for all $\beta \in \Delta(X)$, and the complex 1-dimensional subspace $\{tX'; t \in \mathbb{C}\} \subset g_C$ is the Lie algebra of a closed algebraic subgroup $\mathbb{C}^\times(X') \subset G_C$ isomorphic to $\mathbb{C}^\times$. Fix an isomorphism $\mathbb{C}^\times(X') \simeq \mathbb{C}^\times$ so that the induced isomorphism of Lie algebras $\{tX'; t \in \mathbb{C}\} \simeq \mathbb{C}$ sends $X'$ into an element with nonnegative real part. We apply Theorem 19 to $\mathbb{C}^\times(X')$. Then the set of points in $M$ fixed by $\mathbb{C}^\times(X')$ is just $M_0(X') = M_0(X) = \{x_1, \ldots, x_d\}$, say. Let $O_k \subset M$ denote the attracting set of $x_k$ (instead of $O_{x_k}$).

For instance, if $M$ is the flag variety of $g_C$, then the sets $O_1, \ldots, O_d$ are the orbits of a suitably chosen Borel subgroup containing $T_C(X)$, and the number of orbits $d$ equals the order of the Weyl group of $g_C$.

Since $\mathbb{C}^\times(X')$ is a subgroup of the torus $T_C(X)$, their actions commute, and the action of $T_C(X)$ preserves each $O_k$. Moreover, the proof of Theorem 19 shows that the isomorphism of varieties (10) is $T_C(X)$-equivariant. In particular, the direct sum decomposition (9) is a decomposition of $T_C(X)$-representations.

We define the multiplicity of a complex of sheaves $F \in \mathcal{C}^b_{\mathbb{R} - c} M$ at $x_k$ to be the Euler characteristic

$$m_k(X) = \chi\left( R\Gamma_{\{x_k\}}(F|_{O_k}) \right) = \chi\left( \bigoplus_{j \in \{x_k\} \rightarrow O_k} (F|_{O_k}) \right).$$

The number $m_k(X)$ is an integer which is exactly the local contribution of $x_k$ to the Lefschetz fixed point formula, as generalized to sheaf cohomology by M. Goresky and R. MacPherson [GM].

Now we are ready to state the main result of this article.

**Theorem 20** Let $G_C$ act complex algebraically on a smooth complex projective variety $M$ so that some (hence any) maximal torus $T_C \subset G_C$ acts with isolated fixed points. Suppose that a map $\alpha : g_C \rightarrow \Omega^*(M)$ satisfies Conditions 14. And let $\Lambda \in L^+_\mathcal{X}(M)$ be a $G_\mathbb{R}$-equivariant conic real-Lagrangian cycle in $T^*M$ such that the holomorphic moment map $\mu : T^*M \rightarrow g_C^*$ is proper on the set $\text{supp}(\sigma|_\Lambda)$. Then, if $\varphi$ is a smooth compactly supported differential form on $g'_R$ of top degree,

$$\int_{\Lambda} \mu^*(\varphi \alpha) \wedge e^\sigma = \int_{g'_R} F_\alpha \varphi,$$

where $F_\alpha$ is an $\text{Ad}(G_\mathbb{R} \cap U_\mathbb{R})$-invariant function on $g'_R$ given by the formula

$$F_\alpha(X) = (-2\pi i)^n \sum_{k=1}^d m_k(X) \frac{\alpha(X)|_0 (x_k)}{\text{Den}_{x_k}(X)},$$

where $n = \dim_{\mathbb{C}}(M)$, $\{x_1, \ldots, x_d\} = M_0(X)$ is the set of zeroes of the vector field $X_M$ on $M$, and $m_k(X)$'s are certain integer multiplicities.

To specify the multiplicities, let $F \in \mathcal{C}^b_{\mathbb{R} - c}(M)$ be a bounded complex of $G_\mathbb{R}$-equivariant sheaves on $M$ with $\mathbb{R}$-constructible cohomology such that $\text{Ch}(F) = \Lambda$, then the multiplicities are determined by the formula (11).

We extend the function $F_\alpha$ by zero to a measurable function on $g_\mathbb{R}$. If $F_\alpha$ happens to be locally integrable with respect to the Lebesgue measure on $g_\mathbb{R} \simeq \mathbb{R}^\dim_{\mathbb{C}} g_\mathbb{R}$, then the equation (12) holds for smooth differential forms $\varphi$ of top degree which are compactly supported on $g_\mathbb{R}$ (and not necessarily on $g'_R$).
We divide the argument into two parts and give the proof in sections 5 and 6. We can say more about the multiplicities \( m_k(X) \):

**Proposition 21** For each \( X \in \mathfrak{g}_R \) and each bounded complex of \( G_R \)-equivariant sheaves \( F \in C^b_{\mathbb{R} - c}(M) \) with \( \mathbb{R} \)-constructible cohomology, the multiplicities defined by the local formula (11) can also be given by a global formula

\[
m_k(X) = \chi(M, F^\Omega_{O_k}) = \chi(M, (j_{O_k} \mapsto M! \circ (j_{O_k} \mapsto M)^*)(F)).
\]

Moreover, these multiplicities depend on \( \text{Ch}(F) \) only and not on the complex \( F \).

**Remark 22** In the special case when \( \lambda \) equals \( M \) as oriented cycles, \( \text{Ch}(F) \) is \( U_R \)-invariant, each multiplicity \( m_k(X) \) equals 1 and this theorem can be easily deduced from the classical Berline-Vergne localization formula (Theorem 2).

**Remark 23** Notice that the cycle \( \Lambda \) is invariant with respect to the action of the group \( G_R \) which need not be compact, while the form \( \alpha : \mathfrak{g}_C \to \Omega^*(M) \) is required to be equivariant with respect to a different group \( U_R \) only, and \( U_R \) may not preserve the cycle \( \Lambda \).

The condition of the theorem that the moment map \( \mu \) is proper on the support of the characteristic cycle \( |\Lambda| \) is automatically satisfied when \( \mu \) is proper on the support of the characteristic cycle \( |\Lambda| \).

**Remark 24** Let \( Z(\mathcal{U}(\mathfrak{g}_R)) \) denote the center of the universal enveloping algebra of \( \mathfrak{g}_R \). It is canonically isomorphic to the algebra of conjugate-invariant constant coefficient differential operators on \( \mathfrak{g}_R \). Suppose, in addition, that the distribution \( \Delta \) on \( \mathfrak{g}_R \) defined by

\[
\Delta : \varphi \mapsto \int_{\Lambda} \mu^*(\varphi_\Lambda) \wedge e^\sigma
\]

is \( \text{Ad}(G_R) \)-invariant and is an eigendistribution for \( Z(\mathcal{U}(\mathfrak{g}_R)) \) (i.e. each element of \( Z(\mathcal{U}(\mathfrak{g}_R)) \) acts on \( \Delta \) by multiplication by some scalar). Such situation arises in [SchV2], [L1] and [L2] where the distribution \( \Delta \) is the character of some virtual representation of \( G_R \). Then by Harish-Chandra’s regularity theorem ([HC] or Theorem 3.3 in [A]), the function \( F_\alpha \) from Theorem 20 is an \( \text{Ad}(G_R) \)-invariant, locally \( L^1 \) function on \( \mathfrak{g}_R \) which is represented by a real analytic function on the set of regular semisimple elements \( \mathfrak{g}^{ss}_R \). Hence by the second part of Theorem 20,

\[
\Delta(\varphi) = \int_{\Lambda} \mu^*(\varphi_\Lambda) \wedge e^\sigma = \int_{\mathfrak{g}_R} F_\alpha \varphi
\]

as distributions on \( \mathfrak{g}_R \).

## 5 Deformation of \( \text{Ch}(F) \) in \( T^*M \)

In this section \( F \in C^b_{\mathbb{R} - c}(M) \) is a bounded complex of \( G_R \)-equivariant sheaves on \( M \) with \( \mathbb{R} \)-constructible cohomology and \( \Lambda = \text{Ch}(F) \). Recall that \( B_R \) is an open ball in \( \mathfrak{g}^*_C \) defined by (7). We rewrite the integral (6) as

\[
\int_{\text{Ch}(F)} \mu^*(\varphi_\Lambda) \wedge e^\sigma = \int_{\text{Ch}(F)} \left( \int_{\mathfrak{g}_R} e^{(X,\mu(\zeta))} \varphi(X) \wedge \alpha(X) \wedge e^\sigma \right) = \lim_{R \to \infty} \int_{\mathfrak{g}_R \times (\text{Ch}(F) \cap \mu^{-1}(B_R))} e^{(X,\mu(\zeta))} \varphi(X) \wedge \alpha(X) \wedge e^\sigma.
\]
(Of course, the orientation on $\mathfrak{g}_R' \times (Ch(\mathcal{F}) \cap \mu^{-1}(B_R))$ is induced by the product orientation on $\mathfrak{g}_R \times Ch(\mathcal{F})$.) We will interchange the order of integration: integrate over the characteristic cycle first and only then perform integration over $\mathfrak{g}_R$. By Lemma 16 the integrand in (14) is a closed differential form.

In this section we start with an element $X \in \mathfrak{g}_R'$ and the characteristic cycle $Ch(\mathcal{F})$ of a $G_\mathbb{R}$-equivariant complex of sheaves $\mathcal{F}$ on the projective variety $M$ and use general results of Section 4 in [L1] to deform $Ch(\mathcal{F})$ into a cycle of the form

$$m_1(X)T^*_x M + \cdots + m_d(X)T^*_x M,$$

where $m_1(X), \ldots, m_d(X)$ are the integer multiplicities given by the equations (11) and (13), $x_1, \ldots, x_d$ are the zeroes of the vector field $X_M$ on $M$, and each cotangent space $T^*_x M$ is given some orientation. Moreover, to ensure good behavior of our integral (14), we will stay during the process of deformation inside the set

$$\{\zeta \in T^* M; \text{Re}(\langle X, \mu(\zeta) \rangle) \leq 0\}.$$ (15)

The precise result is stated in Proposition 31. This deformation will help us to calculate the integral (14).

Let $X \in \mathfrak{g}_R'$, and let $\{x_1, \ldots, x_d\} = M_0(X)$ be the set of zeroes of the vector field $X_M$ on $M$. Let $t_C(X) \subseteq G_C$ and $t_R(X) \subseteq G_R$ be the corresponding Cartan subalgebras, and let $T_C(X) = \exp(t_C(X)) \subset G_C$ and $T_R(X) = \exp(t_R(X)) \subset G_R$ be the corresponding connected subgroups. Note that because we require $T_R(X)$ to be connected that may not be a Cartan subgroup of $G_\mathbb{R}$.

As a representation of $T_C(X)$, the tangent space $T^*_x M$ at each zero $x_k$ decomposes into the direct sum (9). The space $T^*_x M$ in turn decomposes into a direct sum of one-dimensional representations:

$$T^*_x M \cong \mathbb{C}\beta_{x_k,i_1} + \cdots + \mathbb{C}\beta_{x_k,i_m}, \quad \{\beta_{x_k,i_1}, \ldots, \beta_{x_k,i_m}\} \subset \{\beta_{x_k,1}, \ldots, \beta_{x_k,n}\}.$$ (16)

By construction,

$$\text{Re} \beta_{x_k,i}(X) < 0 \Rightarrow \beta_{x_k,i} \in \{\beta_{x_k,i_1}, \ldots, \beta_{x_k,i_m}\},$$

$$\text{Re} \beta_{x_k,i}(X) > 0 \Rightarrow \beta_{x_k,i} \notin \{\beta_{x_k,i_1}, \ldots, \beta_{x_k,i_m}\}.$$ (17)

Choose a linear coordinate $z_i : \mathbb{C}\beta_{x_k,i} \xrightarrow{i} \mathbb{C}$ and define an inner product $\langle \cdot, \cdot \rangle_k$ on $T^*_x M$ by

$$\langle (z_1, \ldots, z_n), (z'_1, \ldots, z'_n) \rangle_k = z_1z'_1 + \cdots + z_nz'_n.$$ (19)

Let $\|\cdot\|_k$ be the respective norm on $T^*_x M$:

$$\|(z_1, \ldots, z_n)\|_k = |z_1|^2 + \cdots + |z_n|^2.$$ (20)

Then, using the Bialynicki-Birula decomposition as stated in Theorem 19, we obtain a decomposition of $M$ into smooth locally closed algebraic varieties:

$$M = \prod_{k=1}^d O_k,$$

where each $O_k$ is the attracting set of $x_k$, and we denote by

$$\psi_{X,k} : T^*_x M \xrightarrow{i} O_k$$ (16)
the $T_\mathbb{C}(X)$-equivariant isomorphism of varieties (10).

