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Fixed points of commutative Lüders operations

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Abstract
This paper verifies a conjecture posed in a pair of papers on the fixed point sets for a class of quantum operations. Specifically, it is proved that if a quantum operation has mutually commuting operation elements that are effects forming a resolution of the identity, then the fixed point set of the quantum operation is exactly the commutant of the operation elements.

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1. Introduction
Let $H$ be a complex Hilbert space, $\mathcal{B}(H)$ be the bounded linear operator set on $H$. If $A \in \mathcal{B}(H)$ and $0 \leq A \leq I$, then $A$ is called a quantum effect on $H$. Each quantum effect can be used to represent a yes–no measurement that may be unsharp [1–6]. The set of all quantum effects on $H$ is denoted by $\mathcal{E}(H)$; the set of all orthogonal projection operators on $H$ is denoted by $\mathcal{P}(H)$. Each element $P$ of $\mathcal{P}(H)$ can be used to represent a yes–no measurement that is sharp [1–6]. Let $\mathcal{T}(H)$ be the set of all trace class operators on $H$ and $\mathcal{D}(H)$ be the set of all density operators on $H$, i.e. $\mathcal{D}(H) = \{ \rho : \rho \in \mathcal{T}(H), \rho \geq 0, \text{tr}(\rho) = 1 \}$. Each element $\rho$ of $\mathcal{D}(H)$ represents a state of the quantum system $H$.

Let $\{ E_i \}_{i=1}^n \subseteq \mathcal{E}(H)$ be the quantum measurement, that is $\sum_{i=1}^n E_i^2 = I$ in the strong operator topology, where $1 \leq n \leq \infty$, then the probability of outcome $E_i$ measured in the state $\rho$ is given by $\text{tr}(\rho E_i)$, and the new quantum state after the measurement $\mathcal{A}$ is performed is defined by

$$\Phi(\rho) = \sum_{i=1}^n E_i \rho E_i.$$ 

Note that $\Phi : \rho \to \sum_{i=1}^n E_i \rho E_i$ defined a transformation on the state set $\mathcal{D}(H)$; we call it the Lüders transformation [6, 7]. In physics, the question whether a state $\rho$ is not disturbed

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by the measurement $A = \{E_i\}_{i=1}^n$ becomes equivalent to the fact that $\rho$ is a solution of the equation

$$\Phi(\rho) = \sum_{i=1}^n E_i \rho E_i = \rho.$$ 

It was showed in [8] that the measurement $A = \{E_i\}_{i=1}^2$ does not disturb $\rho$ if and only if $\rho$ commutes with each $E_i$, $i = 1, 2$.

Moreover, if we define the *Luders quantum operation* $\Phi_A$ on $\mathcal{B}(H)$ as

$$\Phi_A : \mathcal{B}(H) \to \mathcal{B}(H), \quad B \to \Phi_A(B) = \sum_{i=1}^n E_i B E_i,$$

then an interesting problem is that if $B \in \mathcal{B}(H)$ is a fixed point of $\Phi_A$, that is, $\Phi_A(B) = \sum_{i=1}^n E_i B E_i = B$, then $B$ commutes with each $E_i$, $i = 1, 2, \ldots, n$.

In [9, 10], we knew the conclusion is true if $H$ is a finite-dimensional complex Hilbert space. In [9–11], it was showed that the conclusion is not true when $n = 5$ or $n = 3$ for infinite-dimensional complex Hilbert spaces. Thus, the general conclusion for infinite-dimensional cases is false. On the other hand, Busch and Singh in [8] showed that for $\mathcal{E}$ the conclusion is true for all complex Hilbert spaces. Note that in this case, $E_1 E_2 = E_2 E_1$, that is, $A = \{E_1, E_2\}$ is commutative. This motivated Arias, Gheonda, Gudder and Nagy to conjecture when $A = \{E_i\}_{i=1}^n \subseteq \mathcal{E}(H)$ is commutative, then the conclusion is true, that is, the fixed point set of $\Phi_A$ is exactly the commutant $A'$ of the operation elements $A = \{E_i\}_{i=1}^n$. Moreover, Nagy in [12] showed that if the conjecture is true, then

$$\Phi_A(E) = \sum_{i=1}^n E_i E E_i = I - E$$

has the unique solution $\frac{1}{2}I$ in $\mathcal{E}(H)$; in physics, it showed that if the measurement $A$ disturbs the quantum effect $E$ completely into its supplement $I - E$, then $E$ has to be $\frac{1}{2}I$.

