TWO DIMENSIONAL EXTENDED HOMOTOPY FIELD THEORIES

KÜŘŞAT SÖZER

Abstract. We define and classify 2-dimensional extended homotopy field theories (E-HFTs) with aspherical targets. For a commutative ring $\mathbb{k}$, oriented E-HFTs taking values in the symmetric monoidal bicategory of $\mathbb{k}$-algebras, bimodules and bimodule maps are classified in terms of certain Frobenius $G$-algebras where $G$ is the fundamental group of a target space. As an application, we verify a special case of the $(G \times SO(2))$-structured cobordism hypothesis due to Lurie by comparing our result with Davidovich’s classification.

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1. Introduction

Extended topological field theories (E-TFTs) are generalizations of TFTs to manifolds with corners and higher categories ([Fr], [La2], [BD]). A different generalization of TFTs is due to V. Turaev ([Tu2]) who introduced homotopy field theories (HFTs). These theories are defined by applying axioms of TFTs to manifolds and cobordisms endowed with maps to a fixed target space. In this paper, we combine these generalizations of TFTs in dimension 2. More precisely, we define and classify 2-dimensional extended homotopy field theories (E-HFTs) with aspherical targets.

We introduce the symmetric monoidal $G$-equivariant oriented bordism bicategory $XBord_d$ which plays a major role in the definition of a 2-dimensional E-HFT with target $X \simeq K(G,1)$. Objects of $XBord_d$ are finitely many oriented points and 1-morphisms are oriented bordisms equipped with homotopy classes of maps to $X$. Two-morphisms are certain type of oriented surfaces with corners equipped with homotopy classes of maps to $X$ and they are considered up to a diffeomorphism relative to boundary. For a symmetric monoidal bicategory $\mathcal{C}$ we define a $\mathcal{C}$-valued oriented 2-dimensional E-HFT with target $X$ as a symmetric monoidal 2-functor from $XBord_d$ to $\mathcal{C}$.

Here we state a special case of our classification of oriented 2-dimensional E-HFTs where $\mathcal{C} = \text{Alg}_k^G$ is the symmetric monoidal bicategory of $k$-algebras, bimodules, and bimodule maps for a commutative ring $k$. The following notions are the main ingredients of our result. For a discrete group $G$, a $G$-algebra is a $G$-graded associative $k$-algebra $A = \oplus_{g \in G} A_g$ with unity such that $A_g A_{g'} \subseteq A_{gg'}$.
for all \(g, g' \in G\). The principal component of \(A\) is \(A_e\) where \(e \in G\) is the identity element. A Frobenius \(G\)-algebra is a \(G\)-algebra \(A = \oplus_{g \in G} A_g\) equipped with a symmetric bilinear form \(\eta\) such that the restriction \(\eta|_{A_g \otimes A_{g^{-1}}} = \eta_A\) is nondegenerate for all \(g \in G\) and zero otherwise.

A biangular \(G\)-algebra \((A, \eta)\), introduced by Turaev [Tu2], is a Frobenius \(G\)-algebra such that the principal component is a separable algebra, each component \(A_g\) is both left and right right one \(A_e\)-module, and \(\eta\) is given by the trace of multiplication map. By allowing different bilinear forms we introduce quasi-biangular \(G\)-algebras (see Section 3.3). We also need \(G\)-graded Morita contexts between \(G\)-algebras which were introduced by P. Boisen [Bo]. We recall their definition and introduce a notion of compatibility with Frobenius structures in Section 3.3. The opposite \(G\)-algebra of \(A\) is \(A^{\text{op}} = \oplus_{g \in G} A_{g^{-1}}\) where the order of multiplication is reversed.

**Theorem 3.5.** Let \(k\) be a commutative ring and let \(X\) be a pointed CW-complex which is an Eilenberg-MacLane space \(K(G, 1)\) for a group \(G\). Then any \(\text{Alg}^2_k\)-valued oriented 2-dimensional \(E\)-HFT with target \(X\) determines a triple \((A, B, \zeta)\) where \(A\) and \(B\) are quasi-biangular \(G\)-algebras and \(\zeta\) is a compatible \(G\)-graded Morita context between \(A\) and \(B^{\text{op}}\). Moreover, any such triple \((A, B, \zeta)\) is realized by an oriented 2-dimensional \(E\)-HFT.

Theorem 3.5 generalizes Schommer-Pries’ classification of \(\text{Alg}^2_k\)-valued oriented 2-dimensional \(E\)-TFTs ([Sc]) which corresponds to taking \(X\) as a point. One direct consequence of Theorem 3.5 is that for a subgroup \(H\) of \(G\), any \(E\)-HFT with target \(K(G, 1)\)-space gives an \(E\)-HFT with target \(K(H, 1)\)-space simply by forgetting the \(G\backslash H\) components of quasi-biangular \(G\)-algebras and a \(G\)-graded Morita context.

Next we upgrade Theorem 3.3 to an equivalence of bicategories. We introduce the bicategory \(\mathcal{E}\)-\(\text{HFT}(X, \text{Alg}^2_k)\) which has \(\text{Alg}^2_k\)-valued oriented \(E\)-HFTs with target \(X \simeq K(G, 1)\) as objects, symmetric monoidal transformations of \(E\)-HFTs as 1-morphisms, and symmetric monoidal modifications as 2-morphisms. We also consider the bicategory \(\text{Frob}^G\) which has quasi-biangular \(G\)-algebras as objects, compatible \(G\)-graded Morita contexts as 1-morphisms, and equivalences of \(G\)-graded Morita contexts as 2-morphisms (see Section 3.3). Finally, we define a forgetting 2-functor \(\mathcal{F}: \mathcal{E}\)-\(\text{HFT}(X, \text{Alg}^2_k) \to \text{Frob}^G\) which assigns the quasi-biangular \(G\)-algebra \(A\) to each oriented \(E\)-HFT determining a triple of the form \((A, B, \zeta)\). On 1-and 2-morphisms the functor \(\mathcal{F}\) similarly forgets the data coming from the last two components of triples. Now we state our main theorem.

**Theorem 3.7.** The forgetting 2-functor \(\mathcal{F}\) is an equivalence of bicategories \(\mathcal{E}\)-\(\text{HFT}(X, \text{Alg}^2_k) \simeq \text{Frob}^G\).

Theorem 3.7 gives a categorical classification of oriented 2-dimensional \(E\)-HFTs. A different approach to categorical classification of such \(E\)-HFTs is given by the structured cobordism hypothesis due to J. Lurie [Lu]. Using \(\infty\)-categories and homotopy fixed points, Lurie [Lu] reformulated the cobordism hypothesis ([BD], [Lu], [AF]) and generalized it to manifolds with structures. It is a well-known fact that an isomorphism class of a principal \(G\)-bundle over a manifold can be described by a homotopy class of a map from manifold to classifying space \(BG\). In this regard, an oriented 2-dimensional \(E\)-HFT with target \(X \simeq K(G, 1)\) is equivalent to a \((G \times \text{SO}(2))\)-structured 2-dimensional \(E\)-TFT. Now we state the main application of Theorem 3.7.

**Corollary 3.8.1** For any finite group \(G\) and any algebraically closed field \(k\) of characteristic zero the \((G \times \text{SO}(2))\)-structured cobordism hypothesis for \(\text{Alg}^2_k\)-valued \((G \times \text{SO}(2))\)-structured \(E\)-TFTs holds true.

This corollary follows from Theorem 3.7 and O. Davidovich’s results [Da] on homotopy \((G \times \text{SO}(2))\)-fixed points in \(\text{Alg}^2_k\).

In our classification of oriented 2-dimensional \(E\)-HFTs we use methods introduced by C. Schommer-Pries in [Sc]. Generalizing the planar decomposition theorem ([Sc]) we prove the \(G\)-planar decomposition theorem which allows us to replace cornered surfaces equipped with homotopy classes of
maps to target space $X \simeq K(G, 1)$ with $G$-planar diagrams. Using diagrams we define a symmetric monoidal bicategory $XB^{PD}$ which is equivalent to $XBord_2$ and freely generated. In other words, $XB^{PD}$ can be freely constructed from a presentation which consists of four sets; generating objects, generating 1-morphisms, generating 2-morphisms, and relations among 2-morphisms. Freely generated symmetric monoidal bicategories satisfy the Cofibrancy theorem (Sc) which states that symmetric monoidal 2-functors out of such bicategories are determined up to an equivalence by the images of generators subject to relations. We obtain a presentation of $XB^{PD}$ using the $G$-planar decomposition theorem and applying the Cofibrancy theorem we classify oriented 2-dimensional E-HFTs.

**Plan of the paper.** This paper consists of two parts. In the first part, we define $G$-linear, $G$-planar, $G$-spatial diagrams and prove the $G$-planar decomposition theorem. In the second part, we introduce symmetric monoidal $G$-equivariant bordism bicategories and obtain their presentations. Then we define and classify oriented 2-dimensional E-HFTs and prove Theorems 3.5 and 3.7 (see Section 3.3). After that we define unoriented 2-dimensional E-HFTs and prove analogous results for them.

**Conventions.** Throughout the paper $G$ is a discrete group with identity element $e$ and the target space is a pointed aspherical CW-complex $(X, x)$ with $\pi_1(X, x) = G$. All manifolds are assumed to be smooth. Closed manifold is a compact manifold without boundary. For smooth manifolds $M$ and $N$ the space of smooth maps $C^\infty(M, N)$ is provided with the Whitney $C^\infty$-topology.

**Acknowledgments.** I would like to thank my advisor Vladimir Turaev for introducing this problem to me and his support throughout this project. I would also like to thank Noah Snyder for fruitful and enlightening discussions on extended field theories, Patrick Chu for helpful discussions, and Alexis Virelizier for his comments on the earlier version of this paper. This work was supported by NSF grant DMS-1664358.

## 2. $G$-Planar Decompositions

### 2.1. Preliminaries

In his study of HFTs Turaev [Tu2] introduced notions of $X$-manifold and $X$-cobordism. An $n$-dimensional $X$-manifold is a tuple $(M, g)$ where $M$ is a closed oriented pointed $n$-manifold and $g \in [(M, \{m_1, \ldots, m_k\}), (X, x)]$ is a pointed homotopy class called the *characteristic map*. An $X$-cobordism between two $X$-manifolds $(M, g)$ and $(M', g')$ is a tuple $(W, P)$ where $W$ is an oriented cobordism between $M$ and $M'$ and $P \in [W, X]$ is a homotopy class restricting to $g$ and $g'$ on the corresponding boundary components. In particular, $P$ carries base points of $M$ and $M'$ to $x \in X$.

In this paper we consider both oriented and unoriented $X$-manifolds and $X$-cobordisms. We also consider compact $X$-manifolds which generalize both $X$-manifolds and $X$-cobordisms. A 1-dimensional compact $X$-manifold is a triple $(M, T, g)$ where $M$ is a compact 1-manifold, $T \subset M$ is a finite set of pairwise distinct points such that $\partial M \subset T$, and $g \in [(M, T), (X, x)]$.

### 2.2. $G$-Linear Diagrams

Linear diagrams were defined by Schommer-Pries [Sc] using the Morse theory of 1-manifolds. We define a $G$-linear diagram by adding linear $G$-data to a linear diagram so that each $G$-linear diagram produces a 1-dimensional compact $X$-manifold.

The main idea of a linear diagram is to represent a compact 1-manifold by a diagram on $\mathbb{R}$ with a combinatorial data. A linear diagram roughly consists of critical values of a fixed Morse function and an open cover of $\mathbb{R}$. Combinatorial data associated to a diagram describes the preimage of each open set under the Morse function. The following definition describes the additional data on a diagram coming from base points and characteristic map of a 1-dimensional compact $X$-manifold.

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1Each connected component of $M$ is equipped with a point.
Definition 2.1. A linear $G$-data $\xi$ is a tuple $(\xi_1, \xi_2)$ where $\xi_1$ is a finite set of points in $\mathbb{R}$ (possibly coinciding) and $\xi_2$ is a finite set of open oriented intervals in $\mathbb{R}$ (possibly overlapping) where each interval is labeled with an element from $G$. For each interval $x \in \xi_2$ we distinguish its boundary components $\partial x = \partial^+ x \sqcup \partial^- x$ by $\partial^+ x > \partial^- x$.

Definition 2.2. A 1-dimensional graphic $\mu$ is a finite set of distinct labeled points in $\mathbb{R}$ where each point is labeled with either cup or cap. A linear $G$-data $\xi$ is said to be compatible with a 1-dimensional graphic $\mu$ if the following conditions hold:

(i) The intersection $\mu \cap \xi_1$ is empty.
(ii) For every $a \in \mu$ there exist two intervals $x_1, x_2 \in \xi_2$ having the same $G$-labels but different orientations such that the intersection $x_1 \cap x_2$ is nonempty and $a \in \partial^+ x_i$ for $i = 1, 2$ if $a$ is labeled with cap and $a \in \partial^- x_i$ for $i = 1, 2$ otherwise.

A 1-dimensional $G$-graphic $\Psi^G = (\mu, \xi)$ is a 1-dimensional graphic $\Psi = \mu$ equipped with a $\Psi$-compatible linear $G$-data $\xi$.

![Figure 1. Singularities of a Morse function on a 1-manifold and their images in $\mathbb{R}$](image)

Let $\Psi^G = (\mu, \xi)$ be a 1-dimensional $G$-graphic. An open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in \mathcal{J}}$ of $\mathbb{R}$ having at most double intersections is said to be $\Psi$-compatible if each $U_\alpha$ contains at most one element from $\mu$ and double intersections are disjoint from $\mu$.

Definition 2.3. Let $\Psi^G = (\mu, \xi)$ be a 1-dimensional $G$-graphic. A chambering set $\Gamma$ for $\Psi^G$ is a set of isolated points in $\mathbb{R}$ disjoint from $\mu \cup \xi_1$. Chambers of $\Gamma$ are the connected components of $\mathbb{R} \setminus (\Gamma \cup \mu \cup \xi_1)$. A chambering set $\Gamma$ is said to be subordinate to an open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in \mathcal{J}}$ of $\mathbb{R}$ if each chamber is a subset of at least one $U_\alpha$ with $\alpha \in \mathcal{J}$.

