Black Markets and Pre-Reform Crises in Former Socialist Economies

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Abstract

Boycko (1992) and others showed that wage increases in a socialist economy result in longer queues and lower output. Beyond certain level of shortages, wage increases may lead to a "near collapse" of the economy. We show that the presence of black markets alleviates this outcome. In particular, wage elasticity of output is always smaller in the framework that includes heterogenous agents and black markets.

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1. Introduction

The second economy or parallel markets arise in response to various restrictions on the official economic activities. While voluntary informal transactions are *ex ante* beneficial to the parties involved, the effect of these transactions on the overall economy may be difficult to ascertain. This issue was particularly important for the study of the pre-reform crises of centrally planned economies (CPE’s), where parallel markets were pervasive\(^1\). Many of the informal discussions of *perestroika* reforms and the pre-reform crisis in the former Soviet Union (FSU) emphasized the role of the second economy in policy outcomes\(^2\). The formal analyses of the pre-reform crises, however, usually did not address the role of parallel markets in a meaningful way\(^3\).

One of the best known examples of the formal analysis of problems that arose in some CPE’s prior to radical market-oriented reforms is Boycko (1992). He modeled an economy that produced one consumer good the monetary price of which was fixed below market clearing level. Rationing of the good was accomplished by

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\(^1\)See Stahl and Alexeev (1985) for a brief survey and references.
\(^2\)For example, Grossman (1977) and Alexeev et al. (1991).
queues, while a representative agent allocated his time between queuing, leisure, and work. Boycko demonstrated that in this queue-rationed economy, an increase in measured real wages leads to longer queues that results in smaller amount of time allocated to production.\textsuperscript{4} This in turn reduces the supply of the good, increasing the queues and output even further. Eventually, the economy may find an equilibrium characterized by longer queues and lower output than prior to the wage increase. Alternatively, under certain conditions, the economy may completely implode\textsuperscript{5}. A similar effect of the rise in wages on output in a queue-rationed CPE was modeled by Osband (1992). Unlike Boycko, Osband allowed for the possibility of resale of goods purchased at the official price. Because his model used a representative agent framework, however, it could not reflect meaningful consequences of the presence of parallel markets. Bennett (1991) contains a model with two types of consumers and parallel markets, but he limited his analysis to a rather special case where one type of consumers worked and did not queue, while the other consumers queued but did not work. Moreover, Bennett simply assumed the existence of market equilibrium and some important characteristics of excess demand functions in both the official and parallel markets.

This paper examines the impact of parallel markets on the Boycko-Osband-

\textsuperscript{4}In the FSU, the annual growth rate of measured real personal income increased from about 2\% in 1987 to 12\% in 1990. We use the term "measured real income" to denote income adjusted by the official price index.

\textsuperscript{5}As Boycko pointed out, the implosion story was unlikely as "consumers would rely more on home production and the second economy" [p. 918] but he did not model these possibilities.
Bennett effect (hereafter, BOB effect). We will evaluate the output effect of the increase in measured real wages in a queue-rationed economy with heterogeneous consumers, and compare the corresponding wage elasticity of output in the CPE’s with and without parallel markets. This comparison is missing from Boycko, Osband, and Bennett’s papers. It will complement their results, addressing one important aspect of the issue posed by Grossman (1985) of whether the second economy was indeed boon or bane for the first economy during the initial reforms.

We find that under a standard assumption of distorted wages (in the sense that wages in a CPE were more egalitarian than productivities), the wage elasticity of output is typically smaller in a CPE with parallel markets than in a pure queue-rationed CPE with the same degree of wage distortion. Numerical examples demonstrate that this result probably holds for a wider range of production technologies than those implied by the above condition on relative wages. Therefore, the presence of parallel markets alleviates the BOB effect. Intuitively, this happens because parallel markets make queuing more efficient by letting low productivity agents queue up instead of high productivity ones. When wages rise, causing an increase in queue length, the output loss in the presence of parallel markets is generally smaller than under pure queue-rationing, because low productivity agents have lower opportunity cost of queuing. The presence of black

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6 Bennet (1990) compared economies with and without black markets but in his model non-price rationing was done costlessly. He called the absence of queues in his model "a significant limitation" to be "remedied in later work." (p. 3)
markets mitigates the decline of total output despite the fact that total queuing time is higher than in the pure queue-rationed CPE.