**Remark 25** Suppose $\mathcal{G}$ is a complex of sheaves on $M$ and $Z$ is a locally closed subset of $M$. Let $i : Z \hookrightarrow M$ be the inclusion. Then M. Kashiwara and P. Schapira introduce in [KSch], Chapter II, a complex $i_! \circ i^*(\mathcal{G})$ denoted by $\mathcal{G}_Z$. If $Z'$ is closed in $Z$, then they prove existence of a distinguished triangle

$$\mathcal{G}_{Z\setminus Z'} \rightarrow \mathcal{G}_Z \rightarrow \mathcal{G}_{Z'}.$$  

**Hence, by the additivity property of characteristic cycles,**

$$Ch(\mathcal{G}_Z) = Ch(\mathcal{G}_{Z\setminus Z'}) + Ch(\mathcal{G}_{Z'}).$$

It follows that, as an element of $K(D^b_{R-\text{c}}(M))$ – the Grothendieck group of $D^b_{R-\text{c}}(M)$, our complex of sheaves $\mathcal{F}$ is equivalent to $\mathcal{F}_{O_1} + \cdots + \mathcal{F}_{O_d}$, and so

$$Ch(\mathcal{F}) = Ch(\mathcal{F}_{O_1}) + \cdots + Ch(\mathcal{F}_{O_d}).$$

The idea is to deform each summand $Ch(\mathcal{F}_{O_k})$ separately. Since $O_k$ is locally closed, there exists an open subvariety $U_k$ of $M$ containing $O_k$ as a closed subvariety. Then by Proposition 4.22 of [DM] or Section 4 of [SchV1] there exists a real-valued semi-algebraic $C^2$-function $f_k$ on $M$ such that $f_k$ is strictly positive on $U_k$ and the boundary $\partial U_k$ is precisely the zero set of $f_k$.

**Lemma 26** There exists an $R > 0$ such that, for each $\zeta \in T^-_{x_k} M \subset T_{x_k} M$ with $\|\zeta\|_k \geq R$, the single-variable function

$$f^\zeta_k(t) = f_k(\psi_{X,k}(t\zeta)), \quad t \in \mathbb{R},$$

is strictly monotone decreasing for $t > 1/2$.

**Proof.** Easily follows from the results on o-minimal structure described in [DM], and in particular the Monotonicity Theorem 4.1. \hfill $\Box$

The dual space to $T^-_{x_k} M$, $(T^-_{x_k} M)^*$, can be regarded as a subspace of the cotangent space at $x_k$:

$$(T^-_{x_k} M)^* \subset T^*_{x_k} M = (T^-_{x_k} M)^* \oplus (T^+_{x_k} M)^*.$$  

Let $B_k$ be the $\psi_{X,k}$-image of the open ball of radius $R$

$$\{\zeta \in (T^-_{x_k} M)^*; \|\zeta\|_k < R\};$$

$B_k$ is an open subset of $O_k$.

According to Remark 25 we have a distinguished triangle:

$$\mathcal{F}_{B_k} \rightarrow \mathcal{F}_{O_k} \rightarrow \mathcal{F}_{O_k \setminus B_k},$$

and hence

$$Ch(\mathcal{F}_{O_k}) = Ch(\mathcal{F}_{B_k}) + Ch(\mathcal{F}_{O_k \setminus B_k}).$$

(17)

Recall that the sheaf $\mathcal{F}$ is $G_\mathbb{R}$-equivariant. In particular, $Ch(\mathcal{F})$ is $T_\mathbb{R}(X)$-invariant, and so

$$\text{Re}(\langle Y, \mu(\zeta) \rangle) = -\text{Re}(\langle Y_M, \zeta \rangle) = 0$$

for all $Y \in t_\mathbb{R}(X)$ and all $\zeta \in |Ch(\mathcal{F})|$.  

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Similarly, because the set \( O_k \) is \( T_\mathbb{R}(X) \)-invariant, the sheaf \( \mathcal{F}_{O_k} \) is \( T_\mathbb{R}(X) \)-equivariant too, its characteristic cycle is \( T_\mathbb{R}(X) \)-invariant, and \( \text{Re}(\langle Y, \mu(\zeta) \rangle) = 0 \) for all \( Y \in t_\mathbb{R}(X) \) and all \( \zeta \in |Ch(\mathcal{F}_{O_k})| \).

On the other hand, \( B_k \) is an open subset of \( O_k \) such that the vector field \( X_M \) is either tangent to the boundary \( \partial B_k \) or points outside \( B_k \), but never points inside \( B_k \). It follows from the Open Embedding Theorem (Theorem 9) and Lemma 26 that \( \text{Re}(\langle \partial C, \mu(\zeta) \rangle) \leq 0 \) for all \( Y \in t_\mathbb{R}(X) \) and all \( \zeta \in |Ch(\mathcal{F}_{B_k})| \). Since \( Ch(\mathcal{F}_{O_k\setminus B_k}) = Ch(\mathcal{F}_{O_k}) - Ch(\mathcal{F}_{B_k}) \), the same is true of \( |Ch(\mathcal{F}_{O_k\setminus B_k})| \).

**Lemma 27** The cycle \( Ch(\mathcal{F}_{O_k\setminus B_k}) \) is homologous to the zero cycle inside the set
\[
\{ \zeta \in T^*M; \text{Re}(\langle X, \mu(\zeta) \rangle) \leq 0 \}.
\]

**Proof.** The sheaf \( \mathcal{F}_{O_k\setminus B_k} \) is the extraordinary direct image of a sheaf on \( U_k \):
\[
\mathcal{F}_{O_k\setminus B_k} = (j_{U_k\to M})! (j_{O_k\setminus B_k\to U_k})!(\mathcal{F}|_{O_k\setminus B_k}).
\]

Recall that \( f_k \) is real-valued semi-algebraic \( C^2 \)-function on \( M \) which is strictly positive on \( U_k \) and its zero set is precisely the boundary \( \partial U_k \). It follows from the equation (5) and Lemma 26 that, for each \( x \in O_k \) with \( \|\psi_{X,k}^{-1}(x)\|_k > R/2 \),
\[
\text{Re}(\langle X, \mu(df_k(x)) \rangle) = -\text{Re}(\langle X_M, df_k(x) \rangle) \geq 0.
\]

By the Open Embedding Theorem (Theorem 9),
\[
Ch(\mathcal{F}_{O_k\setminus B_k}) = Ch((j_{U_k\to M})! (j_{O_k\setminus B_k\to U_k})!(\mathcal{F}|_{O_k\setminus B_k}))
\]
\[
= \lim_{s \to 0^+} Ch((j_{O_k\setminus B_k\to U_k})!(\mathcal{F}|_{O_k\setminus B_k})) - s \frac{df_k}{f_k}.
\]

Let \( C \) be a \((2n+1)\)-chain in \( T^*M \)
\[
C = -\left( Ch((j_{O_k\setminus B_k\to U_k})!(\mathcal{F}|_{O_k\setminus B_k})) - s \frac{df_k}{f_k} \right), \quad s \in (0, \infty).
\]
Then \( C \) is a conic chain, its support \(|C| \) lies inside the set (15) and the boundary of this chain \( \partial C \) is \( Ch(\mathcal{F}_{O_k\setminus B_k}) \) minus another cycle which we call
\[
\lim_{s \to +\infty} Ch((j_{O_k\setminus B_k\to U_k})!(\mathcal{F}|_{O_k\setminus B_k})) - s \frac{df_k}{f_k}.
\]

Notice that the last cycle is a cycle in \( T^*M \) whose support lies completely inside \( T^*U_k \).

Recall the element \( X' \in t_\mathbb{R} \cap g'_\mathbb{R} \) used to define attracting sets \( O_1, \ldots, O_d \). Let \( X'_{T_{x_k}M} \) be the vector field on \( T_{x_k}M \) generated by \( X' \). Define a 1-form \( \eta \) on \( T_{x_k}M \setminus \{0\} \) to be
\[
\eta = \frac{\langle X'_{T_{x_k}M}, \cdot \rangle_k}{\langle X'_{T_{x_k}M}, X'_{T_{x_k}M} \rangle_k}.
\]
We regard \( \eta \) as a section of \( T^*(O_k \setminus \{x_k\}) \) via the isomorphism (16), and let \( \tilde{\eta} \) be any semi-algebraic extension of \( \eta \) to a section of \( T^*M|_{O_k\setminus \{x_k\}} \). Since \( X' \) lies in the same connected component of \( t_\mathbb{R}(X) \cap g'_\mathbb{R} \) as \( X \), it is easy to see that the real part \( \text{Re}\tilde{\eta}(X_M) = -\text{Re}(X, \mu(\tilde{\eta})) \) is strictly positive on \( O_k \setminus \{x_k\} \).
Finally, define a $(2n + 1)$-chain in $T^* M$ 

$$\tilde{C} = - \left( \lim_{s \to +\infty} Ch\left( (j_{O_k \setminus B_k \leftarrow U_k})! (\mathcal{F}|_{O_k \setminus B_k}) \right) - s \frac{df_k}{f_k} \right) + t\tilde{\eta}, \quad t \in [0, \infty).$$

Then its boundary

$$\partial \tilde{C} = \lim_{s \to +\infty} Ch\left( (j_{O_k \setminus B_k \leftarrow U_k})! (\mathcal{F}|_{O_k \setminus B_k}) \right) - s \frac{df_k}{f_k},$$

$\tilde{C}$ is conic and its support $|\tilde{C}|$ lies in the set (15). \hfill \Box

Next we deform $Ch(\mathcal{F}_{B_k})$. We use another distinguished triangle.

**Remark 28** If $\mathcal{G}$ is a complex of sheaves on $M$, $Z$ is a closed subset of $M$, $U = M \setminus Z$ is its complement and $i : Z \hookrightarrow M$, $j : U \hookrightarrow M$ are the inclusion maps, then we have a distinguished triangle

$$(Ri)_* \circ i^!(\mathcal{G}) \to \mathcal{G} \to (Rj)_* \circ j^*(\mathcal{G}).$$

Hence, by the additivity property of characteristic cycles,

$$Ch(\mathcal{G}) = Ch((Ri)_* \circ i^!(\mathcal{G})) + Ch((Rj)_* \circ j^*(\mathcal{G})).$$

We apply this remark with $\mathcal{G} = \mathcal{F}_{B_k} = (j_{B_k \hookrightarrow M})! \circ (j_{B_k \hookrightarrow M})^*(\mathcal{F})$, closed subset $Z = \{ x_k \}$ and its complement $U = M \setminus \{ x_k \}$:

$$Ch(\mathcal{F}_{B_k}) = \quad \quad \quad \quad \quad (18)$$

Using that $B_k$ is an open subset of $O_k$, that the inclusion map $O_k \hookrightarrow M$ is proper on the support of $(j_{B_k \hookrightarrow O_k})!(\mathcal{F}|_{B_k})$, the Cartesian square

$$\begin{array}{ccc}
B_k & \hookrightarrow & O_k \\
\| & \downarrow & \\
B_k & \hookrightarrow & M
\end{array}$$

and Proposition 3.1.9 of [KSch] we can write

$$(j_{\{x_k\} \hookrightarrow M})^!(\mathcal{F}_{B_k}) = (j_{\{x_k\} \hookrightarrow M})^! \circ (j_{B_k \hookrightarrow M})! (\mathcal{F}|_{B_k})$$

$$= (j_{\{x_k\} \hookrightarrow B_k})^! \circ (j_{B_k \hookrightarrow M})^! \circ (Rj_{O_k \leftarrow M})_* \circ (j_{B_k \hookrightarrow O_k})!(\mathcal{F}|_{B_k})$$

$$= (j_{\{x_k\} \hookrightarrow B_k})^! \circ (j_{B_k \hookrightarrow O_k})^* \circ (j_{B_k \hookrightarrow O_k})!(\mathcal{F}|_{B_k}) = (j_{\{x_k\} \hookrightarrow B_k})^! \circ (\mathcal{F}|_{B_k})$$

$$= (j_{\{x_k\} \hookrightarrow B_k})^! \circ (j_{B_k \hookrightarrow O_k})^!(\mathcal{F}|_{O_k}) = (j_{\{x_k\} \hookrightarrow O_k})^!(\mathcal{F}|_{O_k}).$$

Thus we can rewrite the equation (18) as

$$Ch(\mathcal{F}_{B_k}) = \quad \quad \quad \quad \quad (19)$$

The cycle $Ch((Rj_{\{x_k\} \hookrightarrow M})^* \circ (j_{\{x_k\} \hookrightarrow O_k})^!(\mathcal{F}|_{O_k}))$ is the cotangent space $T^*_{x_k} M$ equipped with orientation (20) and multiplicity $m_k(X)$ given by the local formula (11).