Knowing the fact that $\mathbb{R}$ has a covering dimension one it is not hard to see that for any 1-dimensional $G$-graphic $\Psi^G$ there exists a $\Psi$-compatible open cover $\mathcal{U}$ and a chambering set $\Gamma$ subordinate to $\mathcal{U}$.

Example 2.1. Figure 2 shows an example of a 1-dimensional compact $X$-manifold $(M, T, g)$ equipped with a Morse function $f$ where arrows and elements $g', g'', g''' \in \pi_1(X, x) = G$ are induced by $g$. The pair $((M, T, g), f)$ induces a 1-dimensional $G$-graphic $\Psi^G = (\mu, \xi)$ as follows. Red points with their labels form $\mu$ while black points form $\xi_1 = f(T)$. Removing critical points and elements of $T$ divides $M$ into seven connected components whose images under $f$ with their labels and directions form $\xi_2$. An open cover $\mathcal{U} = \{U_0, U_1, U_2, U_3, U_4, U_5\}$ of $\mathbb{R}$ is a $\Psi$-compatible open cover and turquoise points form a chambering set subordinate to $\mathcal{U}$.

Let $\Psi^G = (\mu, \xi)$ be a 1-dimensional $G$-graphic induced from a pair $((M, T, g), f)$ of a 1-dimensional compact $X$-manifold equipped with a Morse function having distinct critical values. Let $\Gamma$ be a chambering set subordinate to a $\Psi$-compatible open cover $\mathcal{U}$. Since $f$ is a Morse function and chambers are disjoint from $\mu$ the preimage of a chamber consists of disjoint union of arcs (possibly empty) each mapping diffeomorphically onto the chamber under $f$. A trivialization of a chamber $V$ is an identification of $f^{-1}(V)$ with $\mathbb{N}_{\leq N} \times V$ for some $N \in \mathbb{N}$ where $\mathbb{N}_{\leq N} = \{a \in \mathbb{N} \mid 0 < a \leq N\}$ if $f^{-1}(V)$ is nonempty and identification with empty set otherwise. In the case of nonempty trivialization each $\{i\} \times V$ is called a sheet.
Two neighboring chambers can be separated by a point from $\Gamma \cup \mu \cup \xi_1$. If two such chambers $V$ and $V'$ are separated by a regular value then both trivializations have the same number of sheets. If $V$ and $V'$ are separated by a critical value $p \in \mu$ then Morse lemma (\[Mi\]) implies that the number of sheets differs by two (see Figure 1).

A sheet data $\mathcal{S}$ for a tuple $(\Psi^G, \Gamma)$ consists of a trivialization of each chamber and an injection between trivializations if two chambers are separated by a point from $\mu$ and a permutation between trivializations if chambers are separated by a point from $\Gamma \cup \xi_1$. Injections and permutations of sheet data are not arbitrary but describe how sheets are glued.

A $G$-sheet data $\mathcal{S}^G$ associated to a tuple $(\Psi^G, \Gamma)$ is a sheet data $\mathcal{S}$ with additional assignments and requirements coming from the linear $G$-data $\xi$ as follows. Each element in $\xi_1$ is lifted to boundaries of two sheets which are identified by a permutation. If a chamber intersects with an interval from $\xi_2$ then the $G$-label and the direction (orientation) of the interval are lifted to one of the sheets. Additionally, permutations and injections coming from the sheet data are required to preserve $G$-labelings and directions of the sheets.

**Definition 2.4.** A $G$-linear diagram is a triple $(\Psi^G, \Gamma, \mathcal{S}^G)$ consisting of a 1-dimensional $G$-graphic $\Psi^G$, a chambering set $\Gamma$ subordinate to $\Psi$-compatible cover $\mathcal{U} = \{U_\alpha\}_\alpha \subseteq \mathcal{J}$ and a $G$-sheet data $\mathcal{S}^G$ associated to $(\Psi^G, \Gamma)$.

Any $G$-linear diagram $(\Psi^G, \Gamma, \mathcal{S}^G)$ produces a compact 1-dimensional $X$-manifold and a Morse function with distinct critical values. A pointed diffeomorphism between (compact) $X$-manifolds is said to be $X$-homeomorphism if it commutes with characteristic maps. Let $(M, g)$ and $(N, h)$ be (compact) $X$-manifolds endowed with Morse functions $f_1 : M \to \mathbb{R}$ and $f_2 : N \to \mathbb{R}$. An $X$-homeomorphism $F : M \to N$ is called an $X$-homeomorphism over $\mathbb{R}$ if $F$ commutes with Morse functions i.e. $f_2 \circ F = f_1$.

**Proposition 2.1.** Let $(M, T, g)$ be a 1-dimensional compact $X$-manifold and $\Psi^G$ be a 1-dimensional $G$-graphic induced by a Morse function $f : M \to \mathbb{R}$ having distinct critical values. Let $\Gamma$ be a chambering set for a $\Psi$-compatible cover $\mathcal{U}$ inducing a $G$-linear diagram $(\Psi^G, \Gamma, \mathcal{S}^G)$. If $(M', T', g')$ is the 1-dimensional compact $X$-manifold constructed from $(\Psi^G, \Gamma, \mathcal{S}^G)$ with a Morse function $f' : M' \to \mathbb{R}$, then there exists an $X$-homeomorphism $F : M \to M'$ over $\mathbb{R}$.

**Proof.** The diffeomorphism $F$ is defined by mapping inverse images of chambers to corresponding trivializations. Since corresponding connected components under $F$ have the same $G$-labelings $F$ is an $X$-homeomorphism. By the construction of the Morse function $f' : M' \to \mathbb{R}$ the critical values of both functions coincide and for any chamber $V$ of $\Gamma$, $f$ and $f' \circ F$ restrict to the same map on $f^{-1}(V)$. □
Remark. If a 1-dimensional compact $X$-manifold $(M, T, g)$ is oriented then a $G$-sheet data can be turned into an oriented $G$-sheet data by assigning an orientation to each sheet and requiring injections and permutations to preserve orientations. This defines oriented $G$-linear diagrams and a version of Proposition 2.1 for oriented compact 1-manifolds and oriented diagrams holds true.

2.3. $G$-planar diagrams. C. Schommer-Pries [Sc] generalized the Morse theory of surfaces to a 2-dimensional Morse theory by stratifying jet spaces $\{ J^k(\Sigma, \mathbb{R}^2) \}_{k \geq 0}$ for surface $\Sigma$ and introduced planar diagrams. We define $G$-planar diagrams by adding planar $G$-data to planar diagrams so that each $G$-planar diagram produces an $X$-cobordism between 1-dimensional compact $X$-manifolds.

In this section and the following section we consider generic maps for certain stratifications of jet spaces. Let $M$ and $N$ be smooth manifolds and let jet spaces $\{ J^k(M, N) \}_{k \in I}$ be equipped with a stratification for $k \in I \subseteq \mathbb{N}$. By a generic map we mean a smooth map $f : M \to N$ whose jet sections $\{ j^k f : M \to J^k(M, N) \}_{k \in I}$ are transverse to each stratum. We call $x \in M$ a singularity of $f$ and $f(x) \in N$ its graphic if $j^k f(x)$ lies in codimension one or two stratum for some $k \in I$.

A generic map for Schommer-Pries’ multijet stratification ([Sc]) can have fold, Morse (cup, cap and saddle’s) and cusp singularities. By the multijet transversality theorem ([GG]) generic maps are dense in $C^\infty(\Sigma, \mathbb{R}^2)$. Some of the singularities of generic maps and their graphics in $\mathbb{R}^2$ are shown in Figure 3. Graphic of any generic map has the following properties. Projection of each fold graphic to the last coordinate is a local diffeomorphism. Intersections of fold graphics are transversal and at most two fold graphics intersect at a point. Fold graphics do not intersect with Morse and cusp graphics (see Section 1.4 in [Sc]).

Observe that changing the folding direction of fold singularity does not affect the graphic. Each such symmetry of a singularity is called index. Different indices either give the same graphic or the symmetry of the graphic. For example, Morse singularity has four indices namely cup, cap, saddle-1 and saddle-2 (see Figure 3). We use numbers to indicate different indices of fold and cusp singularities.

For a closed surface $\Sigma$ we want to encode the homotopy class $P$ of an $X$-cobordism $(\Sigma, P)$ on a graphic of a generic map. To do this we fix points $\sigma_1, \ldots, \sigma_N \in \Sigma$ such that each point lies in a different connected component and choose a representative $\hat{P} \in \{ (\Sigma, \sigma_1, \ldots, \sigma_N), (X, x) \}$ of $P$. Then $P$ can be described by finitely many $G$-labeled based loops on $\Sigma$ where $\sigma_i$-based loops on a fixed connected component are representatives of $\pi_1(\Sigma, \sigma_i)$-generators and their $G$-labels are given by $\hat{P}_* \in \text{Hom}(\pi_1(\Sigma, \sigma_i), G)$. This description of $P$ motivates the following definition.

**Definition 2.5.** A planar $G$-data $\xi$ is a tuple $(\xi_1, \xi_2)$ where $\xi_1 = \{ \sigma_i \}_{i=1}^N$ consists of a finite number of points in $\mathbb{R}^2$ and $\xi_2 = \bigsqcup_{i=1}^N \{ g_{ij}^{ij} \}_{j=1}^R(i)$ consists of a finite number of immersed labeled loops in

![Figure 3. Singularities of Schommer-Pries stratification and their graphics in $\mathbb{R}^2$.](image-url)
\( \mathbb{R}^2 \) such that for a fixed \( i \), each element of \( \{ \alpha_{g_{i,j}}^{R(i)} \}_{j=1} \) is based at \( \sigma_i \), loops are in generic position\(^2\) and each loop \( \alpha_{g_{i,j}}^{x} \) is labeled by \( g_{i,j} \in G \).

**Definition 2.6.** A 2-dimensional \( G \)-graphic \( \Phi^G = (\eta, \mu, \xi) \) is a 2-dimensional graphic \( \Phi = (\eta, \mu) \) \((\mathcal{Sc})\) equipped with a \( \Phi \)-compatible planar \( G \)-data \( \xi \). In other words, it is a diagram in \( \mathbb{R}^2 \) consisting of a finite number of embedded labeled curves (\( \eta \)), a finite number of embedded labeled points (\( \mu \)) and a planar \( G \)-data \( \xi = (\xi_1, \xi_2) \) satisfying the following conditions:

(i) Elements of \( \eta \) can only have transversal intersections and no three or more elements intersect at a point. Each element of \( \eta \) is labeled with either Fold-1 or Fold-2.

(ii) Projections of elements of \( \eta \) to the last coordinate of \( \mathbb{R}^2 \) are local diffeomorphisms.

(iii) Elements of \( \mu \) are isolated and each element is labeled with one of the Cup, Cap, Saddle-1, Saddle-2, Cusp-\( i \) for \( i = 1, 2, 3, 4 \).

(iv) Each element in \( \mu \) has a neighborhood in which two elements of \( \eta \) form one of the Cup, Cap, Saddle-1, Saddle-2, Cusp-\( i \) graphic for \( i = 1, 2, 3, 4 \).

(v) Elements of \( \xi_2 \) are disjoint from \( \mu \) and \( \xi_1 \cap \mu = \emptyset \).

**Proposition 2.2.** Let \( \Sigma \) be a closed surface and \( f : \Sigma \to \mathbb{R}^2 \) be a generic map. Then for any \( X \)-cobordism \((\Sigma, P)\) there exists points \( \{ \sigma_1, \ldots, \sigma_N \} \) and based loops on each connected component of \( \Sigma \) representing generators of \( \pi_1(\Sigma, \sigma_i) \) such that the graphic of \( f \) and the images of these loops under \( f \) form a 2-dimensional \( G \)-graphic in \( \mathbb{R}^2 \).

**Proof.** Assume that \( \Sigma \) is connected. By the properties of the stratification, the graphic of \( f \) gives \( \eta \) and \( \mu \) forming a 2-dimensional graphic \( \Phi = (\eta, \mu) \). Image of any a point \( \sigma \) away from Morse and cusp singularities give \( \xi_1 \). Pick based loops which are in generic position representing \( \pi_1(\Sigma, \sigma) \)-generators. Consider their images in \( \mathbb{R}^2 \) under \( f \). Since \( f \) is a generic map\(^3\) each nontransversal intersection can be changed into a transversal one by perturbing loops in \( \Sigma \) locally without changing their homotopy classes. The homotopy class \( P \) determines the \( G \)-labelings of loops giving \( \xi_2 \). If \( \Sigma \) is not connected, apply this process on each connected component. \( \blacksquare \)

**Example 2.2.** Figure 4 shows an example of a 2-dimensional \( G \)-graphic induced from an \( X \)-torus \((T^2, P)\) equipped with a projection to the page map. In this case the planar \( G \)-data \( \xi = (\xi_1, \xi_2) \) is given by the brown point \( \sigma_1 \) and based brown loops labeled with \( \alpha_{1,1}^{g_{1,j}} \) and \( \alpha_{1,2}^{g_{1,j}} \). The set \( \eta \) consists of four labeled red arcs and the set \( \mu \) consists of four labeled red points.

Loops in planar \( G \)-data cannot be assumed to have transversal intersection with each element of \( \eta \). Figure 5 shows an example of nontransversal intersection of an element of \( \xi_2 \) with a fold graphic.

\(^2\)Loops are self-transversal and have transversal intersection with each other.

\(^3\)In particular, \( f \) is a local diffeomorphism on connected open sets containing no singularity.
Remark. Different finite presentations of $\pi_1(\Sigma, \sigma)$ lead to different 2-dimensional $G$-graphics in $\mathbb{R}^2$. We restrict ourselves to 2-dimensional $G$-graphics whose planar $G$-data come from a fixed finite presentation of $\pi_1(\Sigma, \sigma)$ where $(\Sigma, \sigma)$ is a connected closed surface. If a surface is not connected, the fixed presentation in each connected component is considered. We do not rename this subcollection of 2-dimensional $G$-graphics for fixed presentations and continue to say 2-dimensional $G$-graphics.

Let $\Phi^G = (\eta, \mu, \xi)$ be a 2-dimensional $G$-graphic, an open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in J}$ of $\mathbb{R}^2$ with at most triple intersections is said to be $\Phi$-compatible if each triple intersection is disjoint from $\mu$ and each double intersection is disjoint from $\eta \cup \mu$ or contains a single element from $\eta$. Knowing the fact that $\mathbb{R}^2$ has covering dimension two and the sets $\eta$ and $\mu$ are finite it is not hard to see that $\Phi$-compatible open covers exist for a given 2-dimensional graphic $\Phi$. Schommer-Pries [Sc] used graphs to form open sets. In the following we require these graphs to have transversal intersections with planar $G$-data $\xi$.

