Simplifying Heat Diffusion Problems for Small Heat Transfer

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Abstract

The temperature distribution in a sphere for heat transfer coefficient $h$ equal 0 is presented. The main point is that the functional form is quite different from that for $h$ not equal zero. This is not in the literature. It is a useful approximation for small $h$ and small times and it makes the solution much clearer intuitively. For example recent research in cryobiology thaws cells at ultra-fast rates by suspending the cell in a medium containing carbon black which absorbs near IR laser light, generating a heat source surrounding the cell. It is necessary to know the rate (and uniformity) of warming which the cell experiences due to this indirect heating. Specific results are given.

keywords: heat diffusion, small heat transfer, cryobiology

Statement of Problem

Find the Temperature Distribution $T(r, t)$ for a homogenous sphere of radius $b$ subject to a time-independent heat source and boundary condition $\nabla T(b, t) = 0$ at the surface. The initial distribution $T(r, 0)$ and the heat source are given. The functional form of this solution for heat transfer coefficient $h = 0$ is different from that for $h > 0$ [1-3]. The general solution is applied to a specific research problem in cryobiology [4].

General Solution

The standard approach would be to find a steady-state solution $T_S(r)$ which generates the source and satisfies the boundary condition. Here there is no such solution because the temperature increases indefinitely as long as the heat source exists. We find the asymptotic behavior and subtract this from the desired solution. The difference is amenable to the standard approach. Specifically:

$$c_P \rho \frac{\partial T(r, t)}{\partial t} - k \nabla^2 T(r, t) = q(r)$$

$$T(r, 0) = F(r) \text{ and } \frac{\partial T(b, t)}{\partial r} = 0$$  \hspace{1cm} (1)

The heat source $q(r)$ and the initial temperature distribution $F(r)$ are given.

We reduce the problems to its essentials by using dimensionless variables $r \to rb$ and $t \to t\tau$ where $\tau = \frac{c_P \rho b^2}{k}$ and a heat source $Q(r) = \frac{k^2}{\tau} q(r)$. Then
\[ \frac{\partial T(r,t)}{\partial t} - \nabla^2 T(r,t) = Q(r) \]  
\[ T(r,0) = F(r) \text{ and } \frac{\partial T(1,t)}{\partial r} = 0 \]

**Asymptotic Behavior**

Integrating the basic equation, and using the subscript 'A' to denote the Volume Average gives

\[ \frac{\partial}{\partial t} \frac{1}{V} \int_V T(r,t) \, dV = \frac{1}{V} \int_V [\nabla^2 T(r,t) + Q(r)] \, dV = Q_A \]

\[ \frac{1}{V} \int_V [T(r,t) - Q_A t] \, dV = \text{constant} = \frac{1}{V} \int_V F(r) \, dV = F_A \]

This gives us the asymptotic behavior we need, namely

\[ T_{\infty}(t) = Q_A t + F_A \]  

**Time Independent Portion**

Thus \( \delta T(r,t) = T(r,t) - T_{\infty}(t) \) is amenable to the standard approach.

\[ \frac{\partial \delta T(r,t)}{\partial t} - \nabla^2 \delta T(r,t) = Q(r) - Q_A \equiv \delta Q(r) \]

\[ \frac{\partial \delta T(1,t)}{\partial r} = 0 \]

This has a Time Independent solution satisfying \( \nabla^2 T_0(r) = -\delta Q(r) \) which is found by integration

\[ T_0(r) = -\frac{1}{r} \int_0^r \int_0^{r''} \int_0^{r'} r'' r' \delta Q(r') = - \int_0^r \frac{dr''}{r''^2} \int_0^{r''} r'' r' \delta Q(r') \]  

**Complete Solution**

The desired solution is then:

\[ T(r,t) = T_{\infty}(t) + T_0(r) + A_0 + \sum_{n=1}^{\infty} A_n \frac{\sin(\alpha_n r)}{r} e^{-\lambda_n t} \]

where \( T_{\infty}(t) \) and \( T_0(r) \) are given in Eq.(4) and Eq. (6). The eigenfunctions are given in textbooks [1-3]. The eigenvalues \( \alpha_n \) are solutions of the equation \( \alpha = \tan(\alpha) \) and \( \lambda_n = \alpha_n^2 \). If \( \alpha_n = (n + \frac{1}{2}) \pi - \epsilon_n \), then a rough approximation is \( \epsilon_n \simeq \frac{1}{n + \frac{1}{2}} \pi \) for \( n \geq 1 \). (We use more accurate values.)

