Extended HQFTs in dimension 2

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Overview

1. Review of TQFTs and HQFTs
2. The main theorem and the idea of the proof
3. \((G \times SO(2))\)-structured Cobordism Hypothesis
Topological quantum field theories

Definition ([Atiyah, 1988])

An \((n + 1)\)-dimensional topological quantum field theory (TQFT) is a symmetric monoidal functor

\[ Z : ((n + 1)\text{Cob}, \amalg) \to (\text{Vect}_\mathbb{C}, \otimes). \]

- The symmetric monoidal category \((n + 1)\text{Cob}\) has closed oriented \(n\)-manifolds as objects and diffeomorphism classes of oriented cobordisms relative to boundary as morphisms. The symmetric monoidal product \(\amalg\) is disjoint union of manifolds.

- The symmetric monoidal category \(\text{Vect}_\mathbb{C}\) has finite dimensional complex vector spaces as objects and linear transformations as morphisms. The symmetric monoidal product \(\otimes\) is the tensor product.
Two-dimensional TQFTs

$n = 1$: \[ Z : (\text{2Cob}, \mathbb{I}) \to (\text{Vect}_\mathbb{C}, \otimes) \]
Two-dimensional TQFTs

Functoriality of TQFTs implies that such a linear transformation factors as composition of linear transformations obtained by possible cuttings of the surface along circles.

\[ Z : (2\text{Cob}, \llbracket \rrbracket) \rightarrow (\text{Vect}_\mathbb{C}, \otimes) \]
The category $2\text{TQFT}_k$ of 2-dimensional TQFTs and monoidal natural transformations is equivalent to the category $c\text{Frob}_k$ of commutative Frobenius algebras and Frobenius algebra homomorphisms.

A Frobenius algebra $(A, \beta)$ is a finite dimensional $k$-algebra $A$ equipped with an associative nondegenerate pairing $\beta : A \otimes A \to k$.

Example

$(H^*(\mathbb{C}P^n; \mathbb{C}), \beta) = (\mathbb{C}[\alpha]/\alpha^{n+1}, \beta)$ is a commutative Frobenius algebra with $\beta(a, b) = a \cup b([\mathbb{C}P^n])$. 
Generalizations of TQFTs

There are two different generalizations of TQFTs:

- **Structured TQFTs** where manifolds and cobordisms are equipped with additional structures such as framing or principal $G$-bundles.
- **Extended TQFTs** where manifolds with corners are allowed.

**Definition ([Turaev, 1999])**

Let $(X, x)$ be an aspherical pointed CW-complex with $\pi_1(X, x) = G$. An $n$-dimensional $X$-manifold is a tuple $(M, g)$ where $M$ is a closed oriented pointed $n$-manifold and $g \in [(M, m), (X, x)]$ is a pointed homotopy class.

An $X$-cobordism $(W, P)$ between two $X$-manifolds $(M_1, g_1)$ and $(M_2, g_2)$ is a cobordism between $M_1$ and $M_2$ and $P \in [W, X]$ is a homotopy class restricting to $g_1$ and $g_2$ on boundary components.

$X$-manifolds and $X$-cobordisms form the symmetric monoidal category $((n + 1)X\text{Cob}, \boxtimes)$. 
Homotopy quantum field theories

**Definition ([Turaev, 1999])**

Let $X$ be an aspherical pointed CW-complex. An $(n+1)$-dimensional homotopy quantum field theory (HQFT) with target $X$ is a symmetric monoidal functor $Z : ((n+1)X\text{Cob}, \sqcup) \to (\text{Vect}_\mathbb{C}, \otimes)$.

$n=1$: $Z : (2X\text{Cob}, \sqcup) \to (\text{Vect}_\mathbb{C}, \otimes)$
Theorem ([Turaev, 1999])

The category $2\text{HQFT}$ of 2-dimensional HQFTs with target $X \cong K(G,1)$ and monoidal transformations is equivalent to the category $\mathcal{C}Frob^G$ of crossed Frobenius $G$-algebras and crossed Frobenius $G$-algebra homomorphisms.

Frobenius $G$-algebra $(A, \beta)$ is a $G$-graded algebra $A = \bigoplus_{g \in G} A_g$ equipped with an associative nondegenerate pairing $\beta : A \otimes A \rightarrow \mathbb{k}$. The crossed structure on $A$ is a group homomorphism $\varphi : G \rightarrow \text{Aut}(A)$ where each $\varphi_g$ is conjugation type i.e. $g \mapsto \varphi_g|_{A_g'} : A_g' \rightarrow A_{gg'g^{-1}}$ such that $\varphi_g'(a)b = ba$ for all $a \in A_g$ and $b \in A_{g'}$.