**Definition 2.7.** Let $\Phi^G = (\eta, \mu, \xi)$ be a 2-dimensional $G$-graphic in $\mathbb{R}^2$. A chambering graph $\Gamma$ for $\Phi^G$ is a smoothly embedded graph in $\mathbb{R}^2$ whose vertices are disjoint from $\Phi^G$ and have degree either one or three. Edges of $\Gamma$ are transverse to $\Phi^G$ and satisfy the following conditions. Projection of each edge to the last coordinate of $\mathbb{R}^2$ is a local diffeomorphism. Under this projection at each trivalent vertex at least one edge maps to downward of the vertex and at least one edge maps to upward of the vertex.

**Definition 2.8.** Let $\Phi^G = (\eta, \mu, \xi)$ be a 2-dimensional $G$-graphic in $\mathbb{R}^2$ and let $\Gamma$ be a chambering graph for $\Phi^G$. Chambers of $(\Phi^G, \Gamma)$ are the connected components of $\mathbb{R}^2 \setminus (\Gamma \cup \eta \cup \mu)$. A chambering graph $\Gamma$ is said to be subordinate to an open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in J}$ of $\mathbb{R}^2$ if each chamber is a subset of at least one $U_\alpha$ with $\alpha \in J$.

**Example 2.3.** Figure 6 shows an example of a chambering graph and corresponding chambers.

**Proposition 2.3.** Let $\Phi^G$ be a 2-dimensional $G$-graphic in $\mathbb{R}^2$ and let $\mathcal{U} = \{U_\alpha\}_{\alpha \in J}$ be a $\Phi$-compatible open cover of $\mathbb{R}^2$. Then there exists a chambering graph $\Gamma$ for $\Phi^G$ subordinate to $\mathcal{U}$.

**Proof.** The 2-dimensional graphic version of this proposition is proven in [Sc] (see Proposition 1.46). For a given $\Phi$-compatible cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in J}$ let $\Gamma$ be a chambering graph for the underlying 2-dimensional graphic. If there are nontransversal intersections then edges and vertices of $\Gamma$ can be slightly perturbed to make all intersections transversal while being compatible with $\mathcal{U}$. ■

Let $\Sigma$ be closed surface with a generic map $f : \Sigma \to \mathbb{R}^2$. Let $(\Sigma, P)$ be an $X$-cobordism equipped with base points and $G$-labeled based loops inducing a 2-dimensional $G$-graphic $\Phi^G = (\eta, \mu, \xi)$. To
define a $G$-sheet data we first recall a sheet data $S$ for a tuple $(\Phi, \Gamma)$ where $\Gamma$ is a chambering graph subordinate to $\Phi$-compatible open cover or $\mathbb{R}^2$. Since $f$ is generic for any chamber $U_\beta$ the preimage $f^{-1}(U_\beta)$ consists of disjoint union of open sets (possibly empty) each mapping diffeomorphically onto $U_\beta$. A trivialization of a chamber is the identification of $f^{-1}(U_\beta)$ with $N \leq N \times U_\beta$ for some $N \in \mathbb{N}$ where $N \leq N = \{a \in \mathbb{N} \mid 0 < a \leq N\}$ if $f^{-1}(U_\beta)$ is nonempty and identification with the empty set otherwise. In the case of nonempty trivialization each $\{i\} \times U_\beta$ is called a sheet. Trivialization of each chamber is part of the sheet data.

Let two chambers $U_{\beta_1}$ and $U_{\beta_2}$ be separated by an edge of the chambering graph $\Gamma$. For such neighboring chambers the sheet data contains a permutation between trivializations (see Figure 7) describing how sheets are glued. These permutations are required to satisfy the cocycle condition: if three chambers are separated by edges of a trivalent vertex then the circular composition of permutations is required to be identity permutation.

If two chambers are separated by a fold graphic then the number of sheets in trivializations differ by two. For such neighboring chambers the sheet data contains an injection between trivializations describing fold singularity and how sheets are glued. Sheet data for cup, cap, saddle-1 and saddle-2 graphics are the same as the fold sheet data.

We describe the sheet data for the cusp graphic labeled by Cusp-2. Assume that chambers $U_{\beta_1}$ and $U_{\beta_2}$ are separated by a cusp graphic $\kappa_i \in \mu$ connecting fold graphics $\beta_i, \beta_j \in \eta$ (see Figure 8). Let $N_{\leq N+1}$ and $N_{\leq N+3}$ be the trivializations of the chambers $U_{\beta_2}$ and $U_{\beta_1}$ respectively. Sheet data contains an injection $\sigma_1$ for $\beta_i$. Assume that $\sigma_1$ is given by $\sigma_1|_{N_{\leq N}} = \sigma \in S_N$ (symmetric group) and $\sigma_1(N + 1) = N + 1$. Then sheet data contains an injection $\sigma_2$ for cusp graphic coming from $\beta_j$ as $\sigma_2|_{N_{\leq N}} = \sigma$ and $\sigma_2(N + 1) = N + 3$ (see Figure 8).

A $G$-sheet data $S^G$ consists of a sheet data $S$ and additionally require trivializations of chambers to lift directed $G$-labeled arcs and points coming from planar $G$-data to sheets. Permutations and injections are also required to preserve sheets with directed labeled arcs (see Figure 7).

**Definition 2.9.** A $G$-planar diagram is a triple $(\Phi^G, \Gamma, S^G)$ consisting of a 2-dimensional $G$-graphic $\Phi^G$, a chambering graph $\Gamma$ for $\Phi^G$ subordinate to a $\Phi$-compatible cover $U = \{U_\alpha\}_{\alpha \in J}$ and a $G$-sheet data $S^G$ associated to $(\Phi^G, \Gamma)$. 
Any $G$-planar diagram $(\Phi^G, \Gamma, S^G)$ produces an $X$-cobordism $(\Sigma, P)$ with a generic map $f : \Sigma \to \mathbb{R}^2$ where $P \in [\Sigma, X]$ is determined by the $G$-labeled based loops on $\Sigma$. Let $(\Sigma, P)$ and $(\Sigma', P')$ be $X$-surfaces endowed with generic maps $f_1 : \Sigma \to \mathbb{R}^2$, $f_2 : \Sigma' \to \mathbb{R}^2$. An $X$-homeomorphism $F : \Sigma \to \Sigma'$ is said to be over $\mathbb{R}^2$ if it commutes with generic maps, i.e. $f_2 \circ F = f_1$.

**Proposition 2.4.** Let $\Sigma$ be a closed surface with a generic map $f : \Sigma \to \mathbb{R}^2$. Let $(\Phi^G, \Gamma, S^G)$ be a $G$-planar diagram for an $X$-cobordism $(\Sigma, P)$. Then the $X$-cobordism $(\Sigma', P')$ constructed from $(\Phi^G, \Gamma, S^G)$ is $X$-homeomorphic to $(\Sigma, P)$ over $\mathbb{R}^2$.

**Proof.** The diffeomorphism $F : \Sigma \to \Sigma'$ is defined by sending each open set of $\Sigma$ with no singularity to the union of corresponding sheets and sending each open set containing a singularity to the corresponding sheets mapping to the graph of the singularity under $f'$. Since $G$-labels of the fixed fundamental group generators coincide $F$ is an $X$-homeomorphism. By the construction of the generic map $f' : \Sigma' \to \mathbb{R}^2$, graphics of $f' \circ F$ and $f$ coincide and similarly we have $f = f' \circ F$ on each connected component of $f^{-1}(U_\beta)$ for every chamber $U_\beta$.

The notions and results of this section can be generalized to a general $X$-cobordism $(\Sigma, P)$ as follows. A generic map has the form $f : (\Sigma, \partial \Sigma) \to ([\mathbb{R} \times I, \mathbb{R} \times \{0, 1\}]$ for the relative stratification of jet spaces where Schommer-Pries stratification is considered on $\Sigma \setminus \partial \Sigma$ and the Morse theory\footnote{Morse theory can be formulated using the Thom-Boardman stratification of jet spaces (see Section 1.2.1 in \cite{Sc}).} is considered on $\partial \Sigma$. The relative Thom transversality theorem (\cite{GG}) guarantees that generic maps are dense in $C^\infty([\Sigma, \partial \Sigma], ([\mathbb{R} \times I, \mathbb{R} \times \{0, 1\}]$).

The definition of $2$-dimensional $G$-graphic is modified to include $1$-dimensional $G$-graphics on $\mathbb{R} \times \{0, 1\}$ and elements of $\eta$ and $\xi_2$ are required to have transversal intersection with boundary components. A planar $G$-data $\xi$ has additional $G$-labeled arcs between base points at boundary components. Edges of a chambering diagram are required to end on $\mathbb{R} \times \{0, 1\}$ forming a chambering set subordinate to induced open cover. All vertices of $\Gamma$ lie in interior of $\mathbb{R} \times I$. A $G$-sheet data has additional trivializations of chambers with boundaries where injections and permutations form a $1$-dimensional $G$-sheet data on the boundary.

To classify $2$-dimensional $E$-HFTs we need to consider manifold with corners, more precisely $\langle 2 \rangle$-surfaces endowed with characteristic maps. A $\langle 2 \rangle$-surface is a $2$-dimensional compact manifold with faces $S$ equipped with two submanifold with faces $\partial_h S$ and $\partial_v S$ called horizontal and vertical faces respectively such that $\partial S = \partial_h S \cup \partial_v S$ and $\partial_h S \cap \partial_v S$ is either empty or a face of both. A $\langle 2 \rangle$-surface $S$ is pointed if it equipped with a finite set $R$ consisting of points of $\partial S$ such that $\partial_h S \cap \partial_v S \subset R$ and $\partial_v S \cap R = \partial_h S \cap \partial_v S$. Observe that equipping pointed $\langle 2 \rangle$-surface with pointed homotopy class to $X$ makes horizontal boundary a $1$-dimensional compact $X$-manifold.

**Definition 2.10.** A $\langle 2 \rangle$-surface is a triple $(S, R, P)$ where $(S, R)$ is a pointed $\langle 2 \rangle$-surface and $P \in [(S, R), (X, x)]$ is a homotopy class of pointed maps. A $\langle 2 \rangle$-X-surface $(S, R, P)$ is said to be cobordism type if $\partial_v S$ is a product $X$-manifold with constant characteristic map i.e. $(\partial_h S, P|_{\partial_h S}) = (M \times I, P|_{M \times I})$ where $(M, P|M)$ is a $0$-dimensional compact $X$-manifold and $P|M \times I \in [(M \times I, \partial(M \times I)), (X, x)]$ is the constant homotopy class (see Figure 9).

**Remark.** One can glue two cobordism type $\langle 2 \rangle$-X-surfaces along their common horizontal or vertical faces and obtain a new cobordism type $\langle 2 \rangle$-X-surface by forgetting the points lying in the interior of the resulting $\langle 2 \rangle$-X-surface and in the interior of the vertical face of the resulting surface.

Let $(S, R, P)$ be a cobordism type $\langle 2 \rangle$-X-surface and let $I_{mn} = [m, n]$ be an interval for $m, n \in \mathbb{Z}$. A generic map is a smooth map of the form

$$f : (S, \partial_h S, \partial_v S) \to (I_{mn} \times I, I_{mn} \times \{0, 1\}, \{m, n\} \times I)$$

In order to glue two cobordism type $\langle 2 \rangle$-X-surfaces $(S_1, R_1, P_1)$ and $(S_2, R_2, P_2)$ along their horizontal faces we must have a diffeomorphism $F : (\partial_h S_1, \partial(\partial_h S_1)) \to (\partial_h S_2, \partial(\partial_h S_2))$ and $P_1|_{\partial_h S_1} = P_2|_{\partial_h S_2} \circ [F]$ as a pointed homotopy class.
for the the same stratifications as the compact case. Again by the relative Thom transversality theorem ([GG]) such maps are dense in the space of smooth functions of this form.

In this case a 2-dimensional $G$-graphic lies in $I_{mn} \times I$ and in addition to compact case elements of $\eta$ are required to be disjoint from $\partial I_{mn} \times I$ and transverse to $I_{mn} \times \partial I$. Edges of a chambering graph are additionally required to be disjoint from $\partial I_{mn} \times I$. Since there is no singularity on the vertical boundary $G$-sheet data is similar to compact case producing a cobordism type $(2)$-$X$-manifold. An example of a 2-dimensional $G$-graphic under a projection to the page map and a chambering graph for the cobordism type $(2)$-$X$-manifold is given in Figure 9.

**Remark.** If $(\Sigma, g)$ is an oriented (closed, compact, cobordism type) $X$-cobordism then an oriented $G$-sheet data is defined by assigning compatible orientations to sheets and requiring injections and permutations to preserve orientations. This defines oriented $G$-planar diagrams and a version of Proposition 2.4 for oriented surfaces and oriented diagrams holds true.

2.4. $G$-**spatial diagrams.** C. Schommer-Pries [Sc] introduced spatial diagrams to identify planar diagrams which produce homeomorphic surfaces. We define $G$-spatial diagrams to identify $G$-planar diagrams giving $X$-homeomorphic $X$-cobordisms. Using $G$-spatial diagrams we define an equivalence relation among $G$-planar diagrams and prove the $G$-planar decomposition theorem.

Schommer-Pries [Sc] stratified jet spaces $\{J^k((\Sigma \times I, \Sigma \times \partial I), (\mathbb{R}^2 \times I, \mathbb{R}^2 \times \partial I))\}_{k \geq 0}$ by applying ideas from Cerf theory to generic maps for his stratification of jet spaces $\{J^k(\Sigma, \mathbb{R}^2)\}_{k \geq 0}$. By multijet and relative transversality theorems ([GG]) generic maps are dense in $C^\infty((\Sigma \times I, \Sigma \times \partial I), (\mathbb{R}^2 \times I, \mathbb{R}^2 \times \partial I))$. Graphics of singularities in $\mathbb{R}^2 \times I$ are shown in Figure 10. Graphic of any generic map for Schommer-Pries’ multijet stratification has the following properties. There are only transversal intersections and at most three fold graphics can intersect at a point. Moreover, when two surfaces intersect along an arc the projection of the arc to the last coordinate is a local diffeomorphism except for finitely many points. Thus, the restriction of a graphic to $\mathbb{R}^2 \times \{t\}$ is a 2-dimensional graphic except for finitely many $t \in (0, 1)$ (see Section 1.4 in [Sc]).