It remains to show that the second summand of (19) is homologous to zero. Let $\mathcal{G}$ denote the sheaf $(j_{M \setminus \{x_k\} \hookrightarrow M})^*(\mathcal{F}_{B_k})$ on $M \setminus \{ x_k \}$; it is supported inside the closure of $B_k \setminus \{ x_k \}$ in $M \setminus \{ x_k \}$. Pick any real-valued semi-algebraic $C^2$-function $f_k$ on $M$ such that $f_k$ is strictly positive on $M \setminus \{ x_k \}$ and $f(x_k) = 0$. Similarly to Lemma 26 we have:
Lemma 29 There exists an $R' > 0$ such that, for each $\zeta \in T_{x_k}^*M \subset T_{x_k}M$ with $\|\zeta\|_k \leq R'$, the single-variable function

$$\tilde{f}_k'(t) = \tilde{f}_k(\psi_{X,k}(t\zeta)), \quad t \in \mathbb{R},$$

is strictly monotone increasing for $t \in [0, 2]$.

Since we are free to modify $\tilde{f}_k$ on any compact subset of $M$ which does not contain $x_k$, we may assume that $R' > R$.

Lemma 30 The cycle

$$Ch\left((Rj_{M\setminus\{x_k\}}\to M)^* \circ (j_{M\setminus\{x_k\}}\to M)^* (\mathcal{F}_{B_k})\right) = Ch\left((Rj_{M\setminus\{x_k\}}\to M)^* (\mathcal{G})\right)$$

is homologous to the zero cycle inside the set

$$\{\zeta \in T^*M; \text{Re}(\langle X, \mu(\zeta) \rangle) \leq 0\}.$$ 

Proof. Except for a few obvious modifications, this proof is identical to the proof of Lemma 27. First we observe that because $R'$ from Lemma 29 is bigger than $R$ used to define the set $B_k$, for each $x \in \overline{B_k} \setminus \{x_k\}$, we have $\text{Re}(\langle X, \mu(d\tilde{f}_k(x)) \rangle) = -\text{Re}(\langle X, d\tilde{f}_k(x) \rangle) \leq 0$.

By the Open Embedding Theorem (Theorem 9),

$$Ch\left((Rj_{M\setminus\{x_k\}}\to M)^* (\mathcal{G})\right) = \lim_{s \to 0^+} Ch(\mathcal{G}) + \frac{d\tilde{f}_k}{\tilde{f}_k}.$$ 

Thus we introduce a $(2n + 1)$-chain in $T^*M$

$$C' = -\left(Ch(\mathcal{G}) + s \frac{d\tilde{f}_k}{\tilde{f}_k}\right), \quad s \in (0, \infty).$$

Then $C'$ is a conic chain, its support $|C'|$ lies inside the set (15) and the boundary of this chain $\partial C'$ is $Ch\left((Rj_{M\setminus\{x_k\}}\to M)^* (\mathcal{G})\right)$ minus another cycle which we call

$$\lim_{s \to +\infty} Ch(\mathcal{G}) + s \frac{d\tilde{f}_k}{\tilde{f}_k}.$$ 

Notice that the last cycle is a cycle in $T^*M$ whose support lies completely inside $T^*(M \setminus \{x_k\})$.

Recall the section $\tilde{\eta}$ of $T^*M|_{O_k \setminus \{x_k\}}$ constructed in the proof of Lemma 27. It has the property that $\text{Re}(X, \mu(\tilde{\eta}))$ is strictly negative on $O_k \setminus \{x_k\}$.

Finally, define a $(2n + 1)$-chain in $T^*M$

$$\tilde{C}' = -\left(\lim_{s \to +\infty} Ch(\mathcal{G}) + s \frac{d\tilde{f}_k}{\tilde{f}_k}\right) + t\tilde{\eta}, \quad t \in [0, \infty).$$

Then its boundary

$$\partial \tilde{C}' = \lim_{s \to +\infty} Ch(\mathcal{G}) + s \frac{d\tilde{f}_k}{\tilde{f}_k},$$

$\tilde{C}'$ is conic and its support $|\tilde{C}'|$ lies in the set (15). □

Combining the equations (17), (19) and lemmas 27, 30 we obtain the following key result.
Proposition 31 For each element $X \in g'_R$, there is a Borel-Moore chain $C(X)$ in $T^*M$ of dimension $(2n + 1)$ with the following properties:

1. $C(X)$ is conic, i.e. invariant under the scaling action of the multiplicative group of positive reals $\mathbb{R}^{>0}$ on $T^*M$;

2. The support of $C(X)$ lies in the set $\{\zeta \in T^*M; \text{Re}(\langle X, \mu(\zeta) \rangle) \leq 0\}$;

3. Let $x_1, \ldots, x_d$ be the zeroes of the vector field $X_M$ on $M$, then
   \[
   \partial C(X) = Ch(F) - (m_1(X)T^*_{x_1}M + \cdots + m_d(X)T^*_{x_d}M),
   \]
   where $m_1(X), \ldots, m_d(X)$ are the integer multiplicities determined by the local formula (11) and the orientation of $T^*_{x_k}M$ is chosen so that if we write each $z_l$ as $x_l + iy_l$, then the $\mathbb{R}$-basis
   \[
   \{dx_1, dy_1, \ldots, dx_n, dy_n\}
   \]
   of $T^*_{x_k}M \simeq (C_{\beta_{x_k,1}})^* \oplus \cdots \oplus (C_{\beta_{x_k,n}})^*$ (20)
   is positively oriented;

4. Moreover, if $\tilde{X} \in t_\mathbb{R}(X) \cap g'_R$ lies in the same connected component of $t_\mathbb{R}(X) \cap g'_R$ as $X$, then the same choice of element $X' \in t_\mathbb{R}(\tilde{X}) \cap g'_R$ works for $\tilde{X}$. In this case the chain $C(\tilde{X})$ is identical to $C(X)$.

Remark 32 The holomorphic cotangent space $T^*_{x_k}M$ has a natural orientation coming from its complex structure. This orientation need not agree with the orientation given by (20). In fact, the complex orientation of $T^*_{x_k}M = (-1)^n$ the orientation given by (20).

Next we show that the local formula (11) and the global formula (13) for the coefficient $m_k(X)$ give the same answer.

Proof of Proposition 21. By a generalization of the Hopf Index Theorem (Theorem 11),
   \[
   \chi(M, F_{O_k}) = \#([M] \cap Ch(F_{O_k})).
   \]
Then by Proposition 31, the characteristic cycle $Ch(F_{O_k})$ is homologous to the cycle
   \[
   \chi(R\Gamma_{\{x_k\}}(F_{|O_k})_{x_k}) \cdot T^*_{x_k}M,
   \]
where $T^*_{x_k}M$ is given orientation as described in (20). Since $T^*_{x_k}M$ intersects $M$ transversally, we see that the right hand side of (13) is
   \[
   \chi(M, F_{O_k}) = \#([M] \cap Ch(F_{O_k})) = \chi(R\Gamma_{\{x_k\}}(F_{|O_k})_{x_k}). \quad \square
   \]

1 The holomorphic cotangent bundle $T^*M$ and the $C^\infty$ cotangent bundle $T^*M^\mathbb{R}$ are identified according to Remark 12.
6 Proof of Theorem 20

In this section we compute the integral (14) first under the assumption that the form \( \varphi \) is compactly supported in \( g'_R \) and then in general. First we define a deformation \( \Theta_t(X) : T^*M \to T^*M \), where \( X \in g'_R \), \( t \in [0, 1] \). It has the following purpose. In the classical proof of the Fourier inversion formula

\[
\varphi(X) = \frac{1}{(2\pi i)^{\dim_R g_R}} \int_{\xi \in i g'_R} \hat{\varphi}(\xi) e^{-\langle X, \xi \rangle}
\]

we multiply the integrand by a term like \( e^{-t\|\xi\|^2} \) to make it integrable over \( g_R \times i g_R^* \), and then let \( t \to 0^+ \). The deformation \( \Theta_t(X) \) has a very similar effect – it makes our integrand an \( L^1 \)-object. Proposition 40 says that this substitution is permissible. Its proof is very technical, but the idea is quite simple. The difference between the original integral (14) and the deformed one is expressed by an integral of \( e^{\langle X, \mu(\zeta) \rangle} \varphi(X) \wedge \alpha(X) \wedge e^\sigma \) over a certain cycle \( \tilde{C}(R) \) supported in \( g'_R \times (T^*M \cap \{ \| \mu(\zeta) \|_{g_C^*} = R \}) \) which depends on \( R \) by scaling along the fiber. Recall that the Fourier transform \( \hat{\varphi} \) decays rapidly in the imaginary directions which is shown by integration by parts. We modify this integration by parts argument to prove a similar statement about the behavior of the integrand on the support of \( \tilde{C}(R) \) as \( R \to \infty \). Hence the difference of integrals in question tends to zero.

Pick an element \( X_0 \) lying in the support of \( \varphi \) and let \( t_\mathbb{R}(X_0) \subset g_\mathbb{R} \) be the Cartan subalgebra containing \( X_0 \). There exists an open neighborhood \( \Omega \) of \( X_0 \) in \( g'_R \) and a smooth map \( \omega : \Omega \to G_\mathbb{R} \) with the following three properties:

1. \( \omega|_{\Omega \cap t_\mathbb{R}(X_0)} = e \), the identity element of \( G_\mathbb{R} \);
2. For every \( X \in \Omega \), the conjugate Cartan subalgebra \( \omega(X)t_\mathbb{R}(X_0)\omega(X)^{-1} \) contains \( X \);
3. \( \omega(X) = \omega(Y) \) whenever \( X, Y \in \Omega \) and \( t_C(X) = t_C(Y) \) (i.e. \( [X, Y] = 0 \)).

Notice that if \( X \in \Omega \), then \( \omega(X) \cdot M_0(X_0) = M_0(X) \). Making \( \Omega \) smaller if necessary, we can assume that both \( \Omega \) and \( \Omega \cap t_\mathbb{R}(X_0) \) are connected. Let \( t_C(X_0) = t_\mathbb{R}(X_0) \oplus t_\mathbb{R}(X_0) \subset g_C \) be the complex Cartan subalgebra containing \( X_0 \).

**Remark 33** One cannot deal with the integral (14) “one Cartan algebra at a time” and avoid introducing a map like \( \omega \) because the limit

\[
\lim_{R \to \infty} \int_{t_\mathbb{R} \times (Ch(F) \cap \mu^{-1}(B_R))} e^{\langle X, \mu(\zeta) \rangle} \varphi(X) \wedge \alpha(X) \wedge e^\sigma.
\]

may not exist. (Recall that \( B_R \) is an open ball in \( g_C^* \) defined by (7).)

From now on we assume that the support of \( \varphi \) lies in \( \Omega \). The general case when \( \text{supp}(\varphi) \subset g'_R \) can be reduced to this special case by a partition of unity argument.

Our biggest obstacle to making any deformation argument computing the integral (14) is that the integration takes place over a cycle which is not compactly supported and Stokes’ theorem no longer applies. In order to overcome this obstacle, we construct a deformation \( \Theta_t : \Omega \times T^*M \to \Omega \times T^*M \), \( t \in [0, 1] \), such that \( \Theta_0 \) is the identity map;

\[
\text{Re}(\langle \Theta_t^*(X, \mu(\zeta)) \rangle) = \text{Re}(\langle X, \mu(\zeta) \rangle)
\]
for $t > 0$, $X \in \Omega$ and $\zeta \in T^*M$ which does not lie in the zero section (Lemma 36); $\Theta_t$ essentially commutes with scaling the fiber of $T^*M$ (Lemma 37). The last two properties will imply that the integral

$$\int_{\mathbb{R} \times (\mathbb{C}(\mathcal{F}) \cap \mu^{-1}(B_R))} (\Theta_t)^* (e^{(X,\mu(\zeta)) \varphi(Z)} \wedge \alpha(Z) \wedge e^\sigma)$$

converges absolutely for $t \in (0,1]$. Finally, the most important property of $\Theta_t$ is stated in Proposition 40 which essentially says that we can replace our integrand

$$e^{(X,\mu(\zeta)) \varphi(Z)} \wedge \alpha(Z) \wedge e^\sigma$$

with the pullback

$$(\Theta_t)^*(e^{(X,\mu(\zeta)) \varphi(Z)} \wedge \alpha(Z) \wedge e^\sigma).$$

We restate Theorem 1 of [Su]:

**Proposition 34** There is a projective embedding $\nu : M \to \mathbb{C}P^N$ and a group representation $\rho : G_C \to PGL(N)$ such that $\rho(g) \cdot \nu(x) = \nu(g \cdot x)$ for every $g \in G_C$ and $x \in M$.