The next section replicates the BOB effect in a pure queue-rationed CPE with heterogeneous consumers. In section 3, the model is modified to incorporate parallel markets. Section 4 compares output elasticities between the two models, and illustrates the results with numerical examples. Section 5 concludes.

2. Queue-Rationed Centrally Planned Economy

Households

Consider an economy with two types of individuals, more productive type-A agents and less productive type-B agents. We will denote the values pertaining to each type of agents by subscripts $i = A, B$.

Each individual derives his utility from leisure and consumption of a good. The quantity demanded of the good exceeds the quantity supplied due to the state-controlled price $b \equiv 1$ being fixed at below market-clearing level. In order to purchase one unit of the good from the state retail sector, an agent must spend $q$ units of time standing in line. Thus, the length of queues becomes an endogenous variable that equilibrates the state sector of the economy. The other possible uses of time by the agents include leisure and work. Denote leisure by $h_i$, labor supplied to a state-owned enterprise by $l_i$, and wage rate by $w_i$ (this is a measured real wage, i.e. $w_i \equiv \frac{W_i}{b}$, where $W_i$ is nominal wage). Then, normalizing
total available time to 1 and assuming Cobb-Douglas utility, each individual solves
the following optimization problem:

\[ \text{Max } \ln \{(C_i)^\alpha (h_i)^\beta \} \]  \hspace{1cm} (2.1)

subject to

\[ C_i = w_i l_i, \]  \hspace{1cm} (2.2)

\[ h_i + l_i + qC_i = 1. \]  \hspace{1cm} (2.3)

This problem results in the following optimal choices:

\[ C_i = \frac{\alpha w_i}{(\alpha + \beta)(1 + w_i q)} \]

\[ h_i = \frac{\beta}{\alpha + \beta} \]

\[ l_i = \frac{\alpha}{(\alpha + \beta)(1 + w_i q)}. \]  \hspace{1cm} (2.4)

Note that \((1 + qw_i)\) is the effective price of the consumption good facing each type of agent (i.e. the monetary price plus the "effort" price).

It will be useful to express (2.4) in terms of relative changes:

\[ \hat{C}_i = (1 - \theta_i)\hat{w}_i - \theta_i\hat{q} \]  \hspace{1cm} (2.5)

\[ \hat{l}_i = -\theta_i(\hat{w}_i + \hat{q}), \]

where hat denotes a percentage change \((\hat{x} \equiv dx/x)\), and \(\theta_i \equiv \frac{qw_i}{1 + qw_i}\) is the share of the "effort" price in the total price. We will use the same interpretation of a hat throughout the paper.
State Enterprise and Government

The state enterprise hires all labor supplied by the households at the government-set wage. Technology is of the form

\[ Y = Y(l_A, l_B), \]  

(2.6)

which translates into

\[ \hat{Y} = \gamma l_A + \mu l_B, \]  

(2.7)

where

\[ \gamma \equiv \frac{d \ln Y}{d \ln l_A} \quad \text{and} \quad \mu \equiv \frac{d \ln Y}{d \ln l_B} \]  

(2.8)

are the partial elasticities of output with respect to labor inputs, or the "true" cost shares of the two inputs:

\[ \gamma = \frac{Y_A l_A}{Y} \quad \text{and} \quad \mu = \frac{Y_B l_B}{Y}, \]  

(2.9)

and \( Y_i \equiv \partial Y / \partial l_i \) is the marginal product of agent \( i \), with \( Y_A > Y_B \).

Unlike in a market economy, the labor is not necessarily paid its marginal product in the socialist system (the state enterprise is not a profit-maximizing unit). Instead, the real measured wage (i.e. wage expressed in terms of the controlled monetary price) is greater than the agent’s marginal product (Boycko uses the same assumption of workers’ "overpayment"):

\[ w_i > Y_i \]  

(2.10)
The government in this economy pays the wage bill of the labor force and sets the state price. The government’s budget constraint is

\[ w_A l_A + w_B l_B = Y. \]  

(2.11)

**Market Clearing**

The model closes with the market clearing condition:

\[ C_A + C_B = Y, \]

which, expressed in percentage terms, and upon substituting the optimal values from (2.5), becomes

\[ \chi \hat{C}_A(\hat{q}, \hat{w}_A) + (1 - \chi) \hat{C}_B(\hat{q}, \hat{w}_B) = \gamma \hat{l}_A(\hat{q}, \hat{w}_A) + \mu \hat{l}_B(\hat{q}, \hat{w}_B), \]

(2.12)

where

\[ \chi \equiv \frac{C_A}{Y} = \frac{w_A l_A}{Y}. \]

(2.13)

is the consumption share of type-A agents.