In exactly the same way with the previous section indices of a singularity are the symmetries of the singularity such that different indices either give the same graphic or symmetric graphics. We again use numbers to indicate different indices.

Similar to 1 and 2-dimensional $G$-graphics we add extra data into a graphic of generic map. Recall that to represent the homotopy class $P$ of an $X$-cobordism $(\Sigma, P)$ we choose base points and based loops on each connected component of $\Sigma$. Since we consider a fixed presentation of $\pi_1(\Sigma, \sigma_i)$ for each connected component, different choices of base points and based loops can only change the $G$-labelings on a fixed component by a conjugation. Similarly if there exists an $X$-homeomorphism $F : (\Sigma, P) \to (\Sigma', P')$, then $G$-labels of based loops can only differ by a conjugation.

For both cases we consider $\Sigma \times I$ with base points and based $G$-labeled loops on each connected component of $\partial(\Sigma \times I)$ where $\Sigma \times \{1\}$ is either considered with different choices of points and loops or it is identified with $\Sigma'$ using $F$. Using this approach the conjugating element in a fixed component can be represented by a $G$-labeled representative of a homotopy class in $[(I, 0, 1), (\Sigma \times I, \sigma_i, \sigma_i')]$. 
where $\sigma_i$ and $\sigma'_i$ lies in the fixed connected component of $\Sigma \times I$. A representative of this homotopy class which does not intersect with boundary except endpoints is called a \textit{straight arc}. Now we can define additional data on a graphic of a generic map.

\textbf{Definition 2.11.} Let $\xi^1 = (\xi^i_1, \xi^j_1)$ and $\xi^2 = (\xi^i_2, \xi^j_2)$ be planar $G$-data in $\mathbb{R}^2 \times \{0\}$ and $\mathbb{R}^2 \times \{1\}$ respectively with $\xi^i_1 = \{\sigma_{i,k}\}_{k=1}^{N}$ and $\xi^i_2 = \prod_{j=1}^{N}\{\alpha^{g_{i,j,k}}_{j}\}_{j=1}^{R_k(i)}$ for $k = 1, 2$. A \textit{spatial $G$-data $\tau$} is a quadruple $(\xi^1, \xi^2, \zeta, \rho)$ where $\zeta$ consists of finitely many embedded arcs $\{\gamma^g_i\}_{i=1}^{N}$ in $\mathbb{R}^2 \times I$ each $\gamma^g_i$ is labeled with $g_i \in G$ such that $\partial \gamma^g_i = \{\sigma_{i,1}, \sigma_{\rho(i),2}\} = \gamma^g_i \cap \partial(\mathbb{R}^2 \times I)$ where $\rho \in S_N$ is a permutation and arcs are in generic position. The bijection $p : \xi^1 \to \xi^2$ is defined by $p(\sigma_{i,1}) = \sigma_{\rho(i),2}$ and $p(\alpha^{g_{i,j,1}}_{i,j,1}) = \alpha^{g_{\rho(i),j,2}}_{\rho(i),j,2}$ where $g_{i,j,1} = (g_i)(g_{\rho(i),j,2})(g_i^{-1})$ if $\gamma^g_i$ is directed from $\sigma_{i,1}$ to $\sigma_{\rho(i),2}$ and $g_{i,j,1} = (g_i^{-1})(g_{\rho(i),j,2})(g_i)$ otherwise.

\textbf{Definition 2.12.} A $\text{3-dimensional $G$-graphic} \Delta^G = (\delta, \eta, \mu, \tau)$ is a 3-dimensional graphic $\Delta = (\delta, \eta, \mu)$ equipped with a $\Delta$-compatible spatial $G$-data $\tau = (\xi^1, \xi^2, \zeta, \rho)$ in $\mathbb{R}^2 \times I$ satisfying the following conditions:

(i) Projections of elements of $\delta$ to the last two coordinates are local diffeomorphisms and elements of $\delta$ are labeled with either Fold-1 or Fold-2.

(ii) Projections of elements of $\eta$ to the last coordinate are local diffeomorphisms and elements of $\eta$ are labeled with either Morse-\textit{e} or Cusp-\textit{i} where $i = 1, 2, 3, 4$ indicates the indices.

(iii) Each element of $\eta$ has a neighborhood in which two elements of $\delta$ form either Morse or Cusp graphic.

(iv) Elements of $\mu$ are labeled with one of the following singularities: Morse relation-\textit{i}, Cusp inversion-\textit{j}, Cusp inversion-\textit{i'}, Cusp flip-\textit{j} and Swallowtail-\textit{j} where $i = 1, 2, 3, 4, 5, 6, 7, 8$ and $j = 1, 2, 3, 4$ indicate the indices.

(v) Each element of $\mu$ has a neighborhood in which some elements of $\delta$ and $\eta$ form one of the following graphics: Morse Relation-\textit{i}, Cusp Inversion-\textit{j}, Cusp Inversion-\textit{i'}, Cusp Flip-\textit{j} and Swallowtail-\textit{i} where $i = 1, 2, 3, 4$ and $j = 1, 2$ indicate graphics of different indices.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
Fold & Cusp & Morse & Morse Relation \\
\hline
\includegraphics[width=2cm]{fold.png} & \includegraphics[width=2cm]{cusp.png} & \includegraphics[width=2cm]{morse.png} & \includegraphics[width=2cm]{morse_relation.png} \\
\hline
\includegraphics[width=2cm]{cusp_inversion.png} & \includegraphics[width=2cm]{cusp_inversion_.png} & \includegraphics[width=2cm]{cusp_flip.png} & \includegraphics[width=2cm]{swallowtail.png} \\
\hline
\end{tabular}
\caption{Graphics of the singularities in $\mathbb{R}^2 \times I$}
\end{table}

\textsuperscript{6}Arcs are pairwise disjoint and transversal to $\partial(\mathbb{R}^2 \times I)$.

\textsuperscript{7}An arc $\gamma : [0, 1] \to \mathbb{R}^2 \times I$ is assumed to be directed from 0 to 1.

\textsuperscript{8}Morse singularities are paths of cap, cup, saddle-1 and saddle-2 singularities.
(vi) The restriction of the graphic to boundary components gives 2-dimensional $G$-graphics.
(vii) Elements of $\delta, \eta, \mu$ and $\zeta$ are transversal with respect to each other and to $\mathbb{R}^2 \times \{0, 1\}$. Moreover, when two surfaces intersect along an arc there can only be finitely many points on the arc with tangent space lying in $\langle \partial_x, \partial_y \rangle$ where $(x, y, t)$ is the coordinate for $\mathbb{R}^2 \times I$.

**Proposition 2.5.** Let $\Sigma_1$ be a closed surface with $N$-connected components and $F : \Sigma_1 \times I \to \mathbb{R}^2 \times I$ be a generic map. Let $\mathcal{F} : (\Sigma_1 \times \{1\}, P_1) \to (\Sigma_2, P_2)$ be an $X$-homeomorphism between $X$-cobordisms and $\xi^1$ and $\xi^2$ be planar $G$-data compatible with the graphics of generic maps $F|_{\Sigma_1 \times \{0\}}$ and $F|_{\Sigma_1 \times \{1\}} \circ \mathcal{F}^{-1}$ respectively. Then there exist a bijection $p : \xi^1 \to \xi^2$ and $G$-labeled straight arcs $\{\alpha^g_i\}_{i=1}^N$ such that $(\xi^1, \xi^2, \{F(\alpha^g_i)\}_{i=1}^N, p)$ is a spatial $G$-data compatible with the graphic of $F$.

**Proof.** Image of connected components under the $X$-homeomorphism determines a bijection between points of two planar $G$-data. The induced map $\mathcal{F}_*$ between fundamental groups of connected components determines a bijection between based loops using the (fixed) presentation. Since $F$ is generic there exists a straight arc on each connected component of $\Sigma_1 \times I$ transverse to graphic of $F$. Bijection on based loops determines the $G$-labels of straight arcs. \hfill \blacksquare

**Example 2.4.** Figure 11 shows an example of a 3-dimensional $G$-graphic without the labels of graphics. This graphic is induced from the cylinder of $X$-torus $(T^2, P)$ given in Example 2.2 with the generic map $F = f \times \mathrm{id} : T^2 \times I \to \mathbb{R}^2 \times I$ where $f$ is the projection to the page map while representatives of $\pi_1(T^2)$-generators are different. Thus, we have the following equalities on $G$-labelings $g_{1,1,1} = g_{1,1,2}, g_{1,2,1} = g_{1,2,2}$ and $g_1 = e$. Note that labels of the graphics are the same as in Figure 4 since singularities are paths of fold and Morse singularities.

Let $\Delta^G = (\delta, \eta, \mu, \tau)$ be a 3-dimensional $G$-graphic in $\mathbb{R}^2 \times I$. An open cover of $\mathbb{R}^2 \times I$ with at most 4-fold intersections is said to be $\Delta$-compatible if each 4-fold intersection is disjoint from $\delta \cup \eta \cup \mu$, each triple intersection is disjoint from $\mu \cup \eta$ and contains at most a single component of the surfaces in $\delta$ and each double intersection is disjoint from points in $\mu$. Since $\mathbb{R}^2 \times I$ has covering dimension 3 and there are only finitely many elements in $\delta, \eta$ and $\mu$ it is not hard to see that there exists a $\Delta$-compatible open cover of $\mathbb{R}^2 \times I$. Schommer-Pries [Sc] used 2-dimensional stratified spaces to form open sets. We recall these stratified spaces and require them to be in a generic position with the arcs of spatial $G$-data as follows.

**Definition 2.13.** Let $\Delta^G = (\delta, \eta, \mu, \tau)$ be a 3-dimensional $G$-graphic in $\mathbb{R}^2 \times I$. A chambering foam $\Gamma$ for $\Delta^G$ is a smooth embedding of 2-dimensional locally conical stratified space $\Gamma$ of compact type (see [Sc]) into $\mathbb{R}^2 \times I$ with the following properties. $\Gamma$ is locally conical with respect to the system of local models $I^2, I \times C_1, I \times C_3, CP$ and $CK_4$ shown in Figure 12. Vertices are disjoint from

[9] For the $X$-cobordism $(\Sigma_1 \times \{0\}, P_1)$. 

**Figure 11.** An example of 3-dimensional $G$-graphic without labels of graphics.
Δ^G. Edges can only intersect with a surface from δ and with an arc from ξ_1^2 and ξ_2^2. Faces can only intersect with surfaces from δ and arcs from η and ζ. All intersections are transversal. Γ additionally satisfies the following conditions:

(I) Projection \( p: \Gamma \rightarrow \mathbb{R} \times I \) to the last two coordinates has no singularity and projection of faces to the last coordinate has no singularity.

(II) For every \( t \in I \) satisfying \( \mathbb{R}^2 \times \{t\} \cap \mu = \emptyset \), \( t \) is not a critical value of projection \( p: \Gamma \rightarrow I \) and \( \mathbb{R}^2 \times \{t\} \cap \Gamma \) does not include a vertex of \( \Gamma \), the graph \( \mathbb{R}^2 \times \{t\} \cap \Gamma \) forms a chambering graph for the 2-dimensional graphic \( \Delta \cap \mathbb{R}^2 \times \{t\} \).

(III) Projection of each one of four edges in \( CK_4 \)-model connecting at the cone point to the last coordinate is a local diffeomorphism. Additionally, at least one of them maps to downward of the cone point and at least one of them maps to upward of the cone point.

(IV) Projection of the two edges in \( CP \)-model connecting at the cone point to the last coordinate maps both edges to the same direction with respect to the cone point.

**Definition 2.14.** Let \( \Delta^G = (\delta, \eta, \mu, \tau) \) be a 3-dimensional \( G \)-graphic and let \( \Gamma \) be a chambering foam for \( \Delta^G \). Chambers of \( \Gamma \) are the connected components of \( \mathbb{R}^2 \times I \\setminus (\Gamma \cup \delta \cup \eta \cup \mu) \). A chambering foam \( \Gamma \) is said to be subordinate to an open cover \( \mathcal{O} = \{O_\alpha\}_{\alpha \in J} \) of \( \mathbb{R}^2 \times I \) if each chamber is a subset of at least one \( O_\alpha \) with \( \alpha \in J \).

**Lemma 2.1.** Let \( \Delta^G \) be a 3-dimensional \( G \)-graphic in \( \mathbb{R}^2 \times I \) with a chambering foam \( \Gamma \) inducing 2-dimensional \( G \)-graphics and chambering graphs \( \{\Phi_0^G, \Gamma_0\} \) and \( \{\Phi_1^G, \Gamma_1\} \) on \( \mathbb{R}^2 \times \{0\} \) and \( \mathbb{R}^2 \times \{1\} \) respectively. Let \( \mathcal{O} = \{O_\alpha\}_{\alpha \in J} \) be a \( \Delta \)-compatible open cover of \( \mathbb{R}^2 \times I \) with restrictions \( \mathcal{O}_0 \) and \( \mathcal{O}_1 \) on \( \mathbb{R}^2 \times \{0\} \) and \( \mathbb{R}^2 \times \{1\} \) respectively. Suppose that each \( \Gamma_i \) is a chambering graph subordinate to \( \mathcal{O}_i \) for \( i = 1, 2 \) then there exists a chambering foam \( \Gamma' \) for \( \Delta^G \) which is subordinate to \( \mathcal{O} \) and whose restriction to \( \mathbb{R}^2 \times \{0\} \) and \( \mathbb{R}^2 \times \{1\} \) yield \( \Gamma_0 \) and \( \Gamma_1 \) respectively.

**Proof.** In [Sc] the corresponding statement for 3-dimensional graphic was proven (see Corollary 1.47 in [Sc]). Since \( \Gamma \) is a chambering foam for \( \Delta^G \) its restriction to boundaries give chambering graphs for the associated 2-dimensional graphics. By using the corresponding statement for the underlying 3-dimensional graphic there exists a chambering foam subordinate to \( \mathcal{O} \) whose restriction to \( \mathbb{R}^2 \times \{0\} \) and \( \mathbb{R}^2 \times \{1\} \) gives \( \Gamma_0 \) and \( \Gamma_1 \) respectively. The only possible problem is that arcs in \( \zeta \) connecting base points may not have transversal intersections with \( \Gamma \). However, this can be solved by locally perturbing \( \Gamma \) such that it is still subordinate to \( \mathcal{O} \).