As showed in [13–16], the structures of fixed point sets of quantum operations have important applications in quantum information theory; in particular, in [15, theorem 3], the fixed point set is a matrix algebra which shares an elegant structure, played a central role in identifying the protected structures.

In this paper, by using the spectral theory of self-adjoint operators, we prove the conjecture affirmatively. Moreover, when $A = \{E_i\}_{i=1}^n \subseteq \mathcal{E}(H)$ is commutative and $F = \sum_{i=1}^n E_i^2 < I$, we also obtain a nice conclusion. Note that the von Neumann algebra $\mathcal{N}$ generated by $\{E_i\}_{i=1, \ldots, n}$ is Abelian which can be embed into a maximal Abelian von Neumann algebra. Since a maximal Abelian von Neumann algebra $\mathcal{M}$ on a separable Hilbert space is always a direct sum of $\mathcal{M}_1$ and $\mathcal{M}_2$. Here $\mathcal{M}_1$ is isometric to $\bigoplus_{i=1}^{\infty} C_{i}$ and $\mathcal{M}_2$ is isometric to $L_{\infty}(B)$, where $B$ is a compact subset of the real number set $R$. Thus, $A'$ has the form $\bigoplus_{k=1}^{\infty} M_k \otimes 1_{n_k} \bigoplus L_{\infty}(C)$, where $C$ is a subset of $B$ and $M_k$ is a matrix algebra whose dimension is $k$ and $n_k$ ranges from $0$ to $\infty$ [17]. So our conclusions are analogous with the finite-dimensional cases’ concise shape in theorem 3 in [15].

2. Element lemmas and proofs

Let $1 \leq n < \infty$ and $A = \{E_i\}_{i=1}^n \subseteq \mathcal{E}(H)$ be commutative. Firstly, for each $E_i$, $1 \leq i \leq n$, we have the spectral representation theorem

$$E_i = \int_0^1 \lambda \, dF_{E_i}^{(i)}.$$
where \( \{ F^{(i)}_\lambda \}_{\lambda \in \mathbb{R}} \) is the identity resolution of \( E_i \) satisfying that \( \{ F^{(i)}_\lambda \}_{\lambda \in \mathbb{R}} \) is right continuous in the strong operator topology and \( F^{(i)}_\lambda \) is right continuous for any rational number \( q = \frac{p}{l} \), where \( p, l \) are integers. If \( \frac{p}{l} < 0 \), then \( F^{(i)}_\lambda = 0 \), and if \( \frac{p}{l} > 1 \), then \( F^{(i)}_\lambda = I \). Let \( l > p \geq 0 \), so \( 0 \leq \frac{p}{l} < 1 \). Then \( F^{(i)}_\lambda = PE_i \left( \frac{l - 1}{l}, 0 \right) + PE_i \left( 0, \frac{l}{l} \right) + \ldots + PE_i \left( \frac{l - 1}{l}, \frac{l}{l} \right) \), so we can prove easily that

\[
F^{(i)}_\lambda = \sum_{k_i < p} \left( \sum_{k_{i_1}, \ldots, k_{i_n}, k_{i_{n+1}}, \ldots, k_{i_m}} A \right)
\]

So, for each rational number \( q = \frac{p}{l} \), \( F^{(i)}_\lambda \) commutes with \( B \); note that \( \{ F^{(i)}_\lambda \}_{\lambda \in \mathbb{R}} \) is right continuous in the strong operator topology, so \( B \) commutes with each \( E_i \), \( i = 1, 2, \ldots, n \).

**Lemma 2.1.** Let \( 1 \leq n < \infty \), \( A = \{ E_i \}_{i=1}^n \subseteq \mathcal{E}(H) \) be commutative and \( B \in \mathcal{B}(H) \). If for any integers \( m, k_1, k_2, \ldots, k_n \), \( B \) commutes with \( F^{(i)}_{k_1, \ldots, k_n} \), then \( B \) commutes with each \( E_i \) in \( A = \{ E_i \}_{i=1}^n \).