Let $\{x_1, \ldots, x_d\}$ be the set of zeroes $M_0(X_0)$. For $D > 0$, we denote by $B_D$ the open ball in $\mathbb{C}^n$ of radius $D$:

$$B_D = \{(z_1, \ldots, z_n) \in \mathbb{C}^n; |z_1|^2 + \cdots + |z_n|^2 < D^2\}.$$ 

Using Proposition 34 one can construct a $C^\infty$ diffeomorphism onto an open subset $V_k \subset M$ containing $x_k$

$$\psi_{X_0,k} : B_{4D} \to V_k$$

such that $\psi_{X_0,k}(0) = x_k$ and, for each $Y \in t_C(X_0)$, the tangent map $d\psi_{X_0,k}$ sends the vector field on $\mathbb{C}^n \cong T_0B_{4D}$

$$\beta_{x_k,1}(Y) z_1 \frac{\partial}{\partial z_1} + \cdots + \beta_{x_k,n}(Y) z_n \frac{\partial}{\partial z_n}$$

into $-Y_M$. Note that this condition implies $V_k \cap M_0(X_0) = \{x_k\}$.

On the other hand, each point $x \in M \setminus M_0(X_0)$ has a $C^\infty$ chart

$$\psi_{X_0,x} : B_{4D} \to V_x$$

such that $\psi_{X_0,x}(0) = x$ and

$$d\psi_{X_0,x} \left( \frac{\partial}{\partial z_1} \right) = -(X_0)_M.$$ 

Making $V_x$ smaller if necessary, we can assume that $V_x \cap M_0(X_0) = \emptyset$. For $Y \in \Omega$, let

$$Y^x(z) \frac{\partial}{\partial z_1} + \cdots + Y^n(z) \frac{\partial}{\partial z_n}$$

be the inverse image of the vector field $-Y_M$ under the tangent map $d\psi_{X_0,x}$. By continuity (21) implies that there is an open neighborhood $\Omega_x$ of $X_0$ such that $\text{Re}(Y^x(z)) > 0$ for $z \in B_{4D}$ and $Y \in \Omega_x \cap t_C(X_0)$.

We extend $\{\psi_{X_0,1}, \ldots, \psi_{X_0,d}\}$ to an atlas $\{\psi_{X_0,1}, \ldots, \psi_{X_0,d'}\}$ of $M$ so that, for $d < k \leq d'$, $\psi_{X_0,k} = \psi_{X_0,x_k'}$ for some $x_k' \in M \setminus M_0(X_0)$ and the smaller open sets completely cover $M$:

$$\bigcup_{k=1}^{d'} \psi_{X_0,k}(B_D) = M.$$ 

(22)
Set $V_k = \psi_{X_0,k}(B_{4D})$, $k = 1, \ldots, d'$.

For each $X \in \Omega$ we define maps

$$\psi_{X,k} : B_{4D} \to \omega(X) \cdot V_k, \quad \psi_{X,k}(z) = \omega(X) \cdot \psi_{X_0,k}(z), \quad 1 \leq k \leq d'.$$

Then $\{\psi_{X,1}, \ldots, \psi_{X,d'}\}$ form another atlas of $M$. Note that, for $k = 1, \ldots, d$, $\psi_{X,k}(0) = \omega(X) \cdot x_k$ and, for each $Y \in \mathfrak{t}_C(X) = \omega(X) \mathfrak{t}_C(X_0) \omega(X)^{-1}$, the tangent map $d\psi_{X,k}$ sends the vector field on $T_{x_k}M$

$$\beta_{x_k,1}(\omega(X)^{-1}Y \omega(X)) z_1 \frac{\partial}{\partial z_1} + \cdots + \beta_{x_k,n}(\omega^{-1}(X)Y \omega(X)) z_n \frac{\partial}{\partial z_n}$$

into $-Y_M$. We extend $\beta_{x_k,1}, \ldots, \beta_{x_k,n} \in \mathfrak{t}_C(X_0)$ to $\Omega$ by

$$\beta_{x_k,l}(Y) = \text{def} \beta_{x_k,l}(\omega(Y)^{-1}Y \omega(Y)), \quad Y \in \Omega, \quad l = 1, \ldots, n.$$

This way, for all $X \in \Omega$ and all $Y \in \mathfrak{t}_C(X)$, we can write

$$d\psi_{X,k}(\beta_{x_k,1}(Y) z_1 \frac{\partial}{\partial z_1} + \cdots + \beta_{x_k,n}(Y) z_n \frac{\partial}{\partial z_n}) = -Y_M. \quad (23)$$

If $k = d + 1, \ldots, d'$ and $Y \in \Omega$, let

$$Y_1^k(z) \frac{\partial}{\partial z_1} + \cdots + Y_n^k(z) \frac{\partial}{\partial z_n}$$

be the inverse image of the vector field $-Y_M$ under the tangent map $d\psi_{Y,k}$. Note that

$$d\psi_{X,k}(\frac{\partial}{\partial z_1}) = -X_M, \quad d < k \leq d'.$$

Hence making $\Omega$ smaller if necessary, we can assume that $\text{Re}(Y_1^k(z)) > 0$ for $d < k \leq d'$, $z \in B_{4D}, Y \in \Omega \cap \mathfrak{t}_C(X)$ and all $X \in \Omega$.

Finally, we define maps

$$\psi_k : \Omega \times B_{4D} \to \Omega \times M,$$

$$\psi_k(X, z) = (X, \psi_{X,k}(z)) = (X, \omega(X) \cdot \psi_{X_0,k}(z)), \quad 1 \leq k \leq d'.$$

Each $\psi_k$ is a diffeomorphism onto its image, and their images for $k = 1, \ldots, d'$ cover all of $\Omega \times M$. Thus we obtain an atlas $\{\psi_1, \ldots, \psi_{d'}\}$ of $\Omega \times M$.

Expand $(z_1, \ldots, z_n)$ to a standard coordinate system $(z_1, \ldots, z_n, \xi_1, \ldots, \xi_n)$ on the cotangent space $T^*B_{4D}$ so that every element of $T^*B_{4D} \simeq B_{4D} \times \mathbb{C}^n$ is expressed in these coordinates as

$$(z_1, \ldots, z_n, \xi_1 dz_1 + \cdots + \xi_n dz_n).$$

This gives us a chart

$$\tilde{\psi}_k : (X, z_1, \ldots, z_n, \xi_1, \ldots, \xi_n) \to \Omega \times T^*M$$

and an atlas $\{\tilde{\psi}_1, \ldots, \tilde{\psi}_{d'}\}$ of $\Omega \times T^*M$. For $(z_1, \ldots, z_n, \xi_1, \ldots, \xi_n) \in T^*B_{4D}$, define norms

$$||z|| = \sqrt{|z_1|^2 + \cdots + |z_n|^2} \quad \text{and} \quad ||\xi|| = \sqrt{\xi_1^2 + \cdots + \xi_n^2}.$$

Find an $\varepsilon > 0$ small enough so that for each $k = 1, \ldots, d$

$$\psi_{X_0,k}(B_{\varepsilon}) \cap \bigcup_{l \neq k} \psi_{X_0,l}(B_{3D}) = \emptyset; \quad (24)$$
we also assume that $\varepsilon \leq D/2$.

Since $G_R$ acts on $M$ by complex automorphisms, the symplectic form $\sigma$ in these coordinates is $d\xi_1 \wedge dz_1 + \cdots + d\xi_n \wedge dz_n$.

For $k = 1, \ldots, d$, the equations (5) and (23) say that the exponential part in the chart $\tilde{\psi}_k$ becomes

$$\langle X, \mu(\zeta) \rangle = \beta_{z_k,1}(X)z_1 + \cdots + \beta_{z_k,n}(X)z_n \xi_n. \tag{25}$$

Let $\delta : \mathbb{R} \to [0, 1]$ be a smooth bump function which takes on value 1 on $[-D, D]$, vanishes outside $(-2D, 2D)$, and is nondecreasing on negative reals, non-increasing on positive reals. By making $\varepsilon$ smaller if necessary we may assume that $|2\varepsilon\delta'(x)| < 1$ for all $x$.

Let $\gamma : \mathbb{R}^+ \to (0, 1]$ be another smooth function which is non-increasing, $\gamma([0, 1]) = \{1\}$, $\gamma(x) = \frac{1}{2}$ for $x > 2$, and $\frac{1}{2} \leq \gamma(x) \leq \frac{3}{2}$ for all $x \geq 1$.

And let $\rho : \mathbb{R}^+ \to [0, \infty)$ be a smooth monotone increasing function such that its derivative $\rho'(x) \leq \frac{1}{2}$ for all $x$ and

$$\begin{cases} 
\rho(x) = \frac{1}{4}x^2 & \text{if } x \in [0, 1]; \\
\rho(x) = ax & \text{if } x \geq 2
\end{cases}$$

for some constant $a > 0$.

Note that the derivatives of $\delta$, $\gamma$ and $\rho$ are uniformly bounded on their respective domains.

For each $t \in [0, 1]$ and $k = 1, \ldots, d'$ we define a map $\Theta^k_t : \Omega \times T^*M \to \Omega \times T^*M$. If $Y \in \mathfrak{c}(X_0)$ and $k = 1, \ldots, d$, we define a diffeomorphism $\tilde{\Theta}^k_{Y,t}$ on $T^*B_{4D} \simeq B_{4D} \times \mathbb{C}^n$ by

$$\tilde{\Theta}^k_{Y,t}(z_1, \ldots, z_n, \xi_1, \ldots, \xi_n) = (z'_1, \ldots, z'_n, \xi_1, \ldots, \xi_n) \quad z'_j = z_j - \frac{\beta_{z_k,j}(Y)}{|\beta_{z_k,j}(Y)|}t\varepsilon\delta(||z||)\gamma(t||\xi||)\xi_j, \quad 1 \leq j \leq d,$$

(the requirement $|2\varepsilon\delta'| < 1$ ensures that $\tilde{\Theta}^k_{Y,t}$ is one-to-one). If $k = d + 1, \ldots, d'$, we define a diffeomorphism $\tilde{\Theta}^k_{Y,t}$ on $T^*B_{4D} \simeq B_{4D} \times \mathbb{C}^n$ by

$$\tilde{\Theta}^k_{Y,t}(z_1, \ldots, z_n, \xi_1, \ldots, \xi_n) = (z_1, \ldots, z_n, \xi_1 - \delta(||z||)\rho(t||\xi||), \xi_2, \ldots, \xi_n)$$

(again, the requirement $\rho' \leq \frac{1}{2}$ ensures that $\tilde{\Theta}^k_{Y,t}$ is one-to-one). The map $\tilde{\Theta}^k_{Y,t}$ shifts $(z_1, \ldots, z_n)$ by a vector

$$-t\varepsilon\delta(||z||)\gamma(t||\xi||)\left(\frac{\beta_{z_k,1}(Y)}{|\beta_{z_k,1}(Y)|}\xi_1, \ldots, \frac{\beta_{z_k,n}(Y)}{|\beta_{z_k,n}(Y)|}\xi_n\right) \quad \text{if } k = 1, \ldots, d$$

which has length at most $2\varepsilon \leq D$ (because $\gamma(x) \leq \frac{3}{2}$), and shifts $\xi_1$ by a scalar

$$-\delta(||z||)\rho(t||\xi||) \quad \text{if } k = d + 1, \ldots, d'.$$

Hence the maps $\tilde{\Theta}^k_{Y,t}$ and $(\tilde{\Theta}^k_{Y,t})^{-1}$ leave points outside the set $\{(z, \xi); ||z|| \leq 2D\}$ completely unaffected. Then we use the diffeomorphism between $B_{4D} \times \mathbb{C}^n$ and $T^*V_k \subset T^*M$ induced by the map $\psi_{X_0,k} : B_{4D} \to M$ to regard $\tilde{\Theta}^k_{Y,t}$ as a map on $T^*V_k$. But since $\tilde{\Theta}^k_{Y,t}$ becomes the identity map when the basepoint of $\zeta \in T^*M$ lies away from the compact subset

$$\psi_{X_0,k}(\{z; ||z|| \leq 2D\}) \subset V_k \subset M,$$

$\tilde{\Theta}^k_{Y,t}$ can be extended by identity to a diffeomorphism $T^*M \to T^*M$. 