Let \( \Sigma \) be a closed surface and \( \Delta^G \) a 3-dimensional \( G \)-graphic induced from a generic map \( F: \Sigma \times I \rightarrow \mathbb{R}^2 \times I \) with a compatible spatial \( G \)-data. Let \( \Gamma \) be a chambering foam subordinate to \( \Delta \)-compatible cover \( \mathcal{O} = \{O_\alpha\}_{\alpha \in J} \) of \( \mathbb{R}^2 \times I \). To define a \( G \)-sheet data we first recall a sheet data \( \mathcal{S} \) for a tuple \( (\Delta, \Gamma) \) briefly (see [Sc] for details). Since \( F \) is generic\(^{10} \) for any chamber \( O_\beta \) the preimage \( F^{-1}(O_\beta) \) consists of disjoint union of open sets each mapping diffeomorphically onto \( O_\beta \).

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\(^{10}\)In particular, \( F \) is a local diffeomorphism on open sets not containing any singularity.
under $F$. A trivialization of a chamber is the identification of $F^{-1}(O_{\beta})$ with $\mathbb{N}_{\leq N} \times O_{\beta}$ where $\mathbb{N}_{\leq N} = \{a \in \mathbb{N} \mid 0 < a \leq N\}$ if $F^{-1}(O_{\beta})$ is nonempty and identification with the empty set otherwise. In the case of nonempty trivialization each $\{i\} \times O_{\beta}$ is called a sheet. Trivialization of each chamber is part of the sheet data.

Let two chambers $O_{\beta_1}$ and $O_{\beta_2}$ be separated by a 2-dimensional strata of $\Gamma$. For such chambers the sheet data contains a permutation between trivializations describing how sheets are glued. These permutations are required to satisfy the cocycle condition: in the local models $I \times C_3, CP$ and $CK_4$ circular compositions of three or four permutations are required to be identity. If two chambers are separated by a fold, cusp, Morse, and Morse relation graphic then sheet data is the same as sheet data in the previous section. This is because these graphics are paths of graphics in previous section.

According to properties of multijet stratification transversal double and triple fold intersections are possible. There are four chambers for the double and eight for the triple fold intersection. In both cases the sheet data contains injections between trivializations of neighboring chambers describing how sheets are glued. The sheet data of fold-Morse and fold-cusp intersections such as cusp flip follows similarly from the sheet data of fold, Morse and cusp graphics. The sheet data of cusp inversions consist of multiple injections which satisfy the corresponding cusp sheet data.

We describe the sheet data of Swallowtail-1 graphic shown in Figure 13 where two (blue and green) out of three fold singularities form a double fold crossing. There are three chambers $U_{\beta_1}, U_{\beta_2}$ and $U_{\beta_3}$ as shown in Figure 13. Let $\mathbb{N}_{\leq N}$ be the trivialization of sheets in $U_{\beta_1}$. Cusp singularities determines indices of the fold singularities giving two more sheets in the chamber $U_{\beta_2}$. Assume that the trivializations of chambers $U_{\beta_2}$ and $U_{\beta_3}$ are given by $\mathbb{N}_{\leq N+2}$ and $\mathbb{N}_{\leq N+4}$ respectively and the sheets coming from the fold singularities correspond to $N+1, N+2$ and $N+1, N+2, N+3, N+4$ respectively (see Figure 13). In this case the restrictions of the injections $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ and $\sigma_5$ to $\mathbb{N}_{\leq N}$ are elements of the symmetric group of $N$ elements. Using the sheet data for cusp singularities injections are given as follows:

\[
\begin{align*}
\sigma_2(N+1) &= N+3, & \sigma_2(N+2) &= N+4 \\
\sigma_3(N+1) &= N+1, & \sigma_3(N+2) &= N+2 \\
\sigma_5(N+1) &= N+1, & \sigma_5(N+2) &= N+4.
\end{align*}
\]

In fact, injections $\sigma_2$ and $\sigma_3$ are determined by $\sigma_4$ and $\sigma_1$. Swallowtail singularity with other indices follows from this arguments and the corresponding cusp singularity data.

A $G$-sheet data $\mathcal{S}^G$ for a tuple $(\Delta, \Gamma)$ consists of a sheet data $\mathcal{S}$ and we require trivializations of chambers to lift directed $G$-labeled arcs and points to sheets coming from the spatial $G$-data. Permutations and injections are also required to preserve sheets with directed labeled arcs.

**Definition 2.15.** A $G$-spatial diagram is a triple $(\Delta^G, \Gamma, \mathcal{S}^G)$ consisting of a 3-dimensional $G$-graphic $\Delta^G$, a chambering foam $\Gamma$ for $\Delta^G$ subordinate to a $\Delta$-compatible cover $\mathcal{O} = \{O_{\alpha}\}_{\alpha \in J}$ of $\mathbb{R}^2 \times I$ and a $G$-sheet data $\mathcal{S}^G$ associated to $(\Delta^G, \Gamma)$.

**Proposition 2.6.** Let $(\Phi^G_1, \Gamma_1, \mathcal{S}^G_1)$ and $(\Phi^G_2, \Gamma_2, \mathcal{S}^G_2)$ be $G$-planar diagrams and let $(\Sigma_1, P_1)$ and $(\Sigma_2, P_2)$ be the constructed closed $X$-cobordisms respectively. Then $(\Sigma_1, P_1)$ is $X$-homeomorphic to...
A version of injections and permutations to preserve orientations. This defines oriented types. If Remark.

Now assume that \((\Sigma_2, P_2)\) if and only if there exists a \(G\)-spatial diagram \((\Delta^G, \Gamma, S^G)\) which restricts to \((\Phi^G_1, \Gamma_1, S^G_1)\) and \((\Phi^G_2, \Gamma_2, S^G_2)\) on boundary components.

\[
\{(\Sigma_1, P_1) \to (\Sigma_2, P_2)\text{ is an }X\text{-homeomorphism and } f_i : \Sigma_i \to \mathbb{R}^2 \text{ are generic maps for } i = 1, 2. \text{ Then there exists a generic map } F : \Sigma_1 \times I \to \mathbb{R}^2 \times I \text{ which restricts to generic maps } f_1 \text{ and } f_2 \circ F \text{ on boundary components. Choosing straight arcs (Proposition 2.5) gives a 3-dimensional } G\text{-graphic } \Delta^G \text{ which restricts to } \Phi^G_1 \text{ and } \Phi^G_2 \text{ on } \mathbb{R}^2 \times \{0, 1\}. \text{ Lemma 2.1 states that there exists a chambering foam } \Gamma \text{ restricting to } \Gamma_1 \text{ and } \Gamma_2. \text{ The generic map } F \text{ induces a } G\text{-sheet data } S^G \text{ giving a } G\text{-spatial diagram } (\Delta^G, \Gamma, S^G).
\]

Now assume that \((\Delta^G, \Gamma, S^G)\) is a \(G\)-spatial diagram restricting to \((\Phi^G_1, \Gamma_1, S^G_1)\) and \((\Phi^G_2, \Gamma_2, S^G_2)\) on boundary components. Then the boundary components of the constructed manifold are clearly diffeomorphic and an \(X\)-homeomorphism is defined using the lifts of the spatial \(G\)-data.

We define a relation among \(G\)-planar diagrams by \((\Phi^G_1, \Gamma_1, S^G_1) \sim (\Phi^G_2, \Gamma_2, S^G_2)\) if there exists a \(G\)-spatial diagram \((\Delta^G, \Gamma, S^G)\) restricting to the given \(G\)-planar diagrams on its boundary components. This is an equivalence relation by Proposition 2.6.

**Theorem 2.1 (\(G\)-planar decomposition theorem).** The \(X\)-homeomorphism classes of 2-dimensional closed \(X\)-cobordisms are in bijection with the equivalence classes of \(G\)-planar diagrams.

Proof. Follows from Proposition 2.6.

For general \(X\)-cobordisms and cobordism type \((2)\)-\(X\)-surfaces we need to extend definitions of 3-dimensional \(G\)-graphic, chambering foam and \(G\)-sheet data. We briefly describe these definitions for the cobordism type \((2)\)-\(X\)-surfaces. Let \((S_1, R_1, P_1)\) and \((S_2, R_2, P_2)\) be cobordism type \((2)\)-\(X\)-surfaces which are \(X\)-homeomorphic relative to their boundary. We consider \(S_1 \times I\) whose \(S_1 \times \{1\}\) boundary is identified with \(S_2\) by the \(X\)-homeomorphism. Generic maps restrict to generic map considered in this section on \(S \times (0, 1)\) and they are required to restrict to generic maps for \((2)\)-\(X\)-surfaces on boundary components. By the relative Thom transversality theorem (\([GG]\)) generic maps are dense in

\[
C^\infty((S \times I, \partial_\mu S \times I, \partial_\nu S \times I), (I_{mn} \times I^2, I_{mn} \times \{0, 1\} \times I, \{m, n\} \times I^2)).
\]

A 3-dimensional \(G\)-graphic \(\Phi^G\) is modified as follows: the restriction of \(\Phi^G\) to \(I_{mn} \times I \times \{0, 1\}\) gives 2-dimensional \(G\)-graphics, surfaces in \(\delta\) have transversal intersections with \(I_{mn} \times \{0, 1\} \times I\), \(\mu\) is disjoint from \(\partial(I_{mn} \times I^2)\) and elements of \(\delta, \eta, \text{ and } \mu\) are disjoint from \(\partial I_{mn} \times I^2\).

Chambering foam \(\Gamma\) is generalized as follows: its restriction to \(I_{mn} \times I \times \{0, 1\}\) gives chambering graphs for \((2)\)-\(X\)-surfaces, \(\Gamma\) has transversal intersections with \(I_{mn} \times \{0, 1\} \times I\), \(\Gamma\) is disjoint from \(\partial I_{mn} \times I^2\), and vertices of \(\Gamma\) is disjoint from \(I_{mn} \times I^2\). A \(G\)-sheet data has additional trivializations of chambers with boundary and additional injections and permutations coming from \(G\)-sheet data of \((2)\)-\(X\)-surface.

These additional modifications and conditions allow us to define \(G\)-spatial diagrams for cobordism type \((2)\)-\(X\)-surfaces and previous results in this section extend to such \(X\)-surfaces. In particular, equivalence classes of \(G\)-planar diagrams for cobordism type \((2)\)-\(X\)-surfaces can be defined and there is a bijection between the set of \(X\)-homeomorphism classes (relative to boundary) of cobordism type \((2)\)-\(X\)-surfaces and equivalence classes of \(G\)-planar diagrams for cobordism type \((2)\)-\(X\)-surfaces.

**Remark.** If \((\Sigma \times I, P)\) is an oriented \(X\)-cylinder of (closed, compact, cobordism type) \(X\)-cobordisms then an oriented \(G\)-sheet data is defined by assigning compatible orientations to sheets and requiring injections and permutations to preserve orientations. This defines oriented \(G\)-spatial diagrams and a version of \(G\)-planar decomposition theorem for oriented surfaces and oriented diagrams holds true.

3. Extended homotopy field theories
3.1. \textit{G-equivariant bordism bicategories}. In this section we define extended oriented and unoriented \(G\)-equivariant bordism bicategories using halations introduced by Schommer-Pries in \cite{Sc}. In particular, we define \(X\)-halations using pro-\(X\)-manifold category defined as follows.

**Definition 3.1.** Let \(\text{Man}^{X}\) be the category of smooth \(X\)-manifolds and smooth pointed maps commuting with characteristic maps and let \(I_{1}, I_{2}\) be small cofiltered categories. Then objects of the category pro-\(\text{Man}^{X}\) are functors \(\mathcal{F}_{1} : I_{1} \rightarrow \text{Man}^{X}, \mathcal{F}_{2} : I_{2} \rightarrow \text{Man}^{X}\) called \textit{pro-\(X\)-manifolds} and morphisms are given by

\[
\text{Hom}_{\text{pro-Man}^{X}}(\mathcal{F}_{1}, \mathcal{F}_{2}) = \lim_{\rightarrow} \text{colim}_{\leftarrow} q \text{Hom}_{\text{Man}^{X}}(\mathcal{F}_{1}(q), \mathcal{F}_{2}(p))
\]

where limit and colimit are taken in sets.

Let \((M, g)\) and \((N, h)\) be \(X\)-manifolds possibly with boundary or corners, and let \(\iota : (M, g) \hookrightarrow (N, h)\) be an embedding of \(X\)-manifolds i.e. \(g = h \circ [\iota]\) as elements of \([\{M, m\}, \{X, x\}]\). Consider the following cofiltered directed set consisting of codimension zero \(X\)-submanifolds of \(N\)

\[
I_M = \{(Z, h|_Z) \subset (N, h) \mid \iota(M) \subset Z, \partial Z = \emptyset\}
\]

and let \((\hat{M}, \hat{g}) = (\hat{M} \subset N, \hat{h}|_{\hat{M}})\) be the corresponding pro-\(X\)-manifold for this directed set. An \(X\)-manifold \((M, g)\) is a pro-\(X\)-manifold by the constant directed set. The natural inclusion map between pro-\(X\)-manifolds \((M, g) \hookrightarrow (\hat{M}, \hat{g})\) is called an \textit{\(X\)-halation} and denoted by a triple \((M, \hat{M}, \hat{g})\). An \(X\)-manifold with an \(X\)-halation is called an \textit{\(X\)-haloed manifold}. A map between \(X\)-haloed manifolds \((A, \hat{A}, \hat{g})\) and \((B, \hat{B}, \hat{h})\) is a commutative square of pro-\(X\)-manifolds morphisms.

\[
\begin{array}{c}
(A, g) \longrightarrow (B, h) \\
\downarrow \quad \downarrow \\
(\hat{A}, \hat{g}) \longrightarrow (\hat{B}, \hat{h})
\end{array}
\]

An \(X\)-halation \((M, \hat{M}, \hat{g})\) is said to be \textit{codimension \(k\)} if \(\iota(M) \subset N\) is a codimension \(k\) submanifold. Since \(X\)-halations are defined for \(X\)-manifolds we omit \(X\) and use manifold for brevity.