**Proof.** For each rational number \( q = \frac{p}{l} \), where \( p, l \) are integers. If \( \frac{p}{l} < 0 \), then \( F^{(i)}_\lambda = 0 \), and if \( \frac{p}{l} > 1 \), then \( F^{(i)}_\lambda = I \). Let \( l > p \geq 0 \), so \( 0 \leq \frac{p}{l} < 1 \). Then \( F^{(i)}_\lambda = PE_i \left( \frac{l - 1}{l}, 0 \right) + PE_i \left( 0, \frac{l}{l} \right) + \ldots + PE_i \left( \frac{l - 1}{l}, \frac{l}{l} \right) \); thus, we can prove easily that

\[
F^{(i)}_\lambda = \sum_{k_i < p} \left( \sum_{k_{i_1}, \ldots, k_{i_n}, k_{i_{n+1}}, \ldots, k_{i_m}} B \right)
\]

Let \( \{ E_i \}_{i=1}^n \subseteq \mathcal{E}(H) \) be commutative and \( B \in \mathcal{B}(H) \). If \( B \) does not commute with some \( E_i \) in \( A \), there are integers \( m, k_1, k_2, \ldots, k_n, k'_1, k'_2, \ldots, k'_m \), such that \( k_i \neq k'_i \) for at least one \( i \) and \( F^{(i)}_{k_1, \ldots, k_n} B F^{(i)}_{k'_1, \ldots, k'_n} \neq 0 \).

**Lemma 2.2.** Let \( 1 \leq n < \infty \), \( A = \{ E_i \}_{i=1}^n \subseteq \mathcal{E}(H) \) be commutative and \( B \in \mathcal{B}(H) \). If \( B \) does not commute with some \( E_i \) in \( A \), then there are integers \( m, k_1, k_2, \ldots, k_n, k'_1, k'_2, \ldots, k'_m \), such that \( k_i \neq k'_i \) for at least one \( i \) and \( F^{(i)}_{k_1, \ldots, k_n} B F^{(i)}_{k'_1, \ldots, k'_n} \neq 0 \). In fact, if not, we will get that

\[
F^{(i)}_{k_1, k_2, \ldots, k_n} B = \sum_{k_{i_1}, \ldots, k_{i_n}} F^{(i)}_{k_1, k_2, \ldots, k_n} B F^{(i)}_{k_{i_1}, \ldots, k_{i_n}} = F^{(i)}_{k_1, k_2, \ldots, k_n} B F^{(i)}_{k_{i_1}, \ldots, k_{i_n}}
\]

This is a contradiction. Similarly, if \( B F^{(i)}_{k_1, \ldots, k_n} \neq F^{(i)}_{k_1, \ldots, k_n} B F^{(i)}_{k_{i_1}, \ldots, k_{i_n}} \), we will also get the same conclusion. The lemma is proven.

Moreover, we have a stronger conclusion in the following.

**Lemma 2.3.** Let \( A \in \mathcal{E}(H) \) and \( B \in \mathcal{B}(H) \). If \( B \) does not commute with \( A \), then there exist integers \( m, k \) and \( j \) with \( |k - j| \geq 2 \) such that

\[
P^A \left( \frac{k}{m}, \frac{k + 1}{m} \right) B P^A \left( \frac{j}{m}, \frac{j + 1}{m} \right) = 0.
\]

**Proof.** By lemma 2.2, we can find \( k_1 \neq j_1 \) such that \( C = P^A \left( \frac{k_1}{m}, \frac{k_1 + 1}{m} \right) B P^A \left( \frac{j_1}{m}, \frac{j_1 + 1}{m} \right) \neq 0 \). If \( |k_1 - j_1| \geq 2 \), then we get the \( m, k, j \) satisfy the lemma. If \( j_1 = k_1 + 1 \), we replace \( m \) by
Now we consider \( k_2, k_2 + 1 \) and \( j_2, j_2 + 1 \), if we still cannot take \( |k - j| \geq 2 \) satisfy the conclusion, then

\[
\begin{align*}
&\quad P^A \left( \frac{k_2 + 1}{2m} \right) BP^A \left( \frac{j_2 + 1}{2m} \right) = 0, \\
&\quad P^A \left( \frac{k_2 + 1}{2m} \right) BP^A \left( \frac{j_2 + 1}{2m} \right) = 0, \\
&\quad P^A \left( \frac{k_2 + 1}{2m} \right) BP^A \left( \frac{j_2 + 1}{2m} \right) = 0.
\end{align*}
\]

So we have \( C = P^A \left( \frac{k_2 + 1}{2m} \right) BP^A \left( \frac{j_2 + 1}{2m} \right) \).