The initial condition is used to determine the coefficients \( A_n \) \( \text{n = 0 to } \infty \).

\[ T(r,0) = F(r) = F_A + T_0(r) + A_0 + \sum_{n=1}^{\infty} A_n \frac{\sin(\alpha_n r)}{r} \]

The functions \( \sin(\alpha_n r) \) are orthogonal so the coefficients \( A_n \) are found by integration.

\[ A_0 = -3 \int_0^1 r^2 \, dr \, T_0(r) \]

\[ A_n = \frac{2}{\sin^2(\alpha_n)} \int_0^1 r \, dr \, \sin(\alpha_n r) \left[ \delta F(r) - T_0(r) \right] \quad (n \geq 1) \]
Note: Constants do not contribute to the integral for $A_n; n \geq 1$

That concludes the formal solution to the general problem posed at the beginning. To translate this back into 'real' parameters one simply substitutes $r \rightarrow \frac{r}{b}, t \rightarrow \frac{t}{\tau}$ and $Q(r) \rightarrow \frac{k^2}{k} q(r)$.

**Comment**

A standard textbook problem resembles this one except that the boundary condition is

$$\frac{\partial \delta T(1,t)}{\partial r} = -h T(1,t) \ h > 0$$

(10)

Our solution is an approximation to that problem, valid for small $h$ and small times. However, the limit $h \rightarrow 0$ is by no means trivial and not in the literature [1,3]. The textbook solution is

$$T(r,t) = T_S(r) + \sum_{n=1}^{\infty} A_n \frac{\sin(\alpha_n r)}{r} e^{-\lambda_n t}$$

(11)

The $\alpha_n$ are solutions of the equation $\alpha = (1 - h) \tan(\alpha)$ and have to be found for each $h$. This certainly does not look like our solution. More seriously, as $h \rightarrow 0$, $T_S(r) \rightarrow \frac{Q_A}{3k} \rightarrow \infty$. This is actually cancelled by the behavior of the first term in the infinite series but that is not readily apparent. Our Eq. (7) is precisely the limit $h \rightarrow 0$ of the standard solution Eq. (11).

Furthermore, for small $h$ and small times, where our solution is a reasonable approximation, a linear temperature rise is superimposed on the transient term. That is quite clear in our solution but certainly not obvious from the solution with $h > 0$. The request for a simpler approximation for small $h$ and small times was the motivation for this paper [4].

The natural question is "What is Small?". For the Heat Transfer Coefficient, "Small $h$" ($h<<1$) means, in terms of real parameters, $h << \frac{k}{b}$. "Small Time" means $Q_A t << T_S$. Since $T_S \rightarrow \frac{Q_A}{3k}$ "Small Time" means $t << \frac{1}{h}$. In terms of real parameters this is $t << \tau \frac{h}{k}$ where $\tau$ is the characteristic time for transients to die out, namely $\tau = \frac{c \rho b^2}{k}$.

A more elegant solution to this problem is possible using Green’s Functions. [5]

**Real Life Apparatus**

The apparatus for which this solution was constructed [4] is a sphere of radius $a$ surrounded by a spherical shell whose outer diameter is 1. The inner sphere has no heat source and the outer shell has a uniform heat source. The whole apparatus initially is at a uniform temperature $T_i$ (typically Liquid Nitrogen temperature). In terms of our variables:

$$T(r,0) = F(r) = T_i$$

$$Q(r) = \begin{cases} 0 & 0 \leq r < a \\ Q_0 & a \leq r \leq 1 \end{cases} = Q_0 \Theta(r-a)$$

(12)

**Asymptotic Behavior**

$$Q_A = 3 \int_0^1 r^2 dr \ Q_0 \Theta(r-a) = 3 Q_0 \int_a^1 r^2 dr(r) = Q_0 (1 - a^3)$$

$$T_\infty(t) = Q_0 (1 - a^3) t + T_i$$

(13)
Time Independent Solution

\( T_0(r) \) is determined by \( Q(r) = Q_0 \, \Theta(r - a) \) where \( \Theta(r - a) \) is the Heaviside function; ( 0 for negative argument and 1 for positive argument).

\[
\nabla^2 T_0(r) = -\frac{\delta Q(r)}{Q_0} = -\Theta(r - a) + (1 - a^3) = -a^3 + \Theta(a - r)
\]

The first term \((-a^3)\) gives the Time Independent solution for a uniform heat source; namely \(-\frac{1}{6}a^3r^2\).

The second term gives

\[
\frac{1}{r} \int_0^r dr'' \int_0^{r''} r' dr' \Theta(a - r') = \frac{1}{6} \Theta(a - r) r^2 + \Theta(r - a) \left[ \frac{a^2}{2} - \frac{a^3}{3r} \right]
\]

Thus

\[
T_0(r) = \frac{Q_0}{6} \left( \begin{array}{c}
-a^3r^2 + r^2 & 0 \leq r < a \\
-a^3r^2 + 3a^2 - 2\frac{a^3}{r} & a \leq r \leq 1
\end{array} \right)
\]