Let \((S, R, P)\) be a \((2)\)-surface and \(p : (E, \hat{P}) \rightarrow (S, P)\) be a vector bundle with \(\hat{P}|_{s_{0}(S)} = g\) where \(s_{0}\) is the zero section. A choice of a collar neighborhood (see \cite{La1} for existence) and the directed set \(I_{S}\) with the embedding \(\iota = s_{0}\) gives an \(X\)-halation denoted by \((S, \hat{S}_{E})\). Different choices of collar neighborhoods give (noncanonically) isomorphic \(X\)-halations and all \(X\)-halations are isomorphic to the one of the form \((S, \hat{S}_{E})\) (see \cite{Sc}). A codimension one \(X\)-halation on \(\partial_{h}S\) and \(\partial_{e}S\) or a codimension two \(X\)-halation on \(\partial_{h}S \cap \partial_{e}S\) are restrictions of a codimension zero \(X\)-halation on \(S\) if the corresponding vector bundles are trivial. An isotopy class of a vector bundle trivialization giving an \(X\)-halation is called a \textit{co-orientation}. An \(X\)-halation is called \textit{co-oriented} if it is equipped with such a choice.

\[
\begin{array}{c}
\cdot \\
(N, h) \\
(\hat{N}_{1}, \hat{h}_{1}) \\
(\hat{N}_{2}, \hat{h}_{2}) \\
(A, \hat{A}_{0}, \hat{A}_{1}, T, \hat{p}_{1}) \\
(\hat{M}_{2}, \hat{g}_{2})
\end{array}
\]

**Figure 14.** Co-oriented \(X\)-halations and an \(X\)-haloed 1-bordism

Let \((M, g)\) be a 0-manifold with a pair of co-oriented \(X\)-halations with inclusions \((M, g) \hookrightarrow (\hat{M}_{1}, \hat{g}_{1}) \hookrightarrow (\hat{M}_{2}, \hat{g}_{2})\) where \((M, \hat{M}_{1}, \hat{g}_{1})\) is a codimension one \(X\)-halation and \((M, \hat{M}_{2}, \hat{g}_{2})\) is a codimension two \(X\)-halation. We denote such a pair of co-oriented \(X\)-halations with inclusions by a
quadruple \((M, \hat{M}_1, \hat{M}_2, \hat{g}_2)\). Similarly, let \((N, \hat{N}_1, \hat{N}_2, \hat{h}_2)\) be another such quadruple for a 0-manifold \((N, h)\). A pointed 1-bordism between \((M, g)\) and \((N, h)\) is a 1-dimensional compact manifold \((A, T, p)\) such that \(T\) contains at least two points from each connected component of \(A\). Then an \(X\)-haloed 1-bordism from \((M, \hat{M}_1, \hat{M}_2, \hat{g}_2)\) to \((N, \hat{N}_1, \hat{N}_2, \hat{h}_2)\) is a pointed 1-bordism \((A, T, p)\) with a codimension zero \(X\)-halation \((A, \hat{A}_0, \hat{p}_0)\) and a co-oriented codimension one \(X\)-halation \((A, \hat{A}_1, \hat{p}_1)\) along with a decomposition of the boundary of \((A, T, p)\) as

\[
\partial A = \partial_{\text{in}} A \sqcup \partial_{\text{out}} A
\]

\[
(M, \hat{M}_1, \hat{M}_2, \hat{g}_2) \xrightarrow{\mu} (\partial_{\text{in}} A, \hat{A}_0|_{\partial_{\text{in}} A}, \hat{A}_1|_{\partial_{\text{in}} A}, \hat{p}_1)
\]

\[
(N, \hat{N}_1, \hat{N}_2, \hat{h}_2) \xrightarrow{\nu} (\partial_{\text{out}} A, \hat{A}_0|_{\partial_{\text{out}} A}, \hat{A}_1|_{\partial_{\text{out}} A}, \hat{p}_1)
\]

where \(\mu\) and \(\nu\) are isomorphisms of \(X\)-halations preserving co-orientations (see Figure 14). Here \(\hat{A}_0|_{\partial_{\text{in}} A}\) is co-oriented by an inward pointing normal vector and \(\hat{A}_0|_{\partial_{\text{out}} A}\) is co-oriented by an outward pointing normal vector. We denote such an \(X\)-haloed 1-bordism by quintuple \((A, \hat{A}_0, \hat{A}_1, T, \hat{p}_1)\). Let \((A, \hat{A}_0, \hat{A}_1, T, \hat{p}_1)\) and \((B, \hat{B}_0, \hat{B}_1, Q, \hat{q}_1)\) be \(X\)-haloed 1-bordisms from \((M, \hat{M}_1, \hat{M}_2, \hat{g}_2)\) to \((N, \hat{N}_1, \hat{N}_2, \hat{h}_2)\). An \(X\)-haloed 2-bordism from \((A, \hat{A}_0, \hat{A}_1, T, \hat{p}_1)\) to \((B, \hat{B}_0, \hat{B}_1, Q, \hat{q}_1)\) is a cobordism type \(2\)-manifold \((S, R, F)\) with a codimension zero \(X\)-halation \((S, \hat{S}, \hat{F})\) along with a decomposition \(\partial S = \partial_h S \sqcup \partial_v S\) and isomorphisms of \(X\)-halations (see Figure 15)

\[
(A, \hat{A}_1, \hat{p}_1) \sqcup (B, \hat{B}_1, \hat{q}_1) \xrightarrow{\theta} (\partial_h S, \hat{S}|_{\partial_h S}, \hat{f}|_{\partial_h S})
\]

\[
(M \times I, M \times \mathbb{R}^2, \hat{g}) \sqcup (N \times I, N \times \mathbb{R}^2, \hat{h}) \xrightarrow{\eta} (\partial_v S, \hat{S}|_{\partial_v S}, \hat{f}|_{\partial_v S})
\]

where \((A, \hat{A}_1, \hat{p})\) is co-oriented by an inward pointing normal vector and \((B, \hat{B}_1, \hat{q})\) is co-oriented by an outward pointing normal vector. The \(X\)-halations of \(M \times I\) and \(N \times I\) are induced by their embeddings into \(M \times \mathbb{R}^2\) and \(N \times \mathbb{R}^2\) with constant homotopy classes \(\hat{g}\) and \(\hat{h}\). Co-orientations are given by an inward pointing normal vector for \((M \times I, M \times \mathbb{R}^2, \hat{g})\) and an outward pointing normal vector for \((N \times I, N \times \mathbb{R}^2, \hat{h})\). Note that images of \(T\) and \(Q\) under \(\theta\) form the set \(R\). We denote such an \(X\)-haloed 2-bordism by quadruple \((S, \hat{S}, R, \hat{F})\).
Two $X$-haloed 2-bordisms $(S_0, \hat{S}_0, R_0, F_0), (S_1, \hat{S}_1, R_1, \hat{F}_1)$ are isomorphic if there is an isomorphism of $X$-halations $\xi : (S_0, \hat{S}_0, R_0) \to (S_1, \hat{S}_1, R_1)$ restricting isomorphisms of $X$-halations
\[
\begin{align*}
(\partial_{h, in} S_0, (\hat{S}_0)|_{\partial_{h, in} S_0}, (\hat{F}_0)|_{\partial_{h, in} S_0}) & \to (\partial_{h, in} S_1, (\hat{S}_1)|_{\partial_{h, in} S_1}, (\hat{F}_1)|_{\partial_{h, in} S_1}) \\
(\partial_{h, out} S_0, (\hat{S}_0)|_{\partial_{h, out} S_0}, (\hat{F}_0)|_{\partial_{h, out} S_0}) & \to (\partial_{h, out} S_1, (\hat{S}_1)|_{\partial_{h, out} S_1}, (\hat{F}_1)|_{\partial_{h, out} S_1}) \\
(\partial_{v, in} S_0, (\hat{S}_0)|_{\partial_{v, in} S_0}, (\hat{F}_0)|_{\partial_{v, in} S_0}) & \to (\partial_{v, in} S_1, (\hat{S}_1)|_{\partial_{v, in} S_1}, (\hat{F}_1)|_{\partial_{v, in} S_1}) \\
(\partial_{v, out} S_0, (\hat{S}_0)|_{\partial_{v, out} S_0}, (\hat{F}_0)|_{\partial_{v, out} S_0}) & \to (\partial_{v, out} S_1, (\hat{S}_1)|_{\partial_{v, out} S_1}, (\hat{F}_1)|_{\partial_{v, out} S_1})
\end{align*}
\]
where $\eta \circ \xi = \eta'$, $\theta \circ \xi = \theta'$

where $\theta'$ and $\eta'$ are isomorphisms of co-oriented $X$-halations corresponding to the decomposition $\partial S_1 = \partial_0 S_1 \cup \partial_1 S_1$.

**Definition 3.2.** The $G$-equivariant unoriented bordism bicategory $\text{XBord}_2^\text{un}$ has quadruples consisting of compact 0-manifolds equipped with two co-oriented $X$-halations as objects, $X$-haloed 1-bordisms as 1-morphisms and isomorphism classes of $X$-haloed 2-bordisms as 2-morphisms.

Composition of 1-morphisms in $\text{XBord}_2^\text{un}$ is defined as the pushout\(^1\) of $X$-haloed manifolds. To compose 2-morphisms one needs to choose collar neighborhoods and glue cobordisms. While composing 2-morphisms vertically restrictions of characteristic maps to boundaries are required to match. Note that while composing vertically to obtain a cobordism type (2)-surface we forget the points lying on the identified horizontal faces (see Definition 2.10). The $G$-equivariant oriented bordism bicategory $\text{XBord}_2$ is defined similarly using oriented manifolds equipped with oriented $X$-halations\(^2\).

**Lemma 3.1.** Both $\text{XBord}_2^\text{un}$ and $\text{XBord}_2$ are symmetric monoidal bicategories under disjoint union.

The proof repeats verbatim the proof for $\text{Bord}_2$ (see [Sc]) using M. Shulman’s method [Sh]. Additionally, we need to use constant characteristic maps to form companions of vertical 1-morphisms.

### 3.2. Presentations of the equivariant bordism bicategories

In this section we use the $G$-planar decomposition theorem for cobordism type $(2)$-$X$-surfaces to introduce symmetric monoidal bicategories $\text{XBord}_2^{PD,un}$ and $\text{XBord}_2^{PD,un}$. These bicategories are symmetric monoidally equivalent to $\text{XBord}_2^\text{un}$ and $\text{XBord}_2^{PD,un}$ is a freely generated\(^3\) symmetric monoidal bicategory (see [Sc], [Ps]).

**Definition 3.3.** An object of the $G$-equivariant bordism bicategory with diagrams $\text{XBord}_2^{PD,un}$ is a triple $((M, \hat{M}, \hat{M}_2, \hat{g}_2), \hat{M}, \omega)$ where $(M, \hat{M}, \hat{M}_2, \hat{g}_2)$ is an object of $\text{XBord}_2^\text{un}$, $\hat{M}$ is a finite set of points and $\omega : M \to \hat{M}$ is an isomorphism of sets.

A 1-morphism is a quadruple $((A, \hat{A}, \hat{A}_1, T, \hat{p}_1), \theta, L, \nu)$ where $(A, \hat{A}, \hat{A}_1, T, \hat{p}_1)$ is an $X$-haloed 1-bordism, $\theta : A \to I_{mn}$ is a Morse function with distinct critical values, $L = (\Psi^G, \Gamma, S^G)$ is a $G$-linear diagram whose graphic and sheet data are induced by $\theta$ and $\nu : (A, T, p) \to (\hat{A}, T, \hat{p})$ is an $X$-homeomorphism over $I_{mn}$ with $\nu(T) = T$ where $(\hat{A}, T, \hat{p})$ is the pointed 1-bordism constructed from the $G$-linear diagram.

A 2-morphism is a quadruple $([(S, \hat{S}, R, \hat{F})], \epsilon, P, \kappa)$ where $[(S, \hat{S}, R, \hat{F})]$ is an isomorphism class of $X$-haloed 2-bordism, $\epsilon : S \to I_{mn} \times I$ is a generic map for a representative $(S, \hat{S}, R, \hat{F})$, $P = [(\Phi^G, \Gamma, S^G)]$ is the corresponding equivalence class of cobordism type $G$-planar diagram and $\kappa : (S, R, F) \to (\hat{S}, \hat{R}, \hat{F})$ is an $X$-homeomorphism over $I_{mn} \times I$ with $\kappa(T) = \hat{T}$ where $(\hat{S}, \hat{R}, \hat{F})$ is a

---

1\(^{1}\)See Section 3.2.3 in [Sc] for the existence of colimits in the category of pro-manifolds.

2\(^{2}\)An $X$-halation $(M, g) \to (M, \hat{g})$ is oriented if both manifolds $M$ and $N$ are oriented and the embedding $\iota : (M, g) \to (N, h)$ is orientation preserving.

3\(^{3}\)Computadic symmetric monoidal bicategory in the sense of Schommer-Pries (see Section 2.10 in [Sc]).
cobordism type $\langle 2 \rangle$-$X$-manifold constructed from the representative $(\Phi^G, \Gamma, S^G)$ whose graphic and sheet data are induced by $\epsilon$.

The second bicategory $XB^{PD,un}$ is defined by forgetting $X$-haloed bordisms in $XB_{2}^{PD,un}$ and taking isotopy classes of $G$-linear diagrams. More concretely, its objects are finite set of points, 1-morphisms are isotopy classes of $G$-linear diagrams and 2-morphisms are equivalence classes of $G$-planar diagrams.

Horizontal compositions of $G$-linear and $G$-planar diagrams are given by the horizontal concatenation of diagrams where both $G$-sheet data agree and form a new $G$-sheet data. Since 1-morphisms are isotopy classes of $G$-linear diagrams horizontal composition of 1-morphisms is strictly associative which makes $XB^{PD}$ a strict 2-category. Vertical composition of equivalence classes of $G$-planar diagrams is vertical concatenation of diagrams followed by an isomorphism $I \cup pt I \cong I$ and forgetting the $G$-linear diagram on the face along which two $G$-planar diagrams are concatenated.