Following this, we find the integers \( k, j \) which satisfy the conclusion or we get a sequence \( \{p_i, p_i + 1, 2^{-i-1}m\}_{i=1}^\infty \) such that \( p_i + 1 = 2^{-i}j_1 \) and \( C = P^A \left( \frac{p_i}{2^{i-1}m}, \frac{p_i + 1}{2^{i-1}m} \right) BP^A \left( \frac{p_i + 1}{2^{i-1}m}, \frac{p_i + 2}{2^{i-1}m} \right) \).

If the first case occurs, then we proved the lemma. If the second case occurs, note that \( \lim_{i \to \infty} P^A \left( \frac{p_i}{2^{i-1}m}, \frac{p_i + 1}{2^{i-1}m} \right) = P^A \left( \frac{j_1}{m} \right) \) and \( \lim_{i \to \infty} P^A \left( \frac{p_i + 1}{2^{i-1}m}, \frac{p_i + 2}{2^{i-1}m} \right) = 0 \) in strong operator topology; thus,

\[
\lim_{i \to \infty} P^A \left( \frac{p_i}{2^{i-1}m}, \frac{p_i + 1}{2^{i-1}m} \right) BP^A \left( \frac{p_i + 1}{2^{i-1}m}, \frac{p_i + 2}{2^{i-1}m} \right) = 0
\]

in strong operator topology [17]. But for each positive integer \( i \),

\[
C = P^A \left( \frac{p_i}{2^{i-1}m}, \frac{p_i + 1}{2^{i-1}m} \right) BP^A \left( \frac{p_i + 1}{2^{i-1}m}, \frac{p_i + 2}{2^{i-1}m} \right),
\]

so we get \( C = 0 \); this is a contradiction, and the lemma is proved in this case.

If \( k_1 + 1 = j_1 \), we just need to take all the above calculations in adjoint and interchange the indices \( j \) and \( k \). The proof is similar; thus, we proved the lemma. \( \square \)

**Lemma 2.4.** Let \( 1 \leq n < \infty, A = \{E_i\}_{i=1}^n \subseteq \mathcal{E}(H) \) be commutative and \( \sum_{i=1}^n E_i^2 \leq I \). If \( X \in B(H) \) is not commutative with \( E_i \), then there exists a positive integer \( m \) such that for each positive integer \( p \), there exist projection operators \( P, Q \in A', P Q = 0, Y = P X Q \neq 0 \), and

\[
\frac{||Y|| - ||\Phi_A(Y)||}{||Y||} \geq \frac{p^2 - 4\sqrt{mp} - 2n}{2(pm)^2}.
\]

**Proof.** Since \( X \) does not commute with \( E_i \), it follows from lemma 2.3 that there exist integers \( m, k, j \) such that \( |k - j| \geq 2 \) and \( P^{E_i} (\frac{k}{m}, \frac{k+1}{m}) X P^{E_i} (\frac{j}{m}, \frac{j+1}{m}) \neq 0 \). Note that

\[
p^{E_i} (\frac{k}{m}, \frac{k+1}{m}) X P^{E_i} (\frac{j}{m}, \frac{j+1}{m}) = \sum_{k_1, ..., k_n} \sum_{j_1, ..., j_n} F^{m}_{k, k_1, ..., k_n} X F^{m}_{j, j_1, ..., j_n},
\]
so there exist \( k, k_2, \ldots, k_n \) and \( j, j', k', \ldots, k' \) such that \( |k - j| \geq 2 \) and
\[
F^m_{k, k_2, \ldots, k_n} X F^m_{j, k_2', \ldots, k_n} 
\neq 0.
\]
Let \( P_0 = F^m_{k, k_2, \ldots, k_n}, Q_0 = F^m_{j, k_2', \ldots, k_n}, \) \( Y_0 = P_0 X Q_0 \). Then \( P_0 \) and \( Q_0 \) are projection operators and \( P_0, Q_0 \in \mathcal{A}' \), \( P_0 Q_0 = 0 \), \( Y_0 = P_0 X Q_0 \neq 0 \). Moreover, for each \( i = 1, 2, \ldots, n \), if we denote \( k_i = k, k_i' = j \), then
\[
\| E_i Y_0 E_i \| = \left\| E_i P E_i \left( \frac{k_i}{m}, \frac{k_i + 1}{m} \right) Y_0 P E_i \left( \frac{k_i'}{m}, \frac{k_i' + 1}{m} \right) E_i \right\| 
\leq \left\| E_i P E_i \left( \frac{k_i}{m}, \frac{k_i + 1}{m} \right) Y_0 \right\| \left\| P E_i \left( \frac{k_i'}{m}, \frac{k_i' + 1}{m} \right) E_i \right\| 
\leq \frac{k_i + 1}{m} \| Y_0 \| \frac{k_i' + 1}{m} 
= \frac{k_i + 1}{m} \frac{k_i' + 1}{m} \| Y_0 \|. 
\] (1)