Example

The group algebra $(\mathbb{C}[G], \beta, \varphi)$ is a crossed Frobenius $G$-algebra where $\beta(a, b)$ is the coefficient of $e$ in the expression $ab$ and $\varphi_g$ is the conjugation.
A different generalization of TQFTs is by using manifolds with corners and higher categories. The main motivation for this type of generalization is to be able cut a cobordism along different directions and compute the invariants from the invariants of simpler pieces.
Definition ([Schommer-Pries, 2009])

The symmetric monoidal bordism bicategory $\text{Bord}_2$ has compact oriented 0-manifolds as objects, oriented 1-cobordisms as 1-morphisms and diffeomorphism classes of oriented $\langle 2 \rangle$-surfaces relative to boundary as 2-morphisms.

An example of $\langle 2 \rangle$-surface $S$ is shown below as a 2-morphism $S : A \to B$ where $A, B : M \to N$. 

\begin{center}
\begin{tikzcd}
\vdots \\
M \times I \\
\vdots \\
A \arrow[d, bend left] \arrow[d, bend right] \\
\vdots \\
B \arrow[d, bend left] \arrow[d, bend right] \\
\vdots \\
N \times I \\
\vdots \\
\end{tikzcd}
\end{center}
Definition

A 2-dimensional extended TQFT is a symmetric monoidal 2-functor \( Z : \text{Bord}_2 \rightarrow \text{Alg}_k^2 \) where \( \text{Alg}_k^2 \) is the symmetric monoidal bicategory of \( k \)-algebras, bimodules, and bimodule maps.

Theorem ([Schommer-Pries, 2009])

There is an equivalence of bicategories \( \mathcal{E}\text{-TQFT} \cong \text{Frob} \).

- The bicategory \( \mathcal{E}\text{-TQFT} \) has 2-dimensional E-TQFTs as objects, symmetric monoidal transformations as 1-morphisms, and symmetric monoidal modifications as 2-morphisms.
- The bicategory Frob has separable symmetric Frobenius algebras as objects, Morita equivalences as 1-morphisms, and isomorphisms of Morita equivalences as 2-morphisms.
Summary of classifications

- Commutative Frobenius algebras
  - 2-dimensional TQFTs
  - Crossed Frobenius $G$-algebras
    - 2-dimensional HQFTs
- 2-dimensional E-TQFTs
- Separable symmetric Frobenius algebras

- Restriction to constant homotopy classes.
- Restriction to circles and cobordisms between circles.
The main goal

Commutative Frobenius algebras

\[ \text{2-dimensional TQFTs} \]

Separated symmetric Frobenius algebras

\[ \text{2-dimensional E-TQFTs} \]

Crossed Frobenius $G$-algebras

\[ \text{2-dimensional HQFTs} \]

\[ \text{2-dimensional E-HQFTs} \]
Extended equivariant bordism bicategory

Definition

The symmetric monoidal $G$-equivariant bordism bicategory $\mathcal{X} \text{Bord}_2$ has oriented 0-dimensional compact $X$-manifolds as objects, oriented $X$-cobordisms as 1-morphisms and diffeomorphism classes of $\langle 2 \rangle$-$X$-surfaces relative to boundary as 2-morphisms.

An example of $\langle 2 \rangle$-$X$-surface $(S, f)$ is shown below as a 2-morphism $(S, f) : (A, p) \to (B, q)$ where $(A, p), (B, q) : (M, g) \to (N, h)$.

![Diagram showing an example of a $\langle 2 \rangle$-$X$-surface as a 2-morphism]
Extended homotopy quantum field theories

**Definition**

A 2-dimensional extended HQFT with target $X \simeq K(G, 1)$ is a symmetric monoidal 2-functor

$$Z : XBord_2 \to \text{Alg}_k^2.$$
The main theorem

Theorem (S.)

There is an equivalence of bicategories $\mathcal{E}$-HQFT$(X) \simeq \text{Frob}^G$.

- The bicategory $\mathcal{E}$-HQFT$(X)$ has 2-dimensional E-HQFTs with target $X \simeq K(G,1)$ as objects, symmetric monoidal transformations as 1-morphisms, and symmetric monoidal modifications as 2-morphisms.
- The bicategory $\text{Frob}^G$ has quasi-biangular $G$-algebras as objects, $G$-graded Morita equivalences as 1-morphisms, and isomorphisms of graded Morita equivalences as 2-morphisms.