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Finally, we define \( \Theta^k_t : \Omega \times T^*M \to \Omega \times T^*M \) using the “twisted” product structure of \( \Omega \times T^*M \) induced by \( \omega(X) \). Recall that the group \( G_C \) acts on \( M \) which induces an action on \( T^*M \). For \( g \in G_C \) and \( \zeta \in T^*M \), we denote this action by \( g \cdot \zeta \). Then, for \((X, \zeta) \in \Omega \times T^*M\), we set

\[
\Theta^k_t(X, \zeta) = (X, \omega(X) \cdot (\tilde{\Theta}^k_{Y^t} \omega(X)^{-1} \cdot \zeta)), \quad \text{where } Y = \omega(X)^{-1} X \omega(X) \in \mathfrak{t}_C(X_0).
\]

Inside the chart \( \tilde{\psi}_k \) centered at the point \((X_0, x_k)\), \( \Theta^k_t \) is formally given by the same expression as before:

\[
\Theta^k_t(X, z_1, \ldots, z_n, \xi_1, \ldots, \xi_n) = (X, z'_1, \ldots, z'_n, \xi_1, \ldots, \xi_n)
\]

\[
z'_j = z_j - \frac{\beta_{x_k,j}(X)}{\|\beta_{x_k,j}(X)\|} t \varepsilon \delta(\|z\|) \gamma(t\|\xi\|) \tilde{\xi}_j, \quad 1 \leq j \leq d,
\]

if \( k = 1, \ldots, d \), and

\[
\Theta^k_t(X, z_1, \ldots, z_n, \xi_1, \ldots, \xi_n) = (X, z_1, \ldots, z_n, \xi_1 - \delta(\|z\|) \rho(t\|\xi\|), \xi_2, \ldots, \xi_n)
\]

if \( k = d + 1, \ldots, d' \).

That is we shift

\[
\begin{cases}
(z_1, \ldots, z_n) \text{ by a vector} \\
-\varepsilon \delta(\|z\|) \gamma(t\|\xi\|) \left( \frac{\beta_{x_k,1}(X)}{\|\beta_{x_k,1}(X)\|} \tilde{z}_1, \ldots, \frac{\beta_{x_k,n}(X)}{\|\beta_{x_k,n}(X)\|} \tilde{z}_n \right) \text{ if } k = 1, \ldots, d; \\
\xi_1 \text{ by a scalar} \\
-\delta(\|z\|) \rho(t\|\xi\|) \text{ if } k = d + 1, \ldots, d'.
\end{cases}
\]

(26)

This choice of coefficients \( -\frac{\beta_{x_k,j}(X)}{\|\beta_{x_k,j}(X)\|} \), the property \( \text{Re}(X^k_1(z)) > 0 \) and the equation (25) imply that

\[
\text{Re}((\Theta^k_t)^*(X, \mu(\zeta))) \leq \text{Re}((X, \mu(\zeta))),
\]

(27)

and the equality occurs if and only if \( \Theta^k_t(X, \zeta) = (X, \zeta) \).

We define \( \Theta_t : \Omega \times T^*M \to \Omega \times T^*M \) by

\[
\Theta_t = \Theta^{d'}_t \circ \cdots \circ \Theta^1_t.
\]

Observe that \( \Theta_0 \) is the identity map. The following five lemmas and the proposition are some of the key properties of \( \Theta_t \) that we will use. We do not give their complete proofs (they can be found in [L1]), but rather show the key steps only.

**Lemma 35** For each \( k = 1, \ldots, d \), the maps \( \Theta_t \) and \( \Theta^k_t \) coincide on the set \( \{ \tilde{\psi}_k(X, z, \xi); X \in \Omega, \|z\| < \varepsilon \} \subset \Omega \times T^*M \).

**Proof.** Follows immediately from condition (24). \( \square \)

**Lemma 36** If \( t > 0 \) and \( \zeta \in T^*M \) does not lie in the zero section,

\[
\text{Re}((\Theta_t)^*(X, \mu(\zeta))) < \text{Re}((X, \mu(\zeta))).
\]
Prove. By (27), we have
\[ \text{Re}((\Theta_t)^*(X, \mu(\zeta))) \leq \text{Re}(\langle X, \mu(\zeta) \rangle), \]
and the equality is possible only if \( \Theta_t^*(X, \zeta) = (X, \zeta) \) for all \( k = 1, \ldots, d' \). In presence of the condition (22) it means that the equality is possible only if \( t = 0 \) or \( \zeta \) lies in the zero section.

Moreover, there is an \( \tilde{R}_0 > 0 \), independent of \( t \in (0, 1] \), such that \( R_0 \) can be chosen to be \( \tilde{R}_0/t \).

Proof. Same as the proof of Lemma 18 in [L1]. Recall that \( \Theta_t = \Theta_t^{d'} \circ \cdots \circ \Theta_t^1 \), hence it is sufficient to show by induction on \( k \), \( 1 \leq k \leq d' \), that there exists an \( \tilde{R}_0 > 0 \) such that whenever \( X \in \text{supp}(\varphi) \), \( \zeta \in T^*M \) and \( \|\zeta\|_{T^*M} \geq \tilde{R}_0/t \),
\[ (\Theta_t^k \circ \cdots \circ \Theta_t^1)(X, E\zeta) = E(\Theta_t^k \circ \cdots \circ \Theta_t^1)(X, \zeta) \]
for all \( E \geq 1 \).

Suppose first that \( 1 \leq k \leq d' \). When \( \|\xi\| > 2/t \), \( \gamma(t\|\xi\|) = \frac{1}{t\|\xi\|} \) and the shift vector (26)
\[ -t\varepsilon(\|\xi\|)^2 \gamma(t\|\xi\|) \left( \frac{\beta_{\xi_1,1}(X)}{\|\beta_{\xi_1,1}(X)\|} \xi_1, \ldots, \frac{\beta_{\xi_n,1}(X)}{\|\beta_{\xi_n,1}(X)\|} \xi_n \right) \]
\[ = -\varepsilon(\|\xi\|)^2 \frac{\beta_{\xi_1,1}(X)}{\|\beta_{\xi_1,1}(X)\|} \xi_1, \ldots, \frac{\beta_{\xi_n,1}(X)}{\|\beta_{\xi_n,1}(X)\|} \xi_n \]
stays unchanged if we replace \((\xi_1, \ldots, \xi_n)\) with \((E\xi_1, \ldots, E\xi_n)\), for any real \( E \geq 1 \). Hence in this situation \( \Theta_t^k(X, E\zeta) = E\Theta_t^k(X, \zeta) \).

Now suppose that \( d < k \leq d' \). When \( \|\xi\| > 2/t \), \( \rho(t\|\xi\|) = at\|\xi\| \) and the \( \xi_1 \) coordinate is shifted by
\[ -\delta(\|\xi\|)^2 \rho(t\|\xi\|) = -at\varepsilon(\|\xi\|)^2 \|\xi\|. \]
It follows that \( \Theta_t^k(X, E\zeta) = E\Theta_t^k(X, \zeta) \) whenever \( \|\xi\| > 2/t \) and \( E \geq 1 \).

Set \( (X, \zeta_k) = (\Theta_t^{k-1} \circ \cdots \circ \Theta_t^1)(X, \zeta) \). Then one argues by induction on \( k \) that there exists an \( \tilde{R}_0 > 0 \) such that whenever \( X \in \text{supp}(\varphi) \) and \( \|\zeta\|_{T^*M} \geq \tilde{R}_0/t \) we have \( \|\zeta(X, \zeta_k)\| > 2/t \) which in turn implies
\[ (\Theta_t^k \circ \cdots \circ \Theta_t^1)(X, E\zeta) = \Theta_t^k(X, E\zeta_k) = E(\Theta_t^k \circ \cdots \circ \Theta_t^1)(X, \zeta_k). \]

□

Lemma 38 There exist a smooth bounded function \( \tilde{\kappa}(X, v, t) \) defined on
\[ \Omega \times \{ \zeta \in T^*M; \|\zeta\|_{T^*M} = 1 \} \times [0, 1] \]
and a real number \( \tilde{r}_0 > 0 \) such that, whenever \( t\|\zeta\|_{T^*M} \leq \tilde{r}_0 \),
\[ \langle X, \mu(\zeta) \rangle - (\Theta_t^d \circ \cdots \circ \Theta_t^1)^*(X, \mu(\zeta)) = t\|\zeta\|_{T^*M}^2 \cdot \tilde{\kappa}(X, \frac{\zeta}{\|\zeta\|_{T^*M}}, t). \]
Moreover, \( \text{Re(\tilde{\kappa})} \) is positive and bounded away from zero for \( X \in \text{supp}(\varphi) \).
Proof. Same as the proof of Lemma 22 in \[L1\]. Write

\[
\langle X, \mu(\zeta) \rangle - (\Theta_t^d \cdots \Theta_t^1) \ast \langle X, \mu(\zeta) \rangle = \left( \langle X, \mu(\zeta) \rangle - \langle X, \mu(\Theta_t^1(\zeta)) \rangle \right) \\
+ \cdots + \left( \langle X, \mu((\Theta_t^{k-1} \cdots \Theta_t^1)(\zeta)) \rangle - \langle X, \mu((\Theta_t^k \cdots \Theta_t^1)(\zeta)) \rangle \right) \\
+ \cdots + \left( \langle X, \mu((\Theta_t^{d-1} \cdots \Theta_t^1)(\zeta)) \rangle - \langle X, \mu((\Theta_t^d \cdots \Theta_t^1)(\zeta)) \rangle \right).
\]

Let \((X, \zeta_k) = (\Theta_t^{k-1} \cdots \Theta_t^1)(X, \zeta), z_j = z_j(X, \zeta_k), \xi_j = \xi_j(X, \zeta_k), 1 \leq j \leq n,\) and suppose for the moment \(t\|\xi\| < 1\) so that \(\gamma(t\|\xi\|) = 1\). Then in the coordinate system \(\tilde{\psi}_k\)

\[
\langle X, \mu(\zeta_k) \rangle - \langle X, \mu(\Theta_t^1(X, \zeta_k)) \rangle = t \varepsilon \delta(\|z\|) \gamma(t\|\xi\|) \cdot \left( |\beta_{x_k,1}(X)||\xi_1|^2 + \cdots + |\beta_{x_k,n}(X)||\xi_n|^2 \right) \\
= t\|\xi\|^2_{T^*M} \varepsilon \delta(\|z\|) \frac{|\beta_{x_k,1}(X)||\xi_1|^2 + \cdots + |\beta_{x_k,n}(X)||\xi_n|^2}{\|\zeta\|^2_{T^*M}}.
\]

It is clear that \(\varepsilon \delta(\|z\|)\|\beta_{x_k,1}(X)||\xi_1|^2 + \cdots + |\beta_{x_k,n}(X)||\xi_n|^2\) is positive. Since the set \(\text{supp}(\varphi) \times \{ \zeta \in T^*M; \|\zeta\|_{T^*M} = 1 \} \times [0, 1]\) is compact, this quotient is bounded away from zero on this set. Then one argues that there is a real number \(\tilde{r}_0 > 0\) such that \(t\|\xi\|_{T^*M} \leq \tilde{r}_0\) implies \(t\|\xi\| < 1\).

\square

Similarly we have:

**Lemma 39** There exist a smooth bounded function \(\kappa'(X, v, t)\) defined on

\[\Omega \times \{ \zeta \in T^*M; \|\zeta\|_{T^*M} = 1 \} \times [0, 1]\]

and a real number \(\tilde{r}_0' > 0\) such that, whenever \(t\|\zeta\|_{T^*M} \leq \tilde{r}_0'\),

\[
\langle X, \mu(\zeta) \rangle - (\Theta_t^d \cdots \Theta_t^{d+1}) \ast \langle X, \mu(\zeta) \rangle = t^2\|\zeta\|^2_{T^*M} \cdot \kappa'(X, \frac{\zeta}{\|\zeta\|_{T^*M}}, t).
\]

Moreover, \(\text{Re}(\kappa')\) is positive and bounded away from zero for \(X \in \text{supp}(\varphi)\).