Thus, we have
\[
\sum_{i=1}^{n} E_i Y_0 E_i \leq \sum_{i=1}^{n} \| E_i Y_0 E_i \| \leq \left( \sum_{i=1}^{n} \frac{k_i k_i'}{m^2} + \frac{n}{m^2} \right) \| Y_0 \|. 
\] (2)

Since \( \sum_{i=1}^{n} E_i^2 \leq I \) and
\[
F^m_{k_1, k_2, \ldots, k_n} \left( I - \sum_{i=1}^{n} E_i^2 \right) = F^m_{k_1, k_2, \ldots, k_n} - \sum_{i=1}^{n} E_i^2
\leq F^m_{k_1, k_2, \ldots, k_n} - \sum_{i=1}^{n} \frac{k_i k_i'}{m^2} F^m_{k_1, k_2, \ldots, k_n}
= \left( 1 - \sum_{i=1}^{n} \frac{k_i^2}{m^2} \right) F^m_{k_1, k_2, \ldots, k_n},
\] (3)

so, we have \( \sum_{i=1}^{n} k_i^2 \leq m^2 \). Similarly, we have also \( \sum_{i=1}^{n} k_i^2 \leq m^2 \). Moreover, note that
\[
2m^2 \left( 1 - \sum_{i=1}^{n} \frac{k_i k_i'}{m^2} - \sum_{i=1}^{n} \frac{k_i + k_i'}{m^2} - \frac{n}{m^2} \right) = m^2 + m^2 - 2 \sum_{i=1}^{n} k_i k_i' - 2 \sum_{i=1}^{n} (k_i + k_i') - 2n
\geq \sum_{i=1}^{n} k_i^2 + \sum_{i=1}^{n} k_i^2 - 2 \sum_{i=1}^{n} k_i k_i' - 2 \sum_{i=1}^{n} (k_i + k_i') - 2n
= \sum_{i=1}^{n} (k_i - k_i')^2 - 2 \sum_{i=1}^{n} (k_i + k_i') - 2n
\geq (k_i - k_i')^2 - 2 \sum_{i=1}^{n} (k_i + k_i') - 2n,
\] (4)

and \( (\sum_{i=1}^{n} k_i)^2 \leq n (\sum_{i=1}^{n} k_i^2) \leq nn^2, (\sum_{i=1}^{n} k_i')^2 \leq n (\sum_{i=1}^{n} k_i'^2) \leq nm^2, \) we have
\[
2m^2 \left( 1 - \sum_{i=1}^{n} \frac{k_i k_i'}{m^2} - \sum_{i=1}^{n} \frac{k_i + k_i'}{m^2} - \frac{n}{m^2} \right) \geq (j - k)^2 - 4\sqrt{nm} - 2n.
\] (5)
On the other hand, it follows from
\[ \|Y_0\| - \left\| \sum_{i=1}^{n} E_i Y_0 E_i \right\| \geq \|Y_0\| - \sum_{i=1}^{n} \|E_i Y_0 E_i\| \]
\[ \geq \left[ 1 - \left( \sum_{i=1}^{n} \frac{k_i k'_i}{m^2} + \sum_{i=1}^{n} \frac{k_i + k'_i}{m^2} + \frac{n}{m^2} \right) \right] \|Y_0\| \]
and (5) that
\[ \frac{\|Y_0\| - \|\Phi_A(Y_0)\|}{\|Y_0\|} \geq \frac{(j - k)^2 - 4\sqrt{\eta m} - 2n}{2m^2}. \]