Symmetric monoidal structure on $XB^{PD,un}$ is defined as follows. Let $P_1$ and $P_2$ be two $G$-planar diagrams on $I_{un} \times I$ and on $I_{ab} \times I$ respectively and let $V_{\text{left}}$ be the leftmost chamber of $P_1$ and $V_{\text{right}}$ be the rightmost chamber of $P_2$. Then $P_1 \otimes P_2$ is defined by stretching $V_{\text{left}}$ to the left by $b - a$ units and stretching $V_{\text{right}}$ to the right by $n - m$ units and joining the stretched diagrams (see [Sc]). Symmetric monoidal structure on $G$-linear diagrams can be deduced from this description. It is not hard to see that symmetric monoidal structures of diagrams is compatible with the disjoint union of $X$-haloed manifolds.

**Remark.** Using the oriented $G$-planar decomposition theorem for cobordism type oriented $\langle 2 \rangle$-$X$-surfaces and oriented $X$-halations bicategories $XB_{2}^{PD}$ and $XB^{PD}$ are defined similarly. These bicategories are generalizations of the bicategories $Bord_{2}^{PD}$ and $B^{PD}$ defined by Schommer-Pries ([Sc]) to $X$-manifolds. Similarly, unoriented versions generalize $Bord_{2}^{PD,un}$ and $B^{PD,un}$.

**Lemma 3.2.** Bicategories $XB_{2}^{PD,un}$, $XB_{2}^{PD}$, $XB^{PD,un}$ and $XB^{PD}$ are symmetric monoidal bicategories.

The proof for $XB^{PD,un}$ and $XB^{PD}$ is very similar to the proof of Theorem 3.1. For $XB_{2}^{PD,un}$ and $XB_{2}^{PD}$, our claim follows by the compatibility of symmetric monoidal structures. Considering the results in $G$-planar decompositions section one can naturally ask whether symmetric monoidal bicategories defined by using diagrams are symmetric monoidally equivalent to $G$-equivariant bordism bicategories. Using the following theorem, due to Schommer-Pries [Sc], we give a positive answer without using symmetric monoidal transformations.

**Theorem 3.1** (Whitehead theorem for symmetric monoidal bicategories, [Sc]). Let $\mathcal{B}$ and $\mathcal{C}$ be symmetric monoidal bicategories. A symmetric monoidal 2-functor $F : \mathcal{B} \to \mathcal{C}$ is a symmetric monoidal equivalence if and only if it is an equivalence of underlying bicategories, that is $F$ is essentially surjective on objects, essentially full on 1-morphisms and fully-faithful on 2-morphisms.

**Proposition 3.1.** The forgetful 2-functors $F^{un}, G$ and $G^{un}$ given by forgetting $X$-haloed bordisms and diagrams respectively

\[
XB^{PD} \xrightarrow{F} XB_{2}^{PD} \xrightarrow{G} XB_{2} \quad \xrightarrow{\sim} \quad XB_{2}^{PD,un} \xleftarrow{\sim} XB_{2}^{PD,un} \xrightarrow{G^{un}} XB_{2}^{un}
\]

are symmetric monoidal equivalences.

**Proof.** For any given finite set $W$ of points or a compact 0-manifold with co-oriented codimension two $X$-halation $(Y, Y_0, Y_1, \hat{g})$ there exist objects in $XB_{2}^{PD,un}$ whose images under $F^{un}$ and $G^{un}$ are isomorphic to $W$ and $(Y, Y_0, Y_1, \hat{g})$ respectively. Since for any given $X$-haloed 1-bordism there exists a Morse function with distinct critical values leading to a $G$-linear diagram and any $G$-linear diagram
produces an $X$-haloed 1-bordism. Thus, by Proposition 2.1 each homomorphism is (essentially) full on 1-morphisms. Lastly, by the $G$-planar decomposition theorem for cobordism type $(2)$-$X$-surfaces 2-functors are fully-faithful on 2-morphisms. Oriented case follows in the same way.

Freely generated symmetric monoidal bicategories were introduced by Schommer-Pries [Sc]. They arise from presentations which are also called symmetric monoidal 3-computads. A presentation $P = (G_0, G_1, G_2, R)$ consists of four sets namely generating objects $G_0$, generating 1-morphisms $G_1$, generating 2-morphisms $G_2$ and relations $R$ among 2-morphisms. For a given presentation $P = (G_0, G_1, G_2, R)$ there exists a corresponding free symmetric monoidal bicategory $F(P)$ generated by the $P$ (see Section 2.10 in [Sc] or Appendix A in [Ps] for details). Roughly, objects of $F(P)$ are words in elements of $G_0$, 1-morphisms are either of the form $\beta \sigma : a \to b$ where $\sigma$ is a permutation relating $a$ and $b$, or of the form $\text{id}_a \otimes f \otimes \text{id}_b$ for $f \in G_1$, 2-morphisms are equivalence classes of string diagrams which consist of labeled points, strings and regions (see Figure 16).

Schommer-Pries [Sc] proved that the symmetric monoidal bicategories $B^{PD}, B^{PD, un}$ are freely generated unbiased semistrict symmetric monoidal 2-categories. The key idea of his proof is the correspondence between movie moves of planar diagrams and graphical calculus of string diagrams for unbiased semistrict symmetric monoidal 2-categories (Proposition 3.49 in [Sc]). Since in the definition of $G$-planar diagrams and $G$-spatial diagrams additional $X$-manifold data is not used to define chambers there is still underlying correspondence between movie moves and graphical calculus. Thus, $XB^{PD}$ and $XB^{PD, un}$ are freely generated unbiased semistrict symmetric monoidal 2-categories as well.

We briefly explain the correspondence between planar diagrams and string diagrams for unbiased semistrict symmetric monoidal 2-categories on an example (see [Sc] for details). On the left hand side of Figure 16 we have a 2-morphism in $\text{Bord}_2$ and on the right hand side we have a diagram which can be interpreted both as a planar diagram and a string diagram as follows. Omitting $\beta$-labels leads to a planar diagram whose sheet data is shown on the left hand side. Here labels $F_1, F_2, S_1$ are abbreviations of Fold-1, Fold-2 and Saddle-1 respectively.

As a string diagram regions of Figure 16 are complements of points and red and turquoise line segments. Each region is labeled with objects. Strings of Figure 16 have two forms; red strings are labeled by elements of $G_1$ and turquoise strings are labeled by $\beta^\sigma$ for a permutation $\sigma$. There are three types of points in Figure 16 for different intersections of strings; turquoise points are labeled by $\beta^{\sigma, \sigma'}$, where $\sigma, \sigma'$ are permutations of connecting turquoise strings, red points are labeled by elements of $G_2$, and yellow points are labeled by $\beta^x_2$ where $x \in G_1$.

---

$^{14}$Halation can be encoded into a $G$-sheet data by equipping trivializations of chambers with halations.

$^{15}$Regions with no label are considered as labeled with empty word.

$^{16}$We consider $F_1$-label as cup $\in G_1$ and $F_2$-label as cap $\in G_1$. Note that fold singularities are paths of cup and cap singularities.
Here we do not explain structure morphisms $\beta^g\alpha^r$, $\beta^g$ and do not mention turquoise points for univalent vertices. Reader is referred to [Sc] for the definition of unbiased semistrict symmetric monoidal 2-category and equivalence classes of string diagrams given by local moves which correspond to movie-moves of planar diagrams.

Notational remark: In Figures 17, 18, 23 and 24 each element labeled with $g$ or $g, g'$ is indexed over all $g, g' \in G$. In Figures 17 and 18 signs on the points of oriented (2)-surfaces are omitted.

**Theorem 3.2.** The symmetric monoidal bicategories $XB^{PD}$ and $XB^{PD, un}$ are freely generated unbiased symmetric monoidal 2-categories. The presentation $XP = (XG_0, XG_1, XG_2, XR)$ for $XB^{PD}$ is given in Figures 17 and 18. The presentation $XP^{un} = (X^{un}G_0, X^{un}G_1, X^{un}G_2, X^{un}R)$ for $XB^{PD, un}$ has one generating object $\{\bullet\}$, collection of generating 1-morphisms $\{\cdots, \cdots, \cdots\}_g \in G$, collection of generating 2-morphisms given in Figure 14, 17 and 23 and collection of relations in Figures 18 and 24.

*Proof.* Linear and planar $G$-data turn presentations of $B^{PD}$ and $B^{PD, un}$ given by Schommer-Pries in [Sc] into $G$-labeled versions. They also lead to new generating 1-morphisms and 2-morphisms (last row in Figure 17). New relations among these generators are obtained by the property of characteristic map and diffeomorphism classes of $\langle(2)\rangle$-X-manifolds relative to their boundary. ■

Remark. The list of generating 2-morphisms given in Figure 17 is definitely not minimal. Reader is asked to observe that any generating 2-morphism in the first two rows with $g, g \in G - \{e\}$ labelings can be obtained from composing c-labeled version with a sequence of generating 2-morphisms in the last row (see Figure 21 for the last two generating 2-morphisms in the first row). Similarly, there are obvious relations between these generating 2-morphisms and such compositions (see Figure 21).

3.3. Classification of oriented 2-dimensional E-HFTs. In this section we define and classify 2-dimensional E-HFTs using the cofibrancy theorem of Schommer-Pries [Sc].

**Definition 3.4.** Let $\mathcal{C}$ be a symmetric monoidal bicategory. A $\mathcal{C}$-valued oriented 2-dimensional extended homotopy field theory (E-HFT) with target $X$ is a symmetric monoidal 2-functor from the $G$-equivariant oriented bordism bicategory $XB\text{Bord}_2$ to $\mathcal{C}$.

Definition of symmetric monoidal 2-functor is lengthy (see Definition 2.5 in [Sc]) and it is not needed for the classification of E-HFTs. The key tool for the classification is the cofibrancy property that $XB^{PD}$ satisfy. Schommer-Pries [Sc] proved a coherence theorem for symmetric monoidal 2-functors called the Cofibrancy theorem which states that the bicategory $\text{SymBicat}(F(P), \mathcal{C})$ of symmetric monoidal 2-functors, transformations and modificaitons is equivalent to the bicategory of $P$-data $P(\mathcal{C})$ in $\mathcal{C}$ described below. In particular, a symmetric monoidal 2-functor out of a freely generated symmetric monoidal bicategory $F(P)$ is equivalent to a strict 2-functor.

Let $\mathcal{C}$ be a symmetric monoidal bicategory with a collection of objects $\mathcal{C}_0$, 1-morphisms $\mathcal{C}_1$ and 2-morphisms $\mathcal{C}_2$. Let $F(P)$ be a freely generated symmetric monoidal bicategory with a presentation $P = (G_0, G_1, G_2, R)$. We briefly recall the bicategory of symmetric monoidal $P$-data $P(\mathcal{C})$ in $\mathcal{C}$. A complete definition of $P(\mathcal{C})$ can be found in [Sc] (Section 2.11) or in [Ps] (Appendix A). An object $A$ of $P(\mathcal{C})$ is a collection of objects $A_0(G_0)$, 1-morphisms $A_1(G_1)$ and 2-morphisms $A_2(G_2)$ in $\mathcal{C}$ given by assignments $A_i : G_i \to C_i$ for $i = 0, 1, 2$ such that $A_1$ and $A_2$ are invariant under source and target maps (globular) and the assignment $A_2$ is subject to relations in $R$.

A 1-morphism $\alpha : A \to B$ in $P(\mathcal{C})$ is a collection 1-morphisms $\alpha_0(G_0)$ and 2-morphisms $\alpha_1(G_1)$ given by assignments $\alpha_i : G_i \to C_{i+1}$ for $i = 0, 1$ such that $\alpha_0(a) : A_0(a) \to B_0(a)$ for every $a \in G_0$ and $\alpha_1(f) : \alpha_0(a) \to \alpha_0(b)$ for every $f : a \to b$ in $G_1$. These assignments are required to be natural with respect to generating 2-morphisms i.e. for every $\xi : f_1 \to f_2$ in $G_2$ vertical compositions $A_2(\xi) * \alpha_1(f_1)$ and $\alpha_1(f_2) * B_2(\xi)$ are equal.  

\footnote{Corresponding points, G-linear and G-planar diagrams.}
A 2-morphism $\theta$ of $P(C)$ from $\alpha^1$ to $\alpha^2$ is a collection of 2-morphisms $\theta_0(\mathcal{G}_0)$ given by an assignment $\theta_0 : \mathcal{G}_0 \rightarrow \mathcal{C}_2$ such that $\theta_0(a) : \alpha^1_0(\mathcal{G}_0) \rightarrow \alpha^2_0(\mathcal{G}_0)$ for every $a \in \mathcal{G}_0$ and $\theta_0$ is natural with respect to generating 1-morphisms i.e. for every $f : a \rightarrow b$ in $\mathcal{G}_1$ horizontal compositions $\alpha^1_1(f) \circ \theta_0(a)$ and $\theta_0(b) \circ \alpha^2_1(f)$ are equal.

The bicategory $E\mathcal{HFT}(X, \mathcal{C}) = \text{SymBicat}(\text{XBord}_2, \mathcal{C})$ has oriented E-HFTs with target $X$ as objects, symmetric monoidal transformations as 1-morphisms, symmetric monoidal modifications (see [Sc]) as 2-morphisms.

**Theorem 3.3** (Cofibrancy Theorem, [Sc]). Let $F(P)$ be a freely generated symmetric monoidal bicategory with a presentation $P$ and let $\mathcal{C}$ be a symmetric monoidal bicategory. Then there is an equivalence of bicategories $\text{SymBicat}(F(P), \mathcal{C}) \simeq P(\mathcal{C})$.