Finally, the most important property of \(\Theta_t\) is:

**Proposition 40** For any \(t \in [0, 1]\), we have:

\[
\lim_{R \to \infty} \int_{\Omega \times (Ch(F) \cap \{ \|\zeta\|_{T^*M} \leq R \})} \left( e^{\langle X, \mu(\zeta) \rangle} \varphi(X) \wedge \alpha(X) \wedge e^\sigma \right. \\
- \Theta_t^\ast(e^{\langle X, \mu(\zeta) \rangle} \varphi(X) \wedge \alpha(X) \wedge e^\sigma) \left. \right) = 0.
\]

**Proof.** The proof of Lemma 19 in \[L1\] applies here because it is based on the properties of \(\Theta_t\) stated in lemmas 36, 37, 38, 39 and not on any other properties. It is an integration by parts argument similar to the proof of rapid decay of the Fourier transform \(\hat{\varphi}\) in the imaginary directions.

Since the form \(e^{\langle X, \mu(\zeta) \rangle} \varphi(X) \wedge \alpha(X) \wedge e^\sigma\) is closed, the integral

\[
\int_{\Omega \times (Ch(F) \cap \{ \|\zeta\|_{T^*M} \leq R \})} \left( e^{\langle X, \mu(\zeta) \rangle} \varphi(X) \wedge \alpha(X) \wedge e^\sigma - \Theta_t^\ast(e^{\langle X, \mu(\zeta) \rangle} \varphi(X) \wedge \alpha(X) \wedge e^\sigma) \right)
\]

\[
= \int_{\Omega \times (Ch(F) \cap \{ \|\zeta\|_{T^*M} \leq R \}) - (\Theta_t \ast (\Omega \times (Ch(F) \cap \{ \|\zeta\|_{T^*M} \leq R \})))} e^{\langle X, \mu(\zeta) \rangle} \varphi(X) \wedge \alpha(X) \wedge e^\sigma
\]

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is equal to the integral of $e^{(X,\mu(\zeta))} \varphi(X) \wedge \alpha(X) \wedge e^{\sigma}$ over the chain traced by $(\Theta t)_* \left( \Omega \times \partial(Ch(F) \cap \{ \| \zeta \|_{T'} M \leq R \}) \right)$ as $t'$ varies from $0$ to $t$. We will show that this integral tends to zero as $R \to \infty$.

Since $Ch(F)$ is a cycle in $T^* M$, the chain $\Omega \times \partial(Ch(F) \cap \{ \| \zeta \|_{T'} M \leq R \})$ is supported inside the set $\Omega \times \{ \zeta \in T^* M; \| \zeta \|_{T'} M = R \}$. As $R \to \infty$, we can assume that $R > 0$. Then the chain traced by $(\Theta t)_* \left( \Omega \times \partial(Ch(F) \cap \{ \| \zeta \|_{T'} M \leq R \}) \right)$ as $t'$ varies from $0$ to $t$ lies away from the zero section $\Omega \times T^*_M M$ in $\Omega \times T^* M$. If we regard $\Theta$ as a map $\Omega \times T^*_M M \times [0, 1] \to \Omega \times T^* M$, we get an integral of $\Theta^* \left( e^{(X,\mu(\zeta))} \varphi(X) \wedge \alpha(X) \wedge e^{\sigma} \right)$ over the chain $\Omega \times \partial(Ch(F) \cap \{ \| \zeta \|_{T'} M \leq R \}) \times [0, t]$.

The idea is to integrate out the $\Omega$ variable and check that the result decays faster than any negative power of $R$. Clearly, $\Theta^*(\varphi) = \varphi$ and Lemma 36 says that

$$\Theta^*(X, \mu(\zeta)) = (X, \mu(\zeta)) - \kappa(X, \zeta, t')$$

for some smooth function $\kappa(X, \zeta, t')$ which has positive real part. The integral in question can be rewritten as

$$\int_{\Omega \times \partial(Ch(F) \cap \{ \| \zeta \|_{T'} M \leq R \}) \times [0, t]} e^{(X,\mu(\zeta))} e^{-\kappa(X,\zeta,t')} \varphi(X) \wedge \Theta^*(\alpha(X) \wedge e^{\sigma}).$$

We pick a system of local coordinates $(z_1, \ldots, z_n)$ of $M$ and construct respective local coordinates $(z_1, \ldots, z_n, \xi_1, \ldots, \xi_n)$ of $T^* M$. Suppose that we know that all the partial derivatives of all orders of $e^{-\kappa(X,\zeta,t')}$ and $\Theta^*(\alpha(X) \wedge e^{\sigma})$ with respect to the $X$ variable can be bounded independently of $\zeta$ and $t'$ on the set $\text{supp}(\varphi) \times \{ \zeta \in T^* M; \| \zeta \|_{T^* M} > 0 \} \times [0, t]$. Let $y_1, \ldots, y_m$ be a system of linear coordinates on $g_\mathbb{R}$, write $\mu(\zeta) = \beta_1(\zeta)d y_1 + \cdots + \beta_m(\zeta)d y_m$, then

$$\int_{g_\mathbb{R}} e^{(X,\mu(\zeta))} e^{-\kappa(X,\zeta,t')} \varphi(X) \wedge \Theta^*(\alpha(X) \wedge e^{\sigma})
= - \frac{1}{\beta_1(\zeta)} \int_{g_\mathbb{R}} e^{(X,\mu(\zeta))} \frac{\partial}{\partial y_1} \left( e^{-\kappa(X,\zeta,t')} \varphi(X) \wedge \Theta^*(\alpha(X) \wedge e^{\sigma}) \right),$$

and the last integral can be bounded by a constant multiple of $R^n$. We can keep performing integration by parts to get the desired estimate just like for the ordinary Fourier transform. Thus, after integrating out the $X$-variable, we see that the integrand indeed decays rapidly in the fiber variable of $T^* M$. Hence our integral tends to zero as $R \to \infty$.

To show boundedness of the partial derivatives one follows the proof of Lemma 19 in [L1] which uses lemmas 36, 37, 38 and 39. \qed

Recall the Borel-Moore chain $C(X_0)$ described in Proposition 31. The set $\Omega$ was chosen so that both $\Omega$ and $\Omega \cap t_\mathbb{R}(X_0)$ are connected. Hence, for each $X \in \Omega \cap t_\mathbb{R}(X_0)$, we can choose $C(X)$ equal $C(X_0)$. Moreover, for each $X \in \Omega$, we can choose $C(X)$ equal $\omega(\overline{X})_* C(X_0)$. These chains $C(X)$, $X \in \Omega$, piece together into a Borel-Moore chain in $\Omega \times T^* M$ of dimension $(\dim g_\mathbb{R} + 2n + 1)$ which appears in each chart $\tilde{\psi}_k$ as $\Omega \times C(X_0)$,

$$\partial C = \Omega \times Ch(F) - \Omega \times (m_1(X) T^*_{\overline{X}} - x_1 M + \cdots + m_d(X) T^*_{\overline{X}} - x_d M)$$

and the support of $C$ lies inside $\{(X, \zeta) \in \Omega \times T^* M; \text{Re}(X, \mu(\zeta)) \leq 0 \}$.

Take an $R \geq 1$ and restrict all cycles to the set $\{(X, \zeta) \in \Omega \times T^* M; \| \zeta \|_{T^* M} \leq R \}$. Let $C_{\leq R}$ denote the restriction of the cycle $C$, then it has boundary

$$\partial C_{\leq R} = \Omega \times (Ch(F) \cap \{ \| \zeta \|_{T^* M} \leq R \}) - C'(R) - \Omega \times (m_1(X)(T^*_{\overline{X}} - x_1 M \cap \{ \| \zeta \|_{T^* M} \leq R \}) + \cdots + m_d(T^*_{\overline{X}} - x_d M \cap \{ \| \zeta \|_{T^* M} \leq R \})).$$
where $C'(R)$ is a $(\dim_{\mathbb{R}} g + 2n)$-chain supported in the set
\[
\{(X, \zeta) \in \Omega \times T^* M; \|\zeta\|_{T^* M} = R, \Re(\langle X, \mu(\zeta) \rangle) \leq 0\}.
\]
Because the chain $C$ is conic, the piece of boundary $C'(R)$ depends on $R$ by an appropriate scaling of $C'(1)$ in the fiber direction.

**Lemma 41** For a fixed $t \in (0, 1]$,
\[
\lim_{R \to \infty} \int_{C'(R)} \Theta_t^* \left( e^{(X, \mu(\zeta))} \varphi(X) \wedge \alpha(X) \wedge e^\sigma \right) = 0.
\]

**Proof.** Same as the proof of Lemma 20 in [L1]. Integrating the form $\Theta_t^* \left( e^{(X, \mu(\zeta))} \varphi(X) \wedge \alpha(X) \wedge e^\sigma \right)$ over the chain $C'(R)$ is equivalent to integrating $e^{(X, \mu(\zeta))} \varphi(X) \wedge \alpha(X) \wedge e^\sigma$ over $(\Theta_t)_* C'(R)$. Let $R_0$ be as in Lemma 37, then, for $R \geq R_0$, the chain $(\Theta_t)_* C'(R)$ depends on $R$ by scaling $(\Theta_t)_* C'(R_0)$ in the fiber direction. By Lemma 36, for every $(X, \zeta)$ lying in the support of $C'(R)$, the real part of $\langle X, \mu(\Theta_t(X, \zeta)) \rangle$ is strictly negative. By compactness of $|(\Theta_t)_* C'(R_0)| \cap (\text{supp}(\varphi) \times T^* M)$, there exists an $\varepsilon' > 0$ such that, whenever $(X, \zeta)$ lies in the support of $(\Theta_t)_* C'(R_0)$ and $X$ lies in the support of $\varphi$, we have $\Re(\langle X, \mu(\zeta) \rangle) \leq -\varepsilon'$. Then, for all $R \geq R_0$ and all $(X, \zeta) \in |(\Theta_t)_* C'(R)| \cap (\text{supp}(\varphi) \times T^* X)$, we have $\Re(\langle X, \mu(\zeta) \rangle) \leq -\varepsilon' \frac{R_0}{R}$.

Since the integrand decays exponentially over the support of $(\Theta_t)_* C'(R)$, the integral tends to zero as $R \to \infty$. □

Thus, using Proposition 40,
\[
\int_{Ch(\mathcal{F})} \mu^*(\tilde{\varphi \alpha}) \wedge e^\sigma = \lim_{R \to \infty} \int_{\Omega \times (Ch(\mathcal{F}) \cap \{\|\zeta\|_{T^* M} \leq R\})} e^{(X, \mu(\zeta))} \varphi(X) \wedge \alpha(X) \wedge e^\sigma
\]
\[
= \lim_{R \to \infty} \int_{\Omega \times (Ch(\mathcal{F}) \cap \{\|\zeta\|_{T^* M} \leq R\})} \Theta_t^* \left( e^{(X, \mu(\zeta))} \varphi(X) \wedge \alpha(X) \wedge e^\sigma \right)
\]
\[
= \lim_{R \to \infty} \int_{C'(R) + \Omega \times \left( \hat{\Sigma}_{k=1}^d m_k(X)(T^*_{\omega(X)} \cdot z_k) \cap \{\|\zeta\|_{T^* M} \leq R\} \right)} \Theta_t^* \left( e^{(X, \mu(\zeta))} \varphi(X) \wedge \alpha(X) \wedge e^\sigma \right)
\]
\[
= \lim_{R \to \infty} \int_{\Omega \times \left( \hat{\Sigma}_{k=1}^d m_k(X)(T^*_{\omega(X)} \cdot z_k) \cap \{\|\zeta\|_{T^* M} \leq R\} \right)} \Theta_t^* \left( e^{(X, \mu(\zeta))} \varphi(X) \wedge \alpha(X) \wedge e^\sigma \right),
\]
i.e. the integral over $C'(R)$ can be ignored and we are left with integrals over $m_k(X) \left( \Omega \times (T^*_{\omega(X)} \cdot z_k) \cap \{\|\zeta\|_{T^* M} \leq R\} \right)$, for $k = 1, \ldots, d$. Because the integral converges absolutely, we can let $R \to \infty$ and drop the restriction $\|\zeta\|_{T^* M} \leq R$:
\[
\int_{Ch(\mathcal{F})} \mu^*(\tilde{\varphi \alpha}) \wedge e^\sigma = \int_{\Omega \times (\hat{\Sigma}_{k=1}^d m_k(X)(T^*_{\omega(X)} \cdot z_k) \cap \{\|\zeta\|_{T^* M} \leq R\})} \Theta_t^* \left( e^{(X, \mu(\zeta))} \varphi(X) \wedge \alpha(X) \wedge e^\sigma \right).
\]