For each positive integer \( p \), we replace \( m \) with \( pm \). Note that
\[ Y_0 = \sum_{s_1, s_2, \ldots, s_n} \sum_{s'_1, s'_2, \ldots, s'_n} F^{pm}_{s_1, s_2, \ldots, s_n} Y_0 F^{pm}_{s'_1, s'_2, \ldots, s'_n} \neq 0, \]
so there exist \( s_1, s_2, \ldots, s_n \) and \( s'_1, s'_2, \ldots, s'_n \) such that
\[ Y = F^{pm}_{s_1, s_2, \ldots, s_n} Y_0 F^{pm}_{s'_1, s'_2, \ldots, s'_n} 
eq 0. \]

Thus, it is easily to prove that \( \frac{k_i}{m} \leq \frac{s_i}{pm} \leq \frac{k_i + 1}{m} \) and \( \frac{k'_i}{m} \leq \frac{s'_i}{pm} \leq \frac{k'_i + 1}{m} \). Note that \( k_i = k, k'_i = j \) and \( \frac{j - k}{m} \geq \frac{2}{m} \), we have
\[ \frac{s_i - s'_i}{pm} \geq \frac{k_i + 1 - k'_i}{m} \geq 1/m; \]
thus
\[ \|s_i - s'_i\| \geq p. \]

By the similar analysis methods as (5), we get
\[ 2(pm)^2 \left( 1 - \sum_{i=1}^{n} \frac{s_i s'_i}{(pm)^2} - \sum_{i=1}^{n} \frac{s_i + s'_i}{(pm)^2} - \frac{n}{(pm)^2} \right) \geq p^2 - 4\sqrt{\eta mp} - 2n. \] (6)

On the other hand, we also have
\[ \|Y\| - \left\| \sum_{i=1}^{n} E_i Y E_i \right\| \geq \|Y\| - \sum_{i=1}^{n} \|E_i Y E_i\| \]
\[ \geq \left[ 1 - \left( \sum_{i=1}^{n} \frac{k_i k'_i}{m^2} + \sum_{i=1}^{n} \frac{k_i + k'_i}{m^2} + \frac{n}{m^2} \right) \right] \|Y\|. \]

Let \( P = F^{pm}_{s_1, s_2, \ldots, s_n} P_0 \) and \( Q = Q_0 F^{pm}_{s'_1, s'_2, \ldots, s'_n} \). Then it is clear that \( P, Q \in A' \), \( PQ = 0 \), \( Y = PXQ \neq 0 \), and
\[ \frac{\|Y\| - \|\Phi_A(Y)\|}{\|Y\|} \geq \frac{p^2 - 4\sqrt{\eta m} - 2n}{2(pm)^2}. \]

The lemma is proved. \( \square \)

It follows from the proof of lemma 2.4 that we have the following important conclusion:

**Corollary 2.1.** Let \( 1 \leq n < \infty \), \( A = \{E_i\}_{i=1}^{n} \subseteq \mathcal{E}(H) \) be commutative and \( \sum_{i=1}^{n} E_i^2 \leq 1 \). If \( X \in B(H) \) and there exist integers \( m, k, \) and \( j \) with \( |k - j| \geq 2 \) such that
\[ p E_i \left( \frac{k}{m}, \frac{k + 1}{m} \right) X p E_i \left( \frac{j}{m}, \frac{j + 1}{m} \right) \neq 0, \]

\[ \frac{p E_i \left( \frac{k}{m}, \frac{k + 1}{m} \right) X p E_i \left( \frac{j}{m}, \frac{j + 1}{m} \right)}{\|p E_i \left( \frac{k}{m}, \frac{k + 1}{m} \right) X p E_i \left( \frac{j}{m}, \frac{j + 1}{m} \right)\|} \neq 0, \]

\[ \frac{\|Y\| - \|\Phi_A(Y)\|}{\|Y\|} \geq \frac{p^2 - 4\sqrt{\eta m} - 2n}{2(pm)^2}. \]
then for each positive integer $p$, there exist projection operators $P, Q \in \mathcal{A}$, $PQ = 0$, $Y = PXQ \neq 0$, and
\[
\frac{\|Y\| - \|\Phi_d(Y)\|}{\|Y\|} \geq \frac{p^2 - 4\sqrt{nmp} - 2n}{2(pm)^2}.
\]

3. Main results and proofs

Let $\mathcal{A} = \{E_i\}_{i=1}^n \subseteq \mathcal{E}(H)$ and $\Phi_d$ be the Lüders quantum operation which is decided by $\mathcal{A}$. It is easy to prove that $\|\Phi_d\| = \|\sum_{i=1}^n E_i^2\|$ [9]. Now, we denote $B(H)^{\Phi_d}$ to be the fixed point set of $\Phi_d$ and $\mathcal{A}'$ to be the commutant of $\mathcal{A}$, that is, $B(H)^{\Phi_d} = \{B \in B(H) \mid \Phi_d(B) = B\}$, $\mathcal{A}' = \{B \in B(H) \mid BE_i = E_iB, 1 \leq i \leq n\}$. It is clear that if $\sum_{i=1}^n E_i^2 = I$ in strong operator topology, then $\mathcal{A}' \subseteq B(H)^{\Phi_d}$.

