Modern view of Planetary Orbits

• Based on Kepler’s analysis of Brahe’s data
• Kepler’s 3 laws
• Newton’s 3 laws
• Newton’s version of Kepler’s law
**Nomenclature**

- **Elongation**: angle between sun center and direction to planet (east/west wrt to the sun)
- **Opposition**: elongation of 180°
- **Conjunction**: elongation of 0°
- **Quadrature**: elongation of 90°
- **Superior Planet**: orbital radius larger than Earth’s
- **Inferior Planet**: orbital radius smaller than Earth’s

**Synodic Period**: time to return to the same position on the sky as seen from the Earth.

**Sidereal Period**: time to complete one orbit wrt the stars.
Synodic Periods

Earth moves at an angular rate of $\sim 360^\circ/E$ per day

A planet moves at an angular rate of $\sim 360^\circ/P$ per day

To calculate synodic periods:
For a superior planet, earth completes 1 orbit, but then must complete an additional $S x (360/P)$ in the time $S-E$ to catch up:

$$ (S-E)(360/E) = S(360/P) $$

$$ \Rightarrow \frac{1}{S} = \frac{1}{E} - \frac{1}{P} $$

$S =$ Synodic period
$P =$ Planet sidereal period
$E =$ Earth’s sidereal period
Kepler’s Laws

• The orbit of each planet is an ellipse with the Sun at one focus

• The radius vector sweeps out equal areas in equal intervals of time

• Harmonic law: \( P^2 = k a^3 \)
  
  *\( P = \) sidereal period; \( a = \) semimajor axis*
Newton’s Laws

• The velocity of a body remains constant (in both magnitude and direction) unless a net force acts upon the body.
  – Practical application: hockey puck on ice

• The acceleration is proportional to and in the direction of the force applied and inversely proportional to the mass.
  – Practical application: car acceleration
  – \( \ddot{a} = \frac{F}{m} \)

• For every force acting, there is an equal and opposite force exerted.
  – Practical application: book on a table
Vector review

- Velocity, acceleration, and force are all vector quantities.

\[ \mathbf{v}_1 + \mathbf{v}_2 \]

\[ \mathbf{v}_1 \quad \mathbf{v}_2 \]

\[ \mathbf{v}_1 + \mathbf{v}_2 \]
Circular Motion

• To keep object moving in a circle, you must apply a force:

\[ \Delta \theta = \frac{s}{r} = \frac{v}{r} \Delta t \]

and,

\[ \Delta \theta = \frac{\Delta v}{v} \]

So, \( a = \frac{\Delta v}{\Delta t} = \frac{v^2}{r} \)

\[ F_{\text{cent}} = m \cdot a = m \cdot \frac{v^2}{r} \]
Circular Orbits

• If $P = $ orbital period, then: $v = \frac{2 \pi r}{P}$

• Use Kepler’s laws: $P^2 = k r^3$
  Then, $v = \frac{2 \pi r}{\sqrt{k r^3}}$

\[
\text{and, } F_{\text{cent}} = \frac{(m/r)(2 \pi)^2 r^2}{P^2} = \frac{4 \pi^2 m r^2}{(r k r^3)}
= 4 \pi^2 \frac{m}{k r^2}
\]

• Mutual gravitational force is product of Mm
  $F_{\text{grav}} = \frac{G M m}{r^2}$
  $=> k = \frac{4 \pi^2}{G M}$
Gravity

• $F = m \ a$
• $F_{\text{grav}} = G \ M \ m / r^2$
• $a_{\text{grav}} = G \ M / r^2$

• For Earth, what is the gravitational acceleration?

$\begin{align*}
M &= 5.98 \times 10^{24} \text{ kg} \\
G &= 6.67 \times 10^{-11} \text{ N} \text{ m}^2/\text{kg}^2 \\
R &= 6378 \times 10^3 \text{ m}
\end{align*}$

Answer: $a_{\text{grav}} = 9.805 \text{ m s}^{-2}$

• What we call “weight” = $m \ g$
Gravity continued…

Application of: \( F = \frac{G M m}{r^2} = m \frac{v^2}{r} \)

Satellite of Mars:
- Phobos with \( R = 9.37 \times 10^3 \) km
- \( P = 0.32 \) days

\[
F = \frac{G M m}{R^2} = m \frac{v^2}{R}
\]

\[
v = \frac{2 \pi R}{P} \Rightarrow G \frac{M m}{R^2} = \left(\frac{m}{R}\right)(4 \pi^2 \frac{R^2}{P^2})
\]

\[
M = \frac{4 \pi^2 R^3}{G P}
\]

\[
M = \frac{4 \pi^2 (9.37 \times 10^3 \text{ km} \times 10^3 \text{ m/km})^3}{(6.67 \times 10^{-11} \text{ N m}^2/\text{kg}^2) (0.32 \text{ day} \times 24 \text{ h/day} \times 3600 \text{ s/h})^2}
\]

\[
= 6.4 \times 10^{23} \text{ kg}
\]
Review:

• Kepler (empirical): \( P^2 = k a^3 \)

• Newton (physics): \( \vec{F} = m \dot{\vec{a}} \)
\[
\vec{F} = \frac{G M m}{r^2} \hat{r}
\]

For circular motion: \( \vec{a} = \frac{v^2}{r} \hat{r} \)
\[
\frac{G M m}{r^2} \hat{r} = m \frac{v^2}{r} \hat{r}
\]

Since \( v = 2 \pi \frac{R}{P} \),
\[
\frac{G M m}{r^2} \hat{r} = 4 \pi^2 m \frac{r}{P^2} \hat{r}
\]
By and large, everything we have considered so far is for a small (negligible) mass orbiting a massive (dominant) object. In reality, the gravitational perturbations of the smaller object cause the massive object to orbit as well.
The Two-Body Problem

- Where is center of mass?

\[ \vec{F}_1 = m_1 \frac{v_1^2}{r_1} \hat{r} = 4 \pi^2 \frac{m_1 r_1}{P^2} \hat{r} \]
\[ \vec{F}_2 = m_2 \frac{v_2^2}{r_2} \hat{r} = 4 \pi^2 \frac{m_2 r_2}{P^2} \hat{r} \]

since \( \vec{F}_1 = \vec{F}_2 \)

\[ \Rightarrow \quad \frac{r_1}{r_2} = \frac{m_2}{m_1} \]

More massive object has a smaller radius than low mass object.
Recovering Kepler’s 3rd Law

Total separation: \( a = r_1 + r_2 \)

So, \( r_1 = r_2 \frac{m_2}{m_1} = (a - r_1) \frac{m_2}{m_1} \)

\[ r_1 (1 + \frac{m_2}{m_1}) = a \frac{m_2}{m_1} \]

\[ r_1 (m_1 + m_2) = a m_2 \]

\[ r_1 = a \frac{m_2}{m_1 + m_2} \]

\[ \overrightarrow{F_{grav}} = G \frac{m_1 m_2}{a^2} \]

\[ r = 4 \pi^2 \frac{m_1 r_1}{P^2} \]

So, \( P^2 = 4 \pi^2 \frac{m_1 r_1 a^2}{G m_1 m_2} = 4 \pi^2 \left( \frac{a m_2}{(m_1 + m_2)} \right) a^2 / G m_2 \)

\[ P^2 = \frac{4 \pi^2 a^3}{G (m_1 + m_2)} \]

Newton’s version of Kepler’s 3rd law
Conservation of Energy

• Total energy is conserved:
  \[ \text{T.E.} = \text{K.E.} + \text{P.E.} \]

  • For an object in motion:
    \[ \text{K.E.} = \frac{1}{2} m v^2 \]

  • For gravity:
    \[ \text{P.E.} = - \frac{G M m}{r} \]

  \[ \text{T.E.} = \frac{1}{2} m v^2 - \frac{G M m}{r} \]

Conservation implies:

\[ \left[ \frac{1}{2} m v^2 - \frac{G M m}{r} \right]_{\text{initial}} = \left[ \frac{1}{2} m v^2 - \frac{G M m}{r} \right]_{\text{final}} \]
Escape Velocity

Consider limiting case where $v \to 0$ when $r \to \infty$:

$$\left[ \frac{1}{2} m v^2 - \frac{G M m}{r} \right]_{\text{initial}} = 0 - 0$$

(i.e., total energy = 0)

So, \( \frac{1}{2} m v^2 = \frac{G M m}{r} \)

\[ v_{esc} = \left( \frac{2 G M}{r} \right)^{0.5} \]
Energy

- Kinetic: \( \frac{1}{2} m v^2 \)
- Potential: \(-G \frac{M}{r} m\)

Toss a ball in the air:

(1) K.E. + P.E. < 0 “bound”

For planetary orbits: T.E. = \(-GMm/2a\)

(2) K.E. + P.E. > 0 “unbound”
(3) Intermediate case: minimum velocity needed for an object to escape the system:

\[ v \to 0 \quad \text{as} \quad R \to \infty \]

\[ \text{T.E.} = \frac{1}{2} mv^2 - \frac{GMm}{r} \quad \text{at} \quad R = \infty, \quad \text{T.E.} = 0 \]

Since energy is conserved, \( \frac{1}{2} mv^2 = \frac{GMm}{r} \quad \text{at all points in orbit} \)

Therefore, escape velocity \( v_{\text{esc}} = (2 \frac{GM}{r})^{0.5} \)
Conservation of Energy and Orbital Velocities: 2 body problem