The assumption of workers’ overpayment (equation (2.10)) implies

\[ \gamma < \chi, \]

(2.14)

\[ \mu < 1 - \chi. \]

It follows that in order for markets to clear, the production function must exhibit decreasing returns to scale, as

\[ \gamma + \mu < 1. \]

(2.15)
An equilibrium in this model is defined as (i) a queuing time $q^*$, and (ii) an allocation $(C_i^*, h_i^*, l_i^*)$ such that (a) for each individual, $(C_i^*, h_i^*, l_i^*)$ solves (2.1), (b) no excess demand exists in the goods market, i.e. $q^*$ solves (2.12).

For the sake of simplicity we assume that the government raises wages of both types of agents by the same proportion, i.e. $\hat{w}_A = \hat{w}_B = \hat{w}$. The market-clearing condition (2.12) produces the change in the equilibrium queuing time per unit as a function of the increase in wages

$$\tilde{q}^* = \{-1 + (q[\frac{(\chi - \gamma)w_A}{1 + qw_A} + \frac{(1 - \mu - \chi)w_B}{1 + qw_B}])^{-1}\} \hat{w}. \quad (2.16)$$

Substituting the above equilibrium queuing time into the consumers’ optimal labor supplies, we find the equilibrium wage elasticity of output, $k_1 \equiv \frac{\dot{Y}}{\dot{w}}$, to be

$$k_1 = -\frac{(\gamma w_A + \mu w_B) + (\mu + \gamma)qw_Aw_B}{(\chi - \gamma)w_A + (1 - \chi - \mu)w_B + (1 - \mu - \gamma)qw_Aw_B}. \quad (2.17)$$

Given (2.14) and (2.15), $k_1 < 0$ always. Thus, in a queue-rationed economy, an increase in real measured incomes excarcebates the existing shortages in the goods market, and leads to a higher total queuing time. Hence, labor supply is reduced, and output falls with elasticity $k_1$.

It will prove useful to express the wage elasticity of output in terms of productivities and policy variables. Substituting expressions for $\gamma, \mu$, and $\chi$,

$$k_1 = -\frac{Y_B(1 + \frac{Y_A}{Y_B}\eta)}{(w_B - Y_B) + \eta(w_A - Y_A)}, \quad (2.18)$$

where $\eta \equiv \frac{C_A/(1+qw_A)}{C_B/(1+qw_B)} > 0$ is the ratio of real consumption of the agents.
3. Queue-Rationed CPE with Black Markets

In this section, the framework is modified to incorporate black markets, where agents can costlessly resell goods obtained from the state sector. These prices, denoted by \( p \), are market clearing. Here we consider a situation where type-A agents only work and purchase all their consumption goods in the parallel sector, whereas all the queuing is performed by the type-B agents\(^7\). Two separate cases can be distinguished here: (i) type-B agents work and queue, and (ii) type-B agents only queue.

In both cases, the more productive **A-agents** solve problem (2.1) subject to

\[
\begin{align*}
pC_A &= w_A l_A, \\
h_A + l_A &= 1.
\end{align*}
\]

The optimal choices of these A-agents are

\[
\begin{align*}
C_A &= \frac{\alpha w_A}{(\alpha + \beta)p}, \\
l_A &= \frac{\alpha}{\alpha + \beta}.
\end{align*}
\]

**Case 1.** Type-B agents work and queue.

The **type-B agents** maximize (2.1) subject to

\[
\begin{align*}
C_A + C_B &= w_B l_B + pC_A, \\
h_B + l_B + q(C_A + C_B) &= 1.
\end{align*}
\]

\(^7\)The case where the type-A agents also queue does not affect the qualitative results.
The FOCs for this problem are

\[
\frac{\alpha h_B}{\beta C_B} = \frac{p}{w_B}, \tag{3.6}
\]

\[
p = 1 + qw_B, \tag{3.7}
\]

where the right-hand side of the last equation incorporates the implicit queuing wage earned through resale of goods in the parallel market\(^8\). In this case the queuing wage equals the wage rate \(w_B\) of the less productive workers in the public sector (otherwise, the type-B agents would not engage in both work and queuing.)

The corresponding optimal solutions of type-B agents are as follows:

\[
C_B = \frac{\alpha w_B}{(\alpha + \beta)p},
\]

\[
l_B = \frac{\alpha(1 - qw_A)}{(\alpha + \beta)p}. \tag{3.8}
\]

Note that as long as \(qw_A < 1\), type-B choose both to work and queue.