**Theorem 3.1.** Let $1 \leq n \leq \infty$, $\mathcal{A} = \{E_i\}_{i=1}^n \subseteq \mathcal{E}(H)$ be commutative and $\sum_{i=1}^n E_i^2 = I$ in strong operator topology. Then
\[
B(H)^{\Phi_d} = \left\{B \in B(H) \mid \Phi_d(B) = \sum_{i=1}^n E_iB E_i = B \right\} = \mathcal{A}'.
\]

**Proof.** Since $\mathcal{A}' \subseteq B(H)^{\Phi_d}$, in order to prove the converse containing relation, we suppose that $B \in B(H)^{\Phi_d} \setminus \mathcal{A}'$. Without loss of generality, we can suppose that $B$ is not commutative with $E_1$. By lemma 2.3, there is a triple integer set $(m, j, k)$ such that $|k - j| \geq 2$ and $P E_{\left(\frac{k}{m}, \frac{k+1}{m}\right)} B P E_{\left(\frac{j}{m}, \frac{j+1}{m}\right)} P_q \neq 0$.

For each positive integer $q \leq n$, let $F_q = \sum_{i=1}^n E_i^2$ and $\Phi_q : B(H) \to B(H)$ be defined by $\Phi_q(A) = \sum_{i=1}^n E_i A E_i$. Then $F_q \to I$ in strong operator topology and $\Phi_q$ is a completely positive map. If $P_q = P E_{\left(1 - \frac{2q}{\sqrt{4q^2 - 1}}\right)}$, then $P_q \to I$ in strong operator topology (see [18], P248). Now we show that $P_q P E_{\left(\frac{k}{m}, \frac{k+1}{m}\right)} B P E_{\left(\frac{j}{m}, \frac{j+1}{m}\right)} P_q = 0$. In fact, if not, note that
\[
P E_{\left(\frac{k}{m}, \frac{k+1}{m}\right)} P_q P E_{\left(\frac{j}{m}, \frac{j+1}{m}\right)} B P E_{\left(\frac{j}{m}, \frac{j+1}{m}\right)} P_q \neq 0,
\]
so, by corollary 2.1, for each positive integer $p$, there exist projection operators $P$ and $Q$, $P, Q \in \mathcal{A}'$, $PQ = 0$, such that
\[
Y = PP E_{\left(\frac{k}{m}, \frac{k+1}{m}\right)} B P E_{\left(\frac{j}{m}, \frac{j+1}{m}\right)} P_q Q
\]
and
\[
\frac{\|Y\| - \|\Phi_q(Y)\|}{\|Y\|} \geq \frac{p^2 - 4\sqrt{qmp} - 2q}{2(pm)^2}.
\]
Since
\[
\frac{p^2 - 4\sqrt{qmp} - 2q}{2(pm)^2} \to \frac{1}{2m^2}
\]
as $p \to \infty$. So we can choose $Y$ such that
\[
\frac{\|Y\| - \|\Phi_q(Y)\|}{\|Y\|} \geq \frac{3}{8m^2}.
\]
Note that $P_q E_i = E_i P_q$ and $P_q Y = Y P_q$ for each $1 \leq i \leq n$, $\mathcal{A}_1 = \{ P_q E_i \}_{i=1}^n$ decides a Lüders operation $\Phi_{\mathcal{A}_1}$, and

$$
\|\Phi_{\mathcal{A}_1}\| = \left\| \sum_{i=q+1}^{n} P_q E_i^2 P_q \right\| = \left\| P_q \left( \sum_{i=q+1}^{n} E_i^2 \right) P_q \right\| = \left\| P_q \left( I - \sum_{i=1}^{q} E_i^2 \right) P_q \right\| \leq \frac{1}{4m^2},
$$