• T.E. = K.E. + P.E. = \( \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 - Gm_1m_2/r \)
• From conservation of momentum:
  \( m_1 v_1 = m_2 v_2 \)
• By definition: \( v = v_1 + v_2 \) \( \rightarrow \) \( v_1 = m_2 v/(m_1+m_2) \) and \( v_2 = m_1 v/(m_1+m_2) \)

\[
T.E. = \frac{1}{2} m_1 \left( \frac{m_2 v}{(m_1+m_2)} \right)^2 + \frac{1}{2} m_2 \left( \frac{m_1 v}{(m_1+m_2)} \right)^2 - Gm_1m_2/r \\
= \frac{1}{2} \left[ \frac{m_1 m_2}{(m_1+m_2)} \right] v^2 - Gm_1m_2/r
\]
Conservation of Energy and Orbital Velocities

- At perihelion: T.E. = \(-Gm_1m_2/2a\)
- Plug this into above and:
  \(-Gm_1m_2/2a = \frac{1}{2} \left[ m_1 m_2/(m_1+m_2) \right] v^2 - Gm_1m_2/r\)

\[ v^2 = 2(m_1+m_2)(G/r - G/2a) \]

\[ v^2 = G(m_1+m_2)(2/r - 1/a) \] vis viva equation
Sanity check…

Limit of circular orbit:

\[ v^2 = G \left( m_1 + m_2 \right) \left( \frac{2}{r} - \frac{1}{a} \right) \]  
vis viva equation

For a circular orbit: \( r = a \)

\[ v^2 = G \left( m_1 + m_2 \right) \left( \frac{2}{a} - \frac{1}{a} \right) \]

\[ = G \left( m_1 + m_2 \right) / a \]

\[ v = 2 \pi a / P \quad \Rightarrow \quad v^2 = 4 \pi^2 a^2 / P^2 \]

From \( P^2 = 4 \pi^2 a^3 / G \left( m_1 + m_2 \right) \)

\[ v^2 = \left( 4 \pi^2 a^2 \right) G(m_1 + m_2) / \left( 4 \pi^2 a^3 \right) = G \left( m_1 + m_2 \right) / a \]  
\( \checkmark \)
Equation of an Ellipse

- In an elliptical orbit:

  \[ b^2 = a^2(1 - e^2) \]
  \[ a = \text{semi-major axis} \]
  \[ b = \text{semi-minor axis} \]
  \[ e = \text{eccentricity} \]

For a circle, \( e = 0 \); for a straight line, \( e = 1 \)

Ellipse in polar coords: \[ r = a(1 - e^2) / (1 + e \cos \theta) \]

At perihelion: \( \theta = 0 \) \( \cos \theta = 1 \)
  \[ r = a(1-e)(1+e)/(1+e) = a(1-e) \]

At aphelion: \( \theta = 180 \) \( \cos \theta = -1 \)
  \[ r = a(1-e)(1+e)/(1-e) = a(1+e) \]
Least Energy Orbits

• Total energy is proportional to $1/a$

Least energy orbit: $r_{ap} = $ semi-major axis of superior planet
$r_{peri} = $ semi-major axis of inferior planet

\[
a = \frac{1}{2} (r_{peri} + r_{ap})
\]

\[
v = \left[GM \left(\frac{2}{r} - \frac{1}{a}\right)\right]^{0.5}
\]
Least Energy Orbits cont….

• Launch Speeds: $v = v_{ap}$ for Earth to Venus, Mercury
  $v = v_{peri}$ for Earth to Mars, Jupiter, etc.

Specific case of Earth to Jupiter:

\[
a = \frac{1}{2} (r_{ap} + r_{peri}) = \frac{1}{2} (5.20 \text{ AU} + 1 \text{ AU}) = 3.1 \text{ AU} = 4.64 \times 10^{11} \text{ m}
\]

\[
v = [G M (2/r - 1/a)]^{0.5} = [G M_{\text{sun}}(2/1.496 \times 10^{11} - 1/4.64 \times 10^{11})]^{0.5}
\]

\[
G = 6.67 \times 10^{-11} \text{ N m}^2/\text{kg}^2; \quad M_{\text{sun}} = 1.99 \times 10^{30} \text{ kg}
\]

\[
v = 38.6 \text{ km/s}
\]

BUT, Earth is already moving at $\sim 29.8 \text{ km/s}$

Launch speed = $38.6 - 29.8 \text{ km/s} = 8.8 \text{ km/s}$ in direction of Earth’s orbit.
Least Energy Orbits cont....

- **Travel Time:** \( P^2 = a^3 \) (in years and AU)

Specific case of Earth to Jupiter:

\[
\begin{align*}
P^2 &= (3.1)^3 \\
P &= 5.46 \text{ yr}
\end{align*}
\]

So, \( \text{travel time} = \frac{1}{2} \text{ period} \)

travel time = 2.73 years