Lemma 35 tells us that the maps $\Theta_t$ and $\Theta_t^k$ coincide over $T^*_{\omega(X)} \cdot z_k M$:
\[
\Theta_t |_{T^*_{\omega(X)} \cdot z_k M} = \Theta_t^k |_{T^*_{\omega(X)} \cdot z_k M}.
\]

We also have $\delta(\|z\|) = 1$, and the exponential part $\Theta_t^* \left( \langle X, \mu(\zeta) \rangle \right)$ of our integrand
\[
\Theta_t^* \left( e^{(X, \mu(\zeta))} \varphi(X) \wedge \alpha(X) \wedge e^\sigma \right)
\]
We also have \( \Theta^* \) is identically one, so the exponential part (29) reduces to can be ignored too: 

\[
\lim_{t \to 0^+} \int_{m_k(X)} (|\xi(X,\zeta)| \geq 1/t) \Theta^* (e^{(X \cdot \mu(\zeta))} \varphi(X) \wedge \alpha(X) \wedge e^\sigma) = 0.
\]

**Proof.** When \( ||\xi|| \geq 1/t \), \( \gamma(t||\xi||) \geq \frac{1}{t||\xi||} \) and the exponential part (29) is at most

\[-\frac{\varepsilon}{n} |\beta_{x_k,t}(X)||\xi|| - \frac{\varepsilon}{nt} |\beta_{x_k,t}(X)|.\]

But \( \xi_1 \xi_1 + \cdots + \xi_n \xi_n = ||\xi||^2 \), so at least one of the \( \xi_l \xi_l \geq ||\xi||^2/n \). Thus we get a new estimate of (29) from above:

\[-\frac{\varepsilon}{n} |\beta_{x_k,t}(X)||\xi|| \leq -\frac{\varepsilon}{nt} |\beta_{x_k,t}(X)|.\]

The last expression tends to \(-\infty\) as \( t \to 0^+ \), i.e. the integrand decays exponentially and the lemma follows. \( \square \)

Thus, in the formula (28) the integral over the portion

\[m_k(X) \left( \Omega \times \{T^*_\omega(X) \wedge \xi \geq 1/t\} \right)\]

can be ignored too:

\[
\int_{Ch(F)} \mu^*(\varphi) \wedge e^\sigma = \lim_{t \to 0^+} \int_{\sum_{k=1}^t m_k(X)} (|\xi(X,\zeta)| \geq 1/t) \Theta^* (e^{(X \cdot \mu(\zeta))} \varphi(X) \wedge \alpha(X) \wedge e^\sigma).
\]

Finally, over the portion \( m_k(X) \left( \Omega \times \{T^*_\omega(X) \wedge \xi \leq 1/t\} \right) \), the function \( \gamma(t||\xi||) \) is identically one, so the exponential part (29) reduces to

\[-t\varepsilon (|\beta_{x_k,1}(X)||\xi_1 + \cdots + |\beta_{x_k,n}(X)||\xi_n \xi_n|).
\]

We also have \( \Theta^*_\gamma = \varphi, \Theta^*_\xi(d\xi) = d\xi, \)

\[\Theta^*_\xi(d\zeta) = -d\left(t\varepsilon(t||\xi||) \frac{\beta_{x_k,l}(X)}{|\beta_{x_k,l}(X)|} \xi_l \right) = -t\varepsilon \frac{\beta_{x_k,l}(X)}{|\beta_{x_k,l}(X)|} d\xi_l, \quad \Theta^*_\sigma(d\zeta) = -t\varepsilon \frac{\beta_{x_k,l}(X)}{|\beta_{x_k,l}(X)|} d\xi_l \]

\[
\Theta^*_\xi(\sigma) = -t\varepsilon \frac{\beta_{x_k,1}(X)}{|\beta_{x_k,1}(X)|} d\xi_1 \wedge d\xi_1 - \cdots - t\varepsilon \frac{\beta_{x_k,n}(X)}{|\beta_{x_k,n}(X)|} d\xi_n \wedge d\xi_n.
\]
The form
\[ \tilde{\alpha}(X)_{[2n]} = \left( e^{X \cdot \mu(\zeta) + \sigma} \wedge \pi^* (\alpha(X)) \right)_{[2n]} = e^{X \cdot \mu(\zeta)} \sum_{l=0}^{n} \frac{1}{l!} \sigma^l \wedge \alpha(X)_{[2n-2l]}, \]
and we end up integrating
\[ e^{-t \varepsilon \left( |\beta_{x_k,1}(X)| \xi_1 \xi_1 + \ldots + |\beta_{x_k,n}(X)| \xi_n \xi_n \right)} \cdot \varphi(X) \wedge \left( (-t \varepsilon)^n \Theta_t^* (\pi^* \alpha(X)) \right) \left[ \frac{\beta_{x_k,1}(X)}{|\beta_{x_k,1}(X)|} \right] \ldots \left[ \frac{\beta_{x_k,n}(X)}{|\beta_{x_k,n}(X)|} \right] d\xi_1 \wedge d\xi_1 \wedge \ldots \wedge d\xi_n \wedge d\xi_n + \text{terms containing } \Theta_t^* (\pi^* \alpha(X)_{[2l]}, l > 0) \]
over \( m_k(X) (\Omega \times (T^*_{\omega(X)} - x_k M \cap \| \xi(X, \zeta) \| < 1/t)) \). (Recall that the orientation of this chain is determined by the product orientation on \( \Omega \times T^*_{\omega(X)} x_k M \), and the orientation of \( T^*_{\omega(X)} x_k M \) is given by (20).)

We can write
\[ \Theta_t^* (\pi^* \alpha(X)_{[0]}) = \alpha(X)_{[0]} (\omega(X) \cdot x_k) + t \sum_{a=1}^{n} (\xi_a A_a(X, t \xi_1, \ldots, t \xi_n) + \tilde{\xi}_a B_a(X, t \xi_1, \ldots, t \xi_n)) \]
for some bounded functions \( A_a, B_a \) of \( (X, t \xi_1, \ldots, t \xi_n) \), \( a = 1, \ldots, n \). We can also write
\[ \Theta_t^* (\pi^* \alpha(X)_{[2n]}) = t^2 \sum_{b,c=1}^{n} (C_{b,c}(X, t \xi_1, \ldots, t \xi_n) d\xi_b \wedge d\xi_c + D_{b,c}(X, t \xi_1, \ldots, t \xi_n) d\tilde{\xi}_b \wedge d\xi_c + E_{b,c}(X, t \xi_1, \ldots, t \xi_n) d\xi_b \wedge d\tilde{\xi}_c + F_{b,c}(X, t \xi_1, \ldots, t \xi_n) d\tilde{\xi}_b \wedge d\tilde{\xi}_c), \]
where each of \( C_{b,c}, D_{b,c}, E_{b,c}, F_{b,c} \) is a bounded function in terms of the variables \( (X, t \xi_1, \ldots, t \xi_n) \). Similarly we can express \( \Theta_t^* (\pi^* \alpha(X)_{[2l]}) \) for \( l = 1, \ldots, n \). Then, changing variables \( y_l = \sqrt{t} \xi_l \) for \( l = 1, \ldots, n \), we obtain the following estimate to (14):
\[ (-1)^n \int_{\Omega} m_k(X) \varphi(X) \int_{\|y_1\| \leq \ldots \leq \|y_n\| < \sqrt{t}} e^{-|\beta_{x_k,1}(X)||y_1|^2 - \ldots - |\beta_{x_k,n}(X)||y_n|^2} \left( \alpha(X)_{[0]} (\omega(X) \cdot x_k) \frac{\beta_{x_k,1}(X)}{|\beta_{x_k,1}(X)|} \ldots \frac{\beta_{x_k,n}(X)}{|\beta_{x_k,n}(X)|} d\bar{y}_1 \wedge d\bar{y}_1 \wedge \ldots \wedge d\bar{y}_n \wedge d\bar{y}_n \right) + \sqrt{t} \cdot (\text{bounded terms}). \]

By the Lebesgue dominant convergence theorem this integral tends to
\[ (-1)^n \int_{\Omega} m_k(X) \varphi(X) \alpha(X)_{[0]} (\omega(X) \cdot x_k) \int_{\|y_1\| \leq \ldots \leq \|y_n\| \in \mathbb{C}^n} e^{-|\beta_{x_k,1}(X)||y_1|^2 - \ldots - |\beta_{x_k,n}(X)||y_n|^2} \left( \frac{\beta_{x_k,1}(X)}{|\beta_{x_k,1}(X)|} \ldots \frac{\beta_{x_k,n}(X)}{|\beta_{x_k,n}(X)|} d\bar{y}_1 \wedge d\bar{y}_1 \wedge \ldots \wedge d\bar{y}_n \wedge d\bar{y}_n \right) \]
\[ = (-2\pi i)^n \int_{\Omega} m_k(X) \frac{\alpha(X)_{[0]} (\omega(X) \cdot x_k)}{\beta_{x_k,1}(X) \ldots \beta_{x_k,n}(X)} \varphi(X) \]
\[ \text{for } l = 1, \ldots, n \].
as $t \to 0^+$. The last expression may appear to have an extra factor of $(-1)^n$, but it is correct because of the convention explained in remarks 12 and 32. This proves formula (12) when the form $\varphi$ is supported inside $\Omega$. Then a simple partition of unity argument proves formula (12) when the form $\varphi$ is compactly supported in $g_R$, an open subset of the set of regular semisimple elements in $g_R$ whose complement has measure zero. Since $\Lambda$ is $G_R$-invariant and the form $\alpha$ is $U_R$-equivariant, $F_\alpha$ must be invariant under the adjoint action of $G_R \cap U_R$.

To prove the last statement of Theorem 20 we assume that $F_\alpha$ is a locally $L^1$ function on $g_R$ and drop the assumption that the support of $\varphi$ lies inside $g_R$. Let $\{\varphi_i\}_{i=1}^\infty$ be a partition of unity on $g_R$ subordinate to the covering by those open sets $\Omega$'s. Then $\varphi$ can be realized on $g_R$ as a pointwise convergent series:

$$\varphi = \sum_{i=1}^\infty \varphi_i \varphi.$$

Because $F_\alpha \in L^1_{\text{loc}}(g_R)$, the series $\sum_{i=1}^\infty \int_{g_R} F_\alpha \varphi_i \varphi$ converges absolutely. Hence

$$\int_{Ch(F)} \mu^*(\varphi \alpha) \wedge e^\sigma = \int_{Ch(F)} \left( \int_{g_R} \tilde{\alpha} \wedge \varphi(X) \right)$$

$$= \sum_{i=1}^\infty \int_{Ch(F)} \left( \int_{g_R} \tilde{\alpha} \wedge \varphi_i \varphi(X) \right) = \sum_{i=1}^\infty \int_{g_R} F_\alpha \varphi_i \varphi = \int_{g_R} F_\alpha \varphi,$$

which completes our proof of Theorem 20. □

7 A Gauss-Bonnet Theorem for Constructible Sheaves

In this section we use Theorem 20 to prove a generalization of the Gauss-Bonnet Theorem for constructible sheaves.

As before, let $G_C$ be a connected complex algebraic reductive group which is defined over $\mathbb{R}$, and let $G_R$ be a subgroup of $G_C$ lying between the group of real points $G_C(\mathbb{R})$ and the identity component $G_C(\mathbb{R})^0$. Let $g_C$ and $g_R$ be their respective Lie algebras. This time we require $U_R \subset G_C$ to be a compact real form of $G_C$, and let $u_R$ denote its Lie algebra. As before, $M$ is a smooth complex projective variety with a complex algebraic $G_C$-action on it such that a maximal complex torus $T_C \subset G_C$ acts on $M$ with isolated fixed points, and $F$ is a $G_R$-equivariant sheaf on $M$ with $\mathbb{R}$-constructible cohomology. We assume that the holomorphic moment map $\mu : T^* M \to g_C^*$ is proper on the set $\text{supp}(\sigma|_{Ch(F)})$. Let $n = \dim_C M$.