Expressing variables in (3.3) and (3.8) in terms of changes and proceeding as in the pure queue-rationed model, the wage elasticity of output in this case is:

\[
k_{2.1} = \frac{\mu(w_A + w_B)}{\mu(w_A + w_B) - w_B(1 - qw_A)}, \tag{3.9}
\]

or

\[
k_{2.1} = -\frac{Y_B}{w_B - Y_B}. \tag{3.10}
\]

Given (2.10), output falls when real measured wages are increased: \(k_{2.1} < 0.\)

\(^8\)Stahl and Alexeev (1985) show that this condition is necessary for equilibrium to exist in the CPE with black markets.
Case 2. Type-B agents only queue.

Once the increase in wages reaches the point at which \( qw_A = b ( = 1 ) \), type-B agents would quit their jobs and devote their entire non-leisure time to arbitrage activity presented by the black markets:

\[
I_B = \frac{\alpha (1 - qw_A)}{(\alpha + \beta)p} = 0.
\]

Production in this case will not depend on the wage policy of the government at all, since \( I_A = \frac{\alpha}{\alpha + \beta} \) is constant\(^9\), and hence,

\[
\hat{Y} = \gamma \hat{I}_A = 0. \tag{3.11}
\]

Therefore, output is perfectly inelastic with respect to wages, and

\[
k_{2,2} = 0. \tag{3.12}
\]

4. Comparison of the CPE’s with and without Black Markets

We now compare the wage elasticity of output in the pure queue-rationed CPE with that of the CPE with black markets, discuss the influence of black markets on agents’ welfare, and provide a numerical example.

Compare the two models’ wage elasticity of output (2.18) and (3.10) for given policy parameters. We show that under relatively weak conditions, the wage

\(^9\)In a more general set-up, the amount of labor supplied by agents will depend on the relative risk aversion parameter, which is equal to 1 for the Cobb-Douglas case considered here.
elasticity of output is smaller in the pure queue-rationed CPE than in the CPE with parallel markets, i.e.

\[
\frac{|k_1|}{|k_{2,1}|} > 1. \tag{4.1}
\]

Substituting the expressions in (2.18) and (3.10) into (4.1) yields

\[
\frac{|k_1|}{|k_{2,1}|} = \frac{(Y_B/(w_B - Y_B))^Q}{(Y_B/(w_B - Y_B))^{BM}} \cdot \frac{1 + \frac{Y_A}{Y_B}\eta}{1 + \frac{w_A - Y_A}{w_B - Y_B}\eta}^Q, \tag{4.2}
\]

where superscript \( Q \) denotes the pure queue-rationed CPE, and \( BM \) denotes the CPE with black markets.

If the two models are compared at the point where the marginal product of type-B agents is identical, the above expression simplifies to

\[
\frac{|k_1|}{|k_{2,1}|} = \left[ \frac{1 + \frac{Y_A}{Y_B}\eta}{1 + \frac{w_A - Y_A}{w_B - Y_B}\eta} \right]^Q. \tag{4.3}
\]

The necessary and sufficient condition for this ratio to be greater than 1 is

\[
\frac{Y_A}{Y_B} > \frac{w_A}{w_B}. \tag{4.4}
\]

This condition is likely to be met in a real world CPE, as they were notorious for equalization of workers’ incomes (”uravnilovka”). Thus, when marginal product of type-B agents is the same in the two models, an increase in real measured wages results in a smaller output decline in the economy with black markets than in the pure queue-rationed economy.

In a more general case, when marginal product of type-B agents differs in the
two models, the jointly sufficient conditions for (4.1) to hold are (4.4) and

\[
\left( \frac{Y_B^Q}{w_B - Y_B} \right)_Q \geq \left( \frac{Y_B^{BM}}{w_B - Y_B} \right)_B,
\]  

which means that the marginal product of type-B agents per unit of "overpayment" (or "subsidy") is larger in the pure queue-rationed CPE than in the CPE with black markets. Note that condition (4.5) is only a sufficient condition; a numerical example presented below shows that (4.1) holds at least for some technologies that do not possess the above characteristics.

In the case of complete specialization (i.e. more productive agents only work and less productive only queue and resell goods in the black market), output becomes completely inelastic with respect to changes in wages, as long as this specialization is maintained in equilibrium.