Definition

A Frobenius $G$-algebra $(A = \bigoplus_{g \in G} A_g, \beta)$ is quasi-biangular if each $A_g$ is both left and right rank one $A_e$-module and the principal component $A_e$ is a separable algebra.
The idea of the proof

We generalize the planar decomposition theorem of C. Schommer-Pries to $G$-planar decomposition theorem. In particular, we add $X$-manifold data to linear, planar and spatial diagrams of Schommer-Pries and define

- $G$-linear diagrams
- $G$-planar diagrams
- $G$-spatial diagrams.

Using diagrams we define a symmetric monoidal bicategory $XB^{PD}$ which is equivalent to $XBord_2$ and freely generated on a list of generators and relations. Lastly, using coherence theorems for symmetric monoidal 2-functors developed by Schommer-Pries we classify 2-dimensional $E$-HQFTs.
**Definition**

A $G$-linear diagram is a triple $(\Psi^G, \Gamma, S^G)$ consisting of a 1-dimensional $G$-graphic $\Psi^G$, a compatible chambering set $\Gamma$, and a $G$-sheet data $S^G$.

A 1-dimensional $G$-graphic $\Psi^G$ consists of finitely many isolated points labeled by cup or cap, finitely many points and $G$-labeled directed intervals.
A chambering set $\Gamma$ compatible with $\Psi^G$ consists of isolated points in $\mathbb{R}$ which are disjoint from $\Psi^G$. Any $\Psi^G$-compatible chambering set provides open sets called chambers which are given by the complements of points of $\Psi^G$ and points of $\Gamma$. 
A $G$-sheet data $S^G$ associated to tuple $(\Psi^G, \Gamma)$ consists of

- a trivialization of each chamber by a **finite ordered set** and lifts of $G$-labeled arcs and points using $G$-graphic,
- injections and permutations between trivializations of neighboring chambers describing gluing data.
For a closed surface $\Sigma$, Schommer-Pries stratified jet spaces $J^k(\Sigma, \mathbb{R}^2)$. A generic map $f : \Sigma \to \mathbb{R}^2$ for this stratification can have the following singularities.
**G-planar diagrams**

**Definition**

A $G$-planar diagram is a triple $(\Phi^G, \Gamma, S^G)$ consisting of a 2-dimensional $G$-graphic $\Phi^G$, a compatible chambering graph $\Gamma$, and a $G$-sheet data $S^G$.

A 2-dimensional $G$-graphic $\Phi^G$ consists of embedded labeled arcs and points in $\mathbb{R}^2$, and embedded points and immersed $G$-labeled arcs.
A chambering graph $\Gamma$ compatible with $\Phi^G$ is a smoothly embedded graph in $\mathbb{R}^2$ whose vertices are disjoint from $\Phi^G$ and have degree either 1 or 3 and edges of $\Gamma$ are transversal to $\Phi^G$. Any $\Phi^G$-compatible chambering graph $\Gamma$ provides open sets called chambers which are given by the complement of $\Gamma$ and the embedded arcs and point of $\Phi^G$. 
A $G$-sheet data associated to tuple $(\Psi^G, \Gamma)$ consists of

- a trivialization of each chamber by a finite ordered set and lifts of $G$-labeled arcs and points using $G$-graphic,
- injections and permutations between trivializations of neighboring chambers describing gluing data.

\[ \sigma(1) \quad \sigma(2) \quad \sigma(N) \]

\[ g_i \quad g_i \quad g_j \]

\[ U_{\beta_1} \quad U_{\beta_2} \quad g_j \]
G-spatial diagrams

Recall that in the bordism bicategory 2-morphisms are considered up to a diffeomorphism. To understand how different planar diagrams of the given surface are related Schommer-Pries stratified jet spaces $J^k(\Sigma \times I, \mathbb{R}^2 \times \mathbb{R})$. A generic map for this stratification can have the following graphics in $\mathbb{R}^2 \times \mathbb{R}$.

<table>
<thead>
<tr>
<th></th>
<th>Fold</th>
<th>Cusp</th>
<th>Morse</th>
<th>Morse Relation</th>
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<tbody>
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<td></td>
<td><img src="image1" alt="Fold Graphic" /></td>
<td><img src="image2" alt="Cusp Graphic" /></td>
<td><img src="image3" alt="Morse Graphic" /></td>
<td><img src="image4" alt="Morse Relation Graphic" /></td>
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<tr>
<td>Cusp Inversion</td>
<td><img src="image5" alt="Cusp Inversion Graphic" /></td>
<td><img src="image6" alt="Cusp Inversion' Graphic" /></td>
<td><img src="image7" alt="Cusp Flip Graphic" /></td>
<td><img src="image8" alt="Swallowtail Graphic" /></td>
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**G-spatial diagrams**

**Definition**

A G-spatial diagram is a triple \((\Delta^G, \Gamma, S^G)\) consisting of a 3-dimensional G-graphic \(\Delta^G\), a compatible chambering foam \(\Gamma\), and a G-sheet data \(S^G\).