Pick a $U_R$-invariant connection $\nabla$ on the tangent bundle $TM$. Then N. Berline, E. Getzler and M. Vergne define in Section 7.1 of [BGV] the equivariant connection and the equivariant curvature $F_{u_R}$ associated to $\nabla$. After that they define the equivariant Euler form

$$\chi_{u_R}(\nabla)(X) = \det^{1/2}(-F_{u_R}(X)), \quad X \in u_R.$$

The form $\chi_{u_R}(\nabla)$ is $U_R$-equivariantly closed and its class in equivariant cohomology does not depend on the choice of the $U_R$-invariant connection $\nabla$. It is easy to see that the map $\chi_{u_R}(\nabla) : u_R \to \Omega^*(M)$ is polynomial and extends uniquely to a holomorphic polynomial (but not $G_C$-equivariant) function

$$\chi_{g_C} : g_C \simeq u_R \otimes_\mathbb{R} \mathbb{C} \to \Omega^*(M).$$

We use the following properties of $\chi_{g_C}$:

$$\chi_{g_C}(X)|_{2n} = \text{Euler form of } TM, \quad \forall X \in g_C;$$
\[ \chi_{\mathfrak{c}}(X)[2k] \in \Omega^{(k,k)}(M), \quad \forall k \in \mathbb{N}; \]

if \( p \in M_0(X) \), then
\[ \chi_{\mathfrak{c}}(X)[0](p) = i^n \cdot \text{Den}(p); \]

in particular, \( \chi_{\mathfrak{c}} \) satisfies the Conditions 14.

**Theorem 43** Under the above conditions, if \( \varphi \) is a smooth compactly supported differential form on \( \mathfrak{g}_\mathbb{R} \) of top degree,
\[
(2\pi)^{-\dim_{\mathbb{C}} M} \int_{\text{Ch}(\mathcal{F})} \mu^* (\varphi \chi_{\mathfrak{c}}) \wedge e^\sigma = (2\pi)^{-\dim_{\mathbb{C}} M} \int_{\text{Ch}(\mathcal{F})} \left( \int_{\mathfrak{g}_\mathbb{R}} \chi_{\mathfrak{c}} \wedge \varphi(X) \right) = \chi(M, \mathcal{F}) \cdot \int_{\mathfrak{g}_\mathbb{R}} \varphi,
\]

where
\[ \chi_{\mathfrak{c}}(X) = e^{(X, \mu(\zeta)) + \sigma} \chi_{\mathfrak{c}}(X), \]
\( \chi(M, \mathcal{F}) \) is the Euler characteristic of \( M \) with respect to \( \mathcal{F} \in C^b_{\mathbb{R} - c}(M) \).

**Remark 44** If \( \mathcal{F} \) is the constant sheaf \( \mathbb{C}_M \) on \( M \), then \( \text{Ch}(\mathcal{F}) = [M] \), the moment map \( \mu \) is automatically proper on \( |\text{Ch}(\mathcal{F})| = M \), and we obtain the classical Gauss-Bonnet theorem
\[ \chi(M) = (2\pi)^{-\frac{1}{2} \dim_{\mathbb{R}} M} \int_M \text{Euler class of } TM. \]

Here we do not even need the requirement that a maximal complex torus \( T_C \subset G_C \) acts on \( M \) with isolated fixed points.

**Proof.** First, we assume that the support of the test form \( \varphi \) lies in \( \mathfrak{g}'_\mathbb{R} \). An immediate application of Theorem 20 together with the property (30) show that
\[
(2\pi)^{-\dim_{\mathbb{C}} M} \int_{\text{Ch}(\mathcal{F})} \mu^* (\varphi \chi_{\mathfrak{c}}) \wedge e^\sigma = (2\pi)^{-\dim_{\mathbb{C}} M} \int_{\text{Ch}(\mathcal{F})} \left( \int_{\mathfrak{g}_\mathbb{R}} \chi_{\mathfrak{c}} \wedge \varphi(X) \right) = \int_{\mathfrak{g}_\mathbb{R}} E\varphi,
\]

where, using the global coefficient formula (13),
\[ E(X) = \sum_{x_k \in M_0(X)} m_k(X) = \sum_{x_k \in M_0(X)} \chi(M, \mathcal{F}_{O_k}) = \chi(M, \mathcal{F}). \]

Finally, the constant function \( \chi(M, \mathcal{F}) \) is clearly locally integrable with respect to the Lebesgue measure on \( \mathfrak{g}_\mathbb{R} \), hence the last part of Theorem 20 applies here and this proves Theorem 43 in general. \( \square \)

### 8 Duistermaat-Heckman Measures

As before, \( G_\mathbb{R} \) is a linear real reductive Lie group with complexification \( G_\mathbb{C} \), we denote by \( \mathfrak{g}_\mathbb{R} \) and \( \mathfrak{g}_\mathbb{C} \) their respective Lie algebras. We pick another subgroup \( U_\mathbb{R} \) of \( G_\mathbb{C} \) such that, letting \( \mathfrak{u}_\mathbb{R} \) be the Lie algebra of \( U \), we have an isomorphism \( \mathfrak{u}_\mathbb{R} \otimes_\mathbb{R} \mathbb{C} \simeq \mathfrak{g}_\mathbb{C} \). For instance, \( U_\mathbb{R} \) may equal \( G_\mathbb{R} \), but in most interesting situations \( U_\mathbb{R} \) is a compact real form of \( G_\mathbb{C} \).

Let \( M \) be a smooth complex projective variety equipped with an algebraic action of \( G_\mathbb{C} \) preserving a complex-valued 2-form \( \omega \), and suppose that the restriction of the \( G_\mathbb{C} \)-action to \( U_\mathbb{R} \) is Hamiltonian with respect to \( \omega \). In other words, there exists a moment map \( J : M \to \mathfrak{u}_\mathbb{R}^* \otimes_\mathbb{R} \mathbb{C} \simeq \mathfrak{g}_\mathbb{C}^* \) such that
\[ i(X_M)\omega = dJ(X), \quad \forall X \in \mathfrak{u}_\mathbb{R}. \]
Note that we do not require the 2-form $\omega$ to be symplectic, i.e. $\omega^{\dim_{\mathbb{R}} M/2} \neq 0$. Even the case $\omega = 0$, $J = 0$ is interesting enough, but, of course, symplectic forms are the most interesting ones. We can regard $J : M \to \mathfrak{g}_C^*$ as a map $J : \mathfrak{g}_C \to C^\infty(M)$. Then $\omega + J$ is an equivariantly closed form on $M$ for the action of $U_R$.

Recall that $\sigma$ denotes the canonical complex algebraic holomorphic symplectic form on the holomorphic cotangent bundle $T^*M$ and $\mu : T^*M \to \mathfrak{g}_C^*$ is the ordinary holomorphic moment map. Let $\Lambda \in L_M^+(\mathfrak{g}_C^*)$ be a conic real-Lagrangian $G_R$-invariant cycle in $T^*M$. As before, $n = \dim_{\mathbb{C}} M$. The Liouville form
\[
\frac{(\omega + \sigma)^n}{n!} = (\exp(\omega + \sigma))_{[2n]}
\]
determines a measure $\beta_\Lambda$ on $\Lambda$. We call the pushforward of this measure $(J + \mu)_*(\beta_\Lambda)$ on $\mathfrak{g}_C^*$ the Duistermaat-Heckman measure. That is, for a compactly supported smooth function $f \in C_c^\infty(\mathfrak{g}_C^*)$,
\[
\int_{\mathfrak{g}_C^*} f d(J + \mu)_*(\beta_\Lambda) \quad =_{def} \quad \int_{\Lambda} \frac{(\omega + \sigma)^n}{n!}(f \circ (J + \mu)). \tag{31}
\]
The right hand side of (31) converges whenever the map $J + \mu$ is proper on the set $\supp(\sigma|\Lambda)$. This happens whenever $\mu$ is proper on $\supp(\sigma|\Lambda)$. In particular, the pushforward $(J + \mu)_*(\beta_\Lambda)$ is well-defined when $\mu$ is proper on $|\Lambda|$.

Duistermaat-Heckman measures are important invariants of symplectic manifolds and there are so many papers on this subject that it is impossible to list them all. At first an explicit formula was given by J. J. Duistermaat and G. J. Heckman [DH] using the method of exact stationary phase in the special case when $G$ is a compact torus acting with isolated fixed points. It was extended to compact non-abelian groups by V. Guillemin and E. Prato [GP]. Then it was extended to compact non-abelian groups acting with possibly non-isolated fixed points by L. Jeffrey and F. Kirwan [JK]. Many recent results on Duistermaat-Heckman measures are obtained by computing their Fourier transforms using the integral localization formula and then inverting these Fourier transforms.

Since the cycle $\Lambda$ is real-Lagrangian and $G_R$-invariant, the moment map $\mu$ takes purely imaginary values on its support $|\Lambda|$: $\mu(|\Lambda|) \subset i\mathfrak{g}_R^* \subset \mathfrak{g}_R^* \oplus i\mathfrak{g}_R^* \simeq \mathfrak{g}_C^*$.

Since $M$ is compact, the support of $(J + \mu)_* (\beta_\Lambda)$, which must lie inside $(J + \mu)_* (|\Lambda|)$, is a subset of $\mathfrak{g}_C^* \simeq \mathfrak{g}_R^* + i\mathfrak{g}_R^*$ with bounded real part.

The Fourier transform of the Duistermaat-Heckman measure is a distribution on $\mathfrak{g}_R$, i.e. a continuous linear functional on the space $\Omega^{\text{top}}(\mathfrak{g}_R)$ consisting of differential forms of top degree on $\mathfrak{g}_R$ with compact support. For $\varphi \in \Omega^{\text{top}}(\mathfrak{g}_R)$, its Fourier transform $\hat{\varphi}$ is defined by (4); recall that $\hat{\varphi}(\xi)$ decays rapidly as $\xi \to \infty$ and the real part of $\xi$ stays uniformly bounded. Hence the value of the Fourier transform of $(J + \mu)_* (\beta_\Lambda)$ at $\varphi \in \Omega^{\text{top}}(\mathfrak{g}_R)$ is
\[
(J + \mu)_* (\beta_\Lambda)(\varphi) = \int_{\mathfrak{g}_C^*} \left( \int_{\mathfrak{g}_R} e^{\langle X, \xi \rangle} \varphi(X) \right) d(J + \mu)_* (\beta_\Lambda)
\]
\[
= \int_{\Lambda} \left( \int_{\mathfrak{g}_R} e^{\langle X, (J + \mu) (\zeta) \rangle} \varphi(X) \right) \frac{(\omega + \sigma)^n}{n!}, \quad X \in \mathfrak{g}_R, \ \zeta \in |\Lambda| \subset T^*M. \tag{32}
\]

We introduce a $U_R$-equivariant form $\alpha : \mathfrak{g}_C \to \Omega^*(M)$:
\[
\alpha(X) = \exp(J(X) + \omega),
\]

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then (32) can be rewritten as

\[
(J + \mu) \ast (\beta_{\Lambda})(\varphi) = \int_{\Lambda} \left( \int_{g_{\mathbb{R}}} e^{(X, \mu(\zeta)) + \sigma} \wedge \varphi(X) \wedge \alpha(X) \right)_{[\dim_{\mathbb{R}} M]} \\
= \int_{\Lambda} \left( \int_{g_{\mathbb{R}}} \bar{\alpha} \wedge \varphi(X) \right), \quad X \in g, \ \zeta \in \Lambda \subset T^* M. \quad (33)
\]

This integral is exactly of type (6), hence convergent. The generalized localization formula (12) immediately implies:

**Proposition 45** Suppose there exists a maximal complex torus \( T_{\mathbb{C}} \subset G_{\mathbb{C}} \) acting on \( M \) with finitely many isolated fixed points and that

\[
\omega \in \Omega(2,0)(M) \oplus \Omega(1,1)(M).
\]

Then the restriction of the Fourier transform of the Duistermaat-Heckman measure (33) to \( g_{\mathbb{R}} \) equals

\[
(J + \mu) \ast (\beta_{\Lambda})(\varphi) = \int_{g_{\mathbb{R}}} F_{\omega}(X) \varphi(X),
\]

where \( F_{\omega} \) is an \( \text{Ad}(G_{\mathbb{R}} \cap U_{\mathbb{R}}) \)-invariant function on \( g_{\mathbb{R}} \) given by the formula

\[
F_{\omega}(X) = (-2\pi)^{\dim_{\mathbb{R}} M/2} \sum_{p \in M_0(X)} m_p(X) e^{(X, J(p))} \frac{\mu_p(X)}{\text{Den}_p(X)},
\]

where \( M_0(X) \) is the set of zeros of the vector field \( X_M \) on \( M \), and \( m_p(X) \)'s are certain integer multiplicities given by formula (13).

Note that this formula for \((J + \mu) \ast (\beta_{\Lambda})\) is non-trivial even when \( \omega = 0, J = 0 \).

**References**


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