We have demonstrated the following Proposition.

Proposition. Assume that in the initial equilibrium the marginal products of type-B agents are the same in a pure queue-rationed CPE and in the CPE with black markets \((Y_B^Q = Y_B^{BM})\). Then (4.4) implies that \(|k_2| < |k_1|\). If \(Y_B^Q \neq Y_B^{BM}\), then conditions (4.4) and (4.5) are jointly sufficient for \(|k_2| < |k_1|\).

Numerical Examples

To supplement the above analytical results regarding output response to wage increases, we calibrated the models for two production technologies: in the first example the sufficient conditions (4.4) and (4.5) are satisfied. In the second ex-
ample they are violated. In all calibrations, the government raises wages of both types of workers in the same proportion; the controlled state price $b$ is normalized to 1; and wages of type A agents are kept 50% higher than those of type B:

$$b \equiv 1, \quad \frac{w_A}{w_B} = 1.5.$$

In calibrating the CPE with BM, only case 1 (i.e. type B agents work and queue) is considered. The wage elasticity of output is calculated as

$$k = \frac{\Delta Y/Y}{\Delta w_B/w_B},$$

where bars denote the average values between the two points. Finally, the chosen $w_B$’s are subject to restrictions $q > 0$ and $0 < l_B < 1$.

**Example 1.** The production function is of the form

$$Y = (a l_A + l_B)^{\delta}, \quad (4.6)$$

where $0 \leq \delta \leq 1$. In this case, the relative productivity of the agents, $Y_A/Y_B$, is constant and equals $a$ in the two models. Thus, in this example, the condition (4.4) is satisfied by construction. Let $\alpha = \beta = \delta = 1/2$, and $a = 2$. The results of the simulation are summarized in Table 1.

**Example 2.** In this example, the production function is

$$Y = a l_A^\delta + l_B^\delta. \quad (4.7)$$

Again, we choose $\alpha = \beta = \delta = 1/2$, and $a = 2$. This production function does not generate a constant relative marginal productivity of the two types of agents, and
hence, the condition (4.4) may or may not be satisfied. This simulation’s results are presented in Table 2.

In both examples, output is higher in the presence of black markets. More important, output is less sensitive to wage increases in the CPE with black markets than in the pure queue-rationed CPE, i.e. $|k_1| > |k_{2.1}|$. In the first example, both sufficient conditions (4.4) and (4.5) are satisfied $(2 = \frac{Y_A}{Y_B} > \frac{w_A}{w_B} = 1.5$ and $(\frac{Y_B}{w_B-Y_B})Q > (\frac{Y_B}{w_B-Y_B})^{BM}$). In the second calibration, however, neither (4.4) nor (4.5) hold, as wages rise. For example, for $w_B \in (1.6971, 1.8856)$, $\frac{Y_A}{Y_B} > \frac{w_A}{w_B}$, whereas for $w_B \in (1.8856, 2.0400)$, $\frac{Y_A}{Y_B} < \frac{w_A}{w_B}$, and $(\frac{Y_B}{w_B-Y_B})Q < (\frac{Y_B}{w_B-Y_B})^{BM}$.

5. Conclusions

We have demonstrated that the presence of black markets mitigates the effect of a wage increase on output in a queue-rationed CPEs. While the wage elasticity of output remains negative, under certain natural assumptions it is typically smaller in the framework with black markets. Our simulations suggest that this result most likely holds even when the aforementioned assumptions do not hold. In the model with parallel markets, more productive agents specialize in working and less productive agents specialize in queuing. When wages increase, this specialization partly offsets the negative effect of wage increase on output. Thus, the BOB effect is less pronounced implying that at least in one important respect, the presence of parallel markets alleviates the pre-reform crisis of a CPE.
References

[1] Alexeev M. (1991), ”If market clearing prices are so good then why doesn’t (almost) anybody want them?”. Journal of Comparative Economics, 15, 380-190.