Homogeneous Solution

\[
A_0 = -3 \int_0^1 r^2 dr \, T_0(r) = -\frac{Q_0}{2} a^2 \left[ 1 - \frac{6}{5} a + \frac{1}{5} a^3 \right]
\]

\[
A_n(n \geq 1) = -\frac{2}{sin^2(\alpha_n)} \int_0^1 r \, dr \, T_0(r) =
\]

\[
= -\frac{2}{sin^2(\alpha_n)} \frac{Q_0}{\alpha_n^4} \cos(a \alpha_n) \left[ a \alpha_n - tan(a \alpha_n) \right]
\]

Complete Solution

\[
T(r,t) = T_\infty(t) + T_0(r) + A_0 + \sum_{n=1}^{\infty} A_n \frac{sin(\alpha_n r)}{r} e^{-\lambda_n t}
\]

\[
T_\infty(t) = Q_0(1 - a^3) \, t + T_i
\]

\[
T_0(r) = \frac{Q_0}{6} \left( \begin{array}{c}
-a^3r^2 + r^2 & 0 \leq r < a \\
-a^3r^2 + 3a^2 - 2\frac{a^3}{r} & a \leq r \leq 1
\end{array} \right)
\]

\[
A_0 = -\frac{Q_0}{2} a^2 \left[ 1 - \frac{6}{5} a + \frac{1}{5} a^3 \right]
\]

\[
A_n(n \geq 1) = -\frac{2}{sin^2(\alpha_n)} \frac{Q_0}{\alpha_n^4} \cos(a \alpha_n) \left[ a \alpha_n - tan(a \alpha_n) \right]
\]

As mentioned earlier, for ‘real’ parameters one simply substitutes \( r \rightarrow \frac{r}{a} \), \( t \rightarrow \frac{t}{a} \) and \( Q(r) \rightarrow \frac{Q}{a^3} q(r) \).

Physical Interpretation

The first term, \( T_\infty(t) \) is a flat temperature distribution which starts at the initial temperature \( T_i \) and rises linearly with time, driven by the volume average of the heat source. The Time Independent term describes a sphere with a heat source which cools the inner sphere and heats the outer shell just enough so there is no net heat. Its temperature starts a zero at the center, rises sharply in the inner sphere but more slowly in the outer shell. The Homogeneous term represents the temperature distribution of the sphere with no heat source whose initial distribution exactly cancels the Time Independent term. It contains the transients. However, these decay with time and would approach a small, flat temperature distribution eventually.
Sample Curves

These results were evaluated with Mathematica. The notebooks are available upon request. Some sample curves are shown for typical parameters of the apparatus modeled by this problem [4] which are $T_i = -180^\circ C$, $a = 47.5 \mu m$, $b = 288 \mu m$, $\tau = 43.1 \text{ ms}$. The heat source is adjusted so the outer boundary reaches $0^\circ C$ in $t_0 = 1.2 \text{ ms}$.

![Sample Curves](image1)

Figure 1: Temperature distributions for various times from 0 to cutoff time of 1.2 ms.

![Sample Curves](image2)

Figure 2: Time dependence of the temperature at selected points in the sphere; heat source on.

What’s Next?

The application [4] which motivated this study applies a heat pulse for a short time $t_o$ determined by bringing the outer shell from $T_i$ to $0^\circ C$. After that, the system follows the Homogeneous Equation with the "initial condition" being $T(r, t_o)$. That is a standard problem. For convenience, let us call the Temperature Distribution for times greater than $t_o$ simply $T_2(r, t)$. 

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\[ T_2(r, t) = B_0 + \sum_{n=1}^{\infty} B_n \frac{\sin(\alpha_n r)}{r} e^{-\lambda_n (t-t_o)} \] (20)

The coefficients are found as before with the initial condition being \( T(r, t_o) \).

\[ T(r, t_o) = T_\infty(t_o) + \sum_{n=1}^{\infty} A_n \frac{\sin(\alpha_n r)}{r} (e^{-\lambda_n t_o} - 1) \]

\begin{align*}
B_0 &= Q_A t_o + T_i \\
B_n &= A_n (e^{-\lambda_n t_o} - 1) \quad (n \geq 1)
\end{align*} (21)

For longer times:

Figure 3: Temperature distributions for various times after the heat source is turned off.

Figure 4: Time dependence of the temperature at selected points in the sphere for longer times.
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Nomenclature

\[ h = \text{heat transfer coefficient, } W/m^2K \]
\[ c_p = \text{specific heat at constant pressure, } J/kgK \]
\[ \rho = \text{density, } kg/m^3 \]
\[ k = \text{thermal conductivity, } W/mK \]
\[ q = \text{heat source, } W/m^3 \]

References


[2] Carslaw, H.S., and Jeager, J.C., 1976, Heat Conduction in Solids, 2nd ed. Oxford University Press, Great Britian, pp. 245-246. Chap. IX. Like many references, this discusses the boundary condition \( \frac{\partial T(b,t)}{\partial r} + h T(b,t) = 0 \) for \( h > 0 \). Physically this lets heat escape and allows for a Steady State solution. For \( h \to 0 \), this Time Independent solution blows up like \( \frac{1}{h} \) and is of little help for our problem.