so we have

$$
\|\Phi_{\mathcal{A}}(Y)\| = \left\| \Phi_{\mathcal{A}}(Y) + \sum_{i=q+1}^{n} E_i Y E_i \right\| = \left\| \Phi_{\mathcal{A}}(Y) + \sum_{i=q+1}^{n} P_q E_i Y E_i P_q \right\| \leq \left\| \Phi_{\mathcal{A}}(Y) \right\| + \left\| \sum_{i=q+1}^{n} P_q E_i Y E_i P_q \right\| = \left\| \Phi_{\mathcal{A}}(Y) \right\| + \left\| \Phi_{\mathcal{A}_1}(Y) \right\| \leq \left( 1 - \frac{3}{8m^2} \right) \|Y\| + \frac{1}{4m^2} \|Y\| = \left( 1 - \frac{3}{8m^2} \right) \|Y\|. \tag{7}
$$

On the other hand, we show that $Y = P_q P^E_{\left( \frac{k}{m}, \frac{k+1}{m} \right)} B P^E_{\left( \frac{j}{m}, \frac{j+1}{m} \right)} Q P_q \in \mathcal{B}(H)_{\Phi_{\mathcal{A}}}$. In fact, note that $\{ P_q, P, P^E_{\left( \frac{k}{m}, \frac{k+1}{m} \right)} P^E_{\left( \frac{j}{m}, \frac{j+1}{m} \right)} Q \} \subseteq \mathcal{A}'$ and $\Phi_{\mathcal{A}}(B) = B$, so we have

$$
\Phi_{\mathcal{A}}(Y) = \sum_{i=1}^{n} E_i Y E_i = \sum_{i=1}^{n} E_i P_q P^E_{\left( \frac{k}{m}, \frac{k+1}{m} \right)} B P^E_{\left( \frac{j}{m}, \frac{j+1}{m} \right)} Q P_q E_i
$$

This contradicts (7) and so $P_q P^E_{\left( \frac{k}{m}, \frac{k+1}{m} \right)} B P^E_{\left( \frac{j}{m}, \frac{j+1}{m} \right)} Q P_q = 0$. Note that

$$
P^E_{\left( \frac{k}{m}, \frac{k+1}{m} \right)} B P^E_{\left( \frac{j}{m}, \frac{j+1}{m} \right)} \xrightarrow{q \to \infty} P_q E_{\left( \frac{k}{m}, \frac{k+1}{m} \right)} B P^E_{\left( \frac{j}{m}, \frac{j+1}{m} \right)} P_q
$$

in strong operator topology [17], so

$$
P^E_{\left( \frac{k}{m}, \frac{k+1}{m} \right)} B P^E_{\left( \frac{j}{m}, \frac{j+1}{m} \right)} = 0.
$$

This contradicts $P^E_{\left( \frac{k}{m}, \frac{k+1}{m} \right)} B P^E_{\left( \frac{j}{m}, \frac{j+1}{m} \right)} \neq 0$. So $B \in \mathcal{A}'$. \qed

**Theorem 3.2.** Let $1 \leq n \leq \infty$, $\mathcal{A} = \{ E_i \}_{i=1}^{n} \subseteq \mathcal{C}(H)$ be commutative and $F = \sum_{i=1}^{n} E_i^2 < I$. If $P = P^F\{1\}$, where $P^F$ is the spectral measure of $F$, then

$$
\mathcal{B}(H)_{\Phi_{\mathcal{A}}} = \left\{ B \in \mathcal{B}(H) | \Phi_{\mathcal{A}}(B) = \sum_{i=1}^{n} E_i B E_i = B \right\} = P \mathcal{A}'.
$$
Proof. Firstly, by the spectral representation theorem [17] we have \( PF = FP = P \). Let \( B \in B(H)^{\Phi, \lambda} \). Then as the analysis of theorem 3.1, we have \( B \in \mathcal{A}' \). Let \( Q \in B(H)/\Phi \). Then as the analysis of theorem 3.1, we have \( Q_k \in \mathcal{A}' \). Let \( \Phi_k \) be the completely positive map which is decided by \( \{ E(Q_k) \}_{k=1}^{n} \). Thus, we have \( \| \Phi_k \| \leq 1 - \frac{1}{k} \). Note that \( \mathcal{B}, \mathcal{A}' \) and \( \mathcal{B}(H)/\Phi \); thus, we have \( \mathcal{S}(H)^{\Phi, \lambda} \subseteq \mathcal{B}(H)^{\Phi, \lambda} \), and the theorem is proved.

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