A 3-dimensional G-graphic \(\Delta^G\) consists of labeled embedded surfaces, arcs and points in \(\mathbb{R}^2 \times \mathbb{R}\) and embedded points and immersed G-labeled arcs.
Generalizing diagrams to $\langle 2 \rangle$-$X$-surfaces

A chambering foam compatible with $\Delta^G$ is a certain type of 2-dimensional stratified space and it provides open sets called chambers.

A $G$-sheet data is defined similarly by trivializing chambers of a compatible chambering foam by ordered sets and assignments of injections and permutations between trivializations of neighboring chambers.

$G$-planar and $G$-spatial diagrams can be generalized to compact and $\langle 2 \rangle$-$X$-surfaces using $G$-linear diagrams on the horizontal boundaries.
Define a relation among $G$-planar diagrams $(\Phi^G_1, \Gamma_1, S^G_1) \sim (\Phi^G_2, \Gamma_2, S^G_2)$ if there exists a $G$-spatial diagram $(\Delta^G, \Gamma, S^G)$ whose boundary components give $G$-planar diagrams $(\Phi^G_1, \Gamma_1, S^G_1)$ and $(\Phi^G_2, \Gamma_2, S^G_2)$.

**Theorem (G-planar decomposition)**

There is a bijection between equivalence classes of $G$-planar diagrams and $X$-diffeomorphism classes of surfaces equipped with homotopy class of map to $X$.

**Question:** Why do we need the $G$-planar decomposition theorem to classify 2-dimensional extended HQFTs?

**Answer:** Because $G$-linear and $G$-planar diagrams give generators for the extended $G$-equivariant bordism bicategory $XBord_2$ and $G$-spatial diagrams give relations. In this case the sufficiency of relations corresponds to $G$-planar decomposition theorem.
The list of generators for $XBord_2$

Generating Objects:

\[ + \bullet \quad - \bullet \]

Generating 1-morphisms:

\[ g \]
\[ g' \]

Generating 2-morphisms:
The list of generators for $X\text{Bord}_2$

+reflections
The list of relations for $XBord_2$

Relations among 2-morphisms:

$g\ g' = gg'$

$g\ g' = gg'$

$g\ g' = gg'$

$g\ g' = gg'$

$g\ g' = gg'$

$g\ g' = gg'$

$+\text{reflections}$
The list of relations for $XBord_2$

\[ g' g = g g' = g' g = g g' = g' g = g g' = g' g = g g' = g' g + \text{reflections} \]
Let $\mathcal{C}$ be a symmetric monoidal $(\infty, n)$-category. Then there is a canonical equivalence of $(\infty, n)$-categories

$$\text{Fun}^\otimes \left( \text{Bord}_n^G, \mathcal{C} \right) \sim \left( (\mathcal{C}^{fd})^\sim \right)^{hG}$$

where $\text{Fun}^\otimes$ is the $(\infty, n)$-category of symmetric monoidal functors, $\mathcal{C}^{fd}$ is the subcategory of fully dualizable objects with duality data, $(\mathcal{C}^{fd})^\sim$ is the underlying $\infty$-groupoid and $( (\mathcal{C}^{fd})^\sim )^{hG}$ is the space of homotopy $G$-fixed points given by

$$( (\mathcal{C}^{fd})^\sim )^{hG} = \text{Hom}_G(EG, (\mathcal{C}^{fd})^\sim)$$

where $EG$ is the total space of the universal principle $G$-bundle $p : EG \to BG$. 
(\(G \times SO(2)\))-structured Cobordism Hypothesis

Two-dimensional extended HQFTs with target \(X \simeq K(G, 1)\) are equivalent to \((G \times SO(2))\)-structured (fully-)extended 2-dimensional TQFTs by pulling back the universal bundle \(p : EG \to X = BG\) along homotopy class.

**Theorem ([Davidovich, 2011])]**

Let \(G\) be a finite group and \(k\) be an algebraically closed field of characteristic zero. Then isomorphism classes of homotopy \((G \times SO(2))\)-fixed points in \(((\text{Alg}_{k}^{fd})^{\sim})^{h(G \times SO(2))}\) are \(G\)-equivariant algebras.

**Corollary of the main theorem**

Let \(G\) be a finite group and \(k\) be an algebraically closed field of characteristic zero. Then \((G \times SO(2))\)-structured Cobordism Hypothesis for fully-extended \((G \times SO(2))\)-structured TQFTs with target \(\text{Alg}_{k}^{2}\) holds true.
Example of 1 and 2-dimensional decompositions of $X$-torus

X-torus

1D Decomposition

2D Decomposition
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