**Notation**

\[ k_i = \text{wage elasticity of output in Model } i \]
\[ b = \text{nominal price of the public good (} b \equiv 1) \]
\[ p = \text{black market price} \]
\[ q = \text{queuing time per unit of the public good} \]
\[ l_{A(B)} = \text{labor of more (less) productive workers} \]
\[ Q = \text{total queuing time} \]
\[ Y = \text{total output} \]
\[ U_{A(B)} = \text{utility of more (less) productive workers} \]
\[ U = \text{total utility in the economy} \]
\[ w_{A(B)} = \text{real measured wage of more (less) productive workers} \]
Table 1: Example 1

<table>
<thead>
<tr>
<th>$w_B$</th>
<th>1.0000</th>
<th>1.0500</th>
<th>1.1025</th>
<th>1.1576</th>
<th>1.2155</th>
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<tbody>
<tr>
<td>CPE without BM (Model 1)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$</td>
<td>k_1</td>
<td>$</td>
<td>–</td>
<td>1.0490</td>
<td>1.0451</td>
</tr>
<tr>
<td>$q$</td>
<td>0.0329</td>
<td>0.1122</td>
<td>0.1922</td>
<td>0.2731</td>
<td>0.3551</td>
</tr>
<tr>
<td>$Y$</td>
<td>1.1987</td>
<td>1.1389</td>
<td>1.0823</td>
<td>1.0287</td>
<td>0.9779</td>
</tr>
<tr>
<td>$\frac{y_B}{w_B-y_B}$</td>
<td>0.7156</td>
<td>0.7185</td>
<td>0.7212</td>
<td>0.7238</td>
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<tr>
<td>$U_A$</td>
<td>-0.5145</td>
<td>-0.5474</td>
<td>-0.5796</td>
<td>-0.6113</td>
<td>-0.6424</td>
</tr>
<tr>
<td>$U_B$</td>
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<td>-0.7244</td>
<td>-0.7404</td>
<td>-0.7573</td>
<td>-0.7750</td>
</tr>
<tr>
<td>$U$</td>
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<td>-1.2718</td>
<td>-1.3200</td>
<td>-1.3686</td>
<td>-1.4174</td>
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<tr>
<td>$1+qw_A$</td>
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<td>1.1767</td>
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<td>1.6474</td>
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<tr>
<td>$1+qw_B$</td>
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<td>1.1178</td>
<td>1.2119</td>
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<td>1.4316</td>
</tr>
<tr>
<td>$Q$</td>
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<td>0.1278</td>
<td>0.2080</td>
<td>0.2810</td>
<td>0.3472</td>
</tr>
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</table>

| $|k_{2.1}|$ | –      | 0.6985 | 0.6808 | 0.6629 | 0.6446 |
|---------|--------|--------|--------|--------|--------|
| $q$    | 0.0355 | 0.1190 | 0.2006 | 0.2802 | 0.3579 |
| $Y$    | 1.2071 | 1.1667 | 1.1286 | 1.0926 | 1.0588 |
| $\frac{y_B}{w_B-y_B}$ | 0.0771 | 0.6897 | 0.6718 | 0.6537 | 0.6353 |
| $U_A$  | -0.5079| -0.5249| -0.5415| -0.5577| -0.5734|
| $U_B$  | -0.7106| -0.7276| -0.7443| -0.7604| -0.7761|
| $U$    | -1.2185| -1.2525| -1.2858| -1.3181| -1.3495|
| $p$    | 1.0355 | 1.1250 | 1.2211 | 1.3243 | 1.4350 |
| $Q$    | 0.0429 | 0.1389 | 0.2264 | 0.3061 | 0.3789 |
Table 2: Example 2

<table>
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<th>CPE without BM (Model 1)</th>
<th>CPE with BM (Model 2)</th>
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<td>$w_B$</td>
<td>1.7000</td>
<td>1.8000</td>
</tr>
<tr>
<td>$</td>
<td>k_1</td>
<td>$</td>
</tr>
<tr>
<td>$Y$</td>
<td>$2.1175$</td>
<td>$1.9939$</td>
</tr>
<tr>
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<td>$2.0446$</td>
</tr>
<tr>
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<td>$0.7414$</td>
</tr>
<tr>
<td>$U_A$</td>
<td>$-0.2272$</td>
<td>$-0.2659$</td>
</tr>
<tr>
<td>$U_B$</td>
<td>$-0.4292$</td>
<td>$-0.4466$</td>
</tr>
<tr>
<td>$U$</td>
<td>$-0.6564$</td>
<td>$-0.7125$</td>
</tr>
<tr>
<td>$1 + qw_A$</td>
<td>$1.0041$</td>
<td>$1.1488$</td>
</tr>
<tr>
<td>$1 + qw_B$</td>
<td>$1.0027$</td>
<td>$1.0992$</td>
</tr>
<tr>
<td>$Q$</td>
<td>$0.0034$</td>
<td>$0.1099$</td>
</tr>
</tbody>
</table>