The following theorem gives a classification of oriented 2-dimensional E-HFTs.
Theorem 3.4. Let $\mathcal{C}$ be a symmetric monoidal bicategory and $XP$ be the presentation of $XB^{PD}$ given in Figures 17 and 18 then there is an equivalence of bicategories

$$\mathcal{E}\cdot\mathcal{HFT}(X, \mathcal{C}) \simeq XP(\mathcal{C}).$$

Proof. Proposition 3.1 states the equivalence $XB^{PD} \simeq XBord_2$ and Theorem 3.2 gives a presentation $XP$ of $XB^{PD}$ as a freely generated symmetric monoidal bicategory. By the cofibrancy theorem symmetric monoidal 2-functors out of $XB^{PD}$ are determined up to equivalence by the $XP$-data in $\mathcal{C}$ and precomposition with the equivalence $XB^{PD} \simeq XBord_2$ gives the desired equivalence.
3.3.1. Classification of $\text{Alg}_k^2$-valued oriented 2-dimensional E-HFTs. Every oriented 2-dimensional E-HFT gives a nonextended one by restricting to oriented X-circles and X-cobordisms between them. A natural question is how the classification of oriented 2-dimensional E-HFTs is related to Turaev’s classification ([Tu2]) of 2-dimensional HFTs by crossed Frobenius $G$-algebras. To understand this relation we study oriented E-HFTs taking values in $\text{Alg}_k^2$ which has $k$-algebras as objects, bimodules as 1-morphisms and bimodule maps as 2-morphisms for a commutative ring $k$.

The symmetric monoidal structure of $\text{Alg}_k^2$ is given by tensoring over $k$. We denote $(C, E)$-bimodule $D$ with $CD_E$ and omit the symbol $k$ when either $C$ or $E$ is $k$. We regard $CD_E$ as a 1-morphism from $E$ to $C$ which is in line with composition in $X\text{Bord}_2$ (see Figure [15]). Composition of 1-morphisms $CD_E$ and $AB_C$ is the tensor product $A(B \otimes_C D)_E$.

Before studying $\text{Alg}_k^2$-valued oriented 2-dimensional E-HFTs we recall necessary algebraic notions and introduce quasi-biangular $G$-algebras. Recall that a $G$-algebra over a ring $k$ is an associative algebra $K$ over $k$ equipped with a decomposition $K = \oplus_{g \in G} K_g$ such that $K_gK_h \subset K_{gh}$ for any $g, h \in G$. In this case, $K_e$ is called principal component of $K$ where $e \in G$ is the identity element.

**Definition 3.5. ([Tu1])** Let $K = \oplus_{g \in G} K_g$ be a $G$-algebra over a commutative ring $k$. An inner product on $K$ is a symmetric bilinear form $\eta : K \otimes K \to k$ satisfying $\eta(ab, c) = \eta(a, bc)$ for any $a, b, c \in K$ such that $\eta|_{K_g \otimes K_h}$ is nondegenerate when $gh = e$ and zero otherwise. A Frobenius $G$-algebra is a $G$-algebra $K$ with an inner product $\eta$ and components of $K$ are finitely generated projective $k$-modules.

Let $(K = \oplus_{g \in G} K_g, \eta)$ be a Frobenius $G$-algebra over $k$. Each nondegenerate form $\eta|_{K_g \otimes K_{g^{-1}}}$ yields an element $\eta_g^- = \sum_{i \in I_g} p_i^g \otimes q_i^{g^{-1}} \in K_g \otimes K_{g^{-1}}$, called an inner product element, where $I_g$ is finite and $\eta_g^-$ is characterized by $a = \sum_{i \in I_g} (a, q_i^{g^{-1}}) p_i^g$ for any $a \in K_g$. Since $\eta$ is symmetric we have $\sum_i p_i^g \otimes q_i^{g^{-1}} = \sum_i q_i^g \otimes p_i^g$ for all $g \in G$ and the equality

$$\eta(a, b) = \eta\left( a, \sum_i \eta(p_i^g, b) q_i^{g^{-1}} \right) = \sum_i \eta(a, q_i^g) \eta(p_i^g, b).$$

Recall that an associative $k$-algebra $A$ is separable if there exists an element $a = \sum_{i=1}^n p_i \otimes q_i \in A \otimes_k A^{op}$ called separability idempotent such that $\sum_{i=1}^n p_i q_i = 1$ and $ab = ba$ for all $b \in A$. A separable algebra $A$ is called strongly separable if the separability idempotent is symmetric i.e. $a = \sum_{i=1}^n p_i \otimes q_i = \sum_{i=1}^n q_i \otimes p_i$.

**Lemma 3.3. ([Tu4])** Let $(K = \oplus_{g \in G} K_g, \eta)$ be a Frobenius $G$-algebra with inner product elements $\{\eta_g^- = \sum_i p_i^g \otimes q_i^{g^{-1}}\}_{g \in G}$ and a central element $z \in K_e$ i.e. $az = za$ for all $a \in K$. Then for any $g, h \in G$ and $b \in K_{g^{-1}}$ we have

$$\sum_i p_i^h \otimes z q_i^{h^{-1}} b = \sum_j b p_j^{gh} \otimes z q_j^{gh}.$$  

In particular, for any $b \in K$ and $c \in K_{h^{-1}}$ we have $\sum_j p_j^h b z q_j^{h^{-1}} c = \sum_k c p_k^{h^{-1}} b z q_k^{h^{-1}}$.

**Proof.** Since both sides belong to $K_h \otimes K_{h^{-1}}$ it is enough to check that they give the same functionals on the dual $k$-module $K_{h^{-1}} \otimes K_{gh}$. For any $x \in K_{h^{-1}}$ and $y \in K_{gh}$ applying $x \otimes y$ to the left hand side of equation [1] and using cyclic symmetry property of $\eta$ we obtain

$$\sum_i \eta(p_i^h, x) \eta(z q_i^{h^{-1}} b, y) = \sum_i \eta(x, p_i^h) \eta(q_i^{h^{-1}} b y z, h) = \eta(x, \sum_i \eta(b y z, q_i^{h^{-1}} p_i^h)) = \eta(x, b y z).$$
Similarly applying $x \otimes y$ to the right hand side of the equation \([1]\) we have

$$
\sum_{j} \eta(bp_{j}^{gh}, x) \eta(zq_{j}^{gh}, y) = \sum_{j} \eta(xb, p_{j}^{gh}) \eta(q_{j}^{gh}, yz) = \eta(xb, \sum_{j} \eta(yz, q_{j}^{gh}) p_{j}^{gh}) = \eta(xb, yz).
$$

We now generalize biangular $G$-algebras which are introduced by Turaev \([Tu2]\) in his study of lattice HFTs.

**Definition 3.6.** A Frobenius $G$-algebra $(K, \eta)$ is called quasi-biangular if there exists a central element $z \in K_e$ such that for some collection of inner product elements $\{ \sum_{i} p_{i}^{g} \otimes q_{i}^{g} \}_{g \in G}$ equations $\sum_{i} z_{q_{i}^{g}} = 1$ hold for all $g \in G$ and each component $K_{g}$ is both left and right free of rank one $K_{e}$-module.

**Remark.** By the Lemma \([3.3]\) the principal component of a quasi-biangular $G$-algebra is a separable algebra with separability idempotent $\sum_{i} p_{i}^{g} \otimes z_{q_{i}^{g}}$. A biangular $G$-algebra is a quasi-biangular $G$-algebra with $z = 1 \in K_{e}$. Similarly, the principal component of a biangular $G$-algebra is a strongly separable algebra.

One way of studying an algebra is to study the category of modules over that algebra. Recall that Morita equivalence of algebras is the equivalence of categories of modules. In the case of a graded algebra it is natural to consider the category of graded modules. An equivalence of such categories is called a graded Morita equivalence which is introduced by P. Boisen \([Bo]\) as follows.

**Definition 3.7.** (\([Bo]\)) A $G$-graded Morita equivalence $\zeta$ between $G$-algebras $K = \oplus_{g \in G} K_{g}$ and $L = \oplus_{g \in G} L_{g}$ is a quadruple $(L M_{K}, K \otimes_{L} M_{K}, \tau, \mu)$ where $L M_{K} = \oplus_{g \in G} M_{g}$ is a graded $(L, K)$-bimodule that is $L_{g} M_{h} K_{g'} \subset M_{ghg'}$, $K_{N_{L}} = \oplus_{g \in G} N_{g}$ is a graded $(K, L)$-bimodule, and $\tau : K K_{K} \rightarrow K N \otimes_{L} M_{K}$ and $\mu : L M \otimes_{K} N_{L} \rightarrow L L_{L}$ are graded $(K, K)$ and $(L, L)$ bimodule maps respectively such that the following compositions

$L M_{K} \rightarrow L M \otimes_{K} K_{K} \xrightarrow{id \otimes \tau} L M \otimes_{K} (N \otimes_{L} M_{K}) \xrightarrow{(L M \otimes_{K} N) \otimes_{L} M_{K} \mu \otimes id} L L \otimes_{L} M_{K} \rightarrow L M_{K}$

$K N_{L} \rightarrow K K \otimes_{K} N_{L} \xrightarrow{\tau \otimes id} (K N \otimes_{L} M) \otimes_{K} N_{L} \rightarrow K N \otimes_{L} (M \otimes_{K} N_{L}) \xrightarrow{id \otimes \mu} K N \otimes_{L} L L \rightarrow K N_{L}$

are $id_{M}$ and $id_{N}$ respectively. When $\tau$ and $\epsilon$ are invertible as $G$-graded bimodule maps it is called a $G$-graded Morita context.

Let $\zeta = (L M_{K}, K \otimes_{L} M_{K}, \tau, \mu)$ be a $G$-graded Morita equivalence. For any subgroup $H$ of $G$ we have $H$-graded algebras $K_{H} = \oplus_{h \in H} K_{h}$, $L_{H} = \oplus_{h \in H} L_{h}$ and $H$-graded bimodules $N_{H} = \oplus_{h \in H} N_{h}$, $M_{h} = \oplus_{h \in H} M_{h}$. Then there is a natural $(L H, L H)$-bimodule map $M_{H} \otimes_{K H} N_{H} \rightarrow M_{H} \otimes_{K H} N_{H}$ given by $a \otimes_{K H} b \rightarrow a \otimes_{H} b$. Define $\mu_{H}$ as the composition of this map with $\mu$ and similarly define $\tau_{H}$. Then $\zeta_{H} = (M_{H}, N_{H}, \tau_{H}, \mu_{H})$ is an $H$-graded Morita context. It is not hard to see that for $G = \{ e \}$ the graded Morita context coincides with the ordinary Morita context.

**Definition 3.8.** Let $\zeta_{1} = (L M_{K}, K \otimes_{L} M_{K}, \tau, \mu)$ and $\zeta_{2} = (L X_{K}, K Y_{L}, \tau', \mu')$ be two $G$-graded Morita equivalences. An equivalence between $G$-graded Morita equivalences $\zeta_{1}$ and $\zeta_{2}$ is a $G$-graded bimodule maps $\xi : L M_{K} \rightarrow L X_{K}$ and $\rho : K N_{L} \rightarrow K Y_{L}$ such that $\mu = \mu' \circ (\xi \otimes \rho)$ and $\tau' = (\rho \otimes \xi) \circ \tau$.

A $G$-algebra $K = \oplus_{g \in G} K_{g}$ is called strongly graded if $K_{g} K_{h} = K_{gh}$ for all $g, h \in G$. Quasi-biangular algebras are strongly graded because each component is rank 1 module over the principal component. Haefner \([Ha]\) showed that the strongly graded property of a $G$-algebra is a graded Morita invariant. Now following \([Sc]\) we transfer the inner product of one Frobenius $G$-algebra to another using a graded Morita context between them.
Lemma 3.4. ([S]) Any Morita context \( \zeta = (LM_K, K N_L, \tau, \eta) \) between \( k \)-algebras \( K \) and \( L \) induces a canonical isomorphism of \( k \)-modules

\[
\zeta_\star: K/[K, K] \to L/[L, L].
\]

The inner product \( \eta \) of a Frobenius \( G \)-algebra \((K, \eta)\) is determined at its principal component by \( \eta(a, b - 1) = \eta(ab, 1) \). This allows us to denote \((K, \eta)\) by \((K, \Lambda)\) where \( \Lambda: K_e \to k \) is a nondegenerate trace. Since \( \eta \) is symmetric \( \Lambda \) factors through \( K_e/[K_e, K_e] \). Lemma 3.3 implies that for a symmetric Frobenius algebra \((K_e, \Lambda_e)\) an inner product element \( \sum_i p_i^e \otimes q_i^e \) can be considered as the image of \( 1 \otimes 1 \) under a bimodule map \( \xi: K_1^e(K_e) K_2^e \otimes K_3^e(K_e) K_4^e = K_1^e(K_e) K_2^e \otimes K_3^e(K_e) K_4^e \) where numbers indicate module actions i.e. \( K_i = K_e \) for \( i = 1, 2, 3, 4 \). In the case of a quasi-biangular \( G \)-algebra \((K = \oplus_{g \in G} K_g, \Lambda)\) inner product elements \( \{ \sum_i p_i^g \otimes q_i^g \}_{g \in G \setminus \{e\}} \) are the image of \( 1 \otimes 1 \) under the following composition

\[
(2) \quad K_1^e(K_e) K_2^e \otimes K_3^e(K_e) K_4^e = K_1^e(K_e) K_2^e \otimes K_3^e(K_g^{-1} \otimes K_e) K_4^e \to K_1^e(K_e) K_2^e \otimes K_3^e(K_g^{-1} \otimes K_e) K_4^e \to K_1^e(K_g) K_2^e \otimes K_3^e(K_g^{-1}) K_4^e
\]

where the second homomorphism is identity on \( K_g^{-1} \) and \( K_g \), and \( \xi \) on \( K_e \otimes K_e \). In the following we consider inner product elements as the images of \( 1 \otimes 1 \) under above bimodule maps.

Definition 3.9. Let \((K, \Lambda_K)\) and \((L, \Lambda_L)\) be quasi-biangular \( G \)-algebras over \( k \) with collection of inner product elements \( \{ \eta_g^K \}_{g \in G} \) and \( \{ \eta_g^L \}_{g \in G} \) respectively. A \( G \)-graded Morita context \( \zeta = (LM_K, K N_L, \tau, \mu) \) between \( K \) and \( L \) is said to be compatible if \( \Lambda_L = (\zeta(e), \Lambda_K \) and \( \eta_g^L = (\zeta(e))_* (\eta_g^K) \) for all \( g \in G \) where \((\zeta(e))_* (\eta_g^K) \) consists of inner product elements for \((L, (\zeta(e))_* \Lambda_K)\). Inner product element \((\zeta(e))_* (\eta_g^K) \) is given by \( \xi'(1 \otimes 1) \) under the commutative diagram

\[
\begin{array}{ccc}
L_e(M_e \otimes K_e \otimes N_e)_{L_e} \otimes_k L_e(M_e \otimes K_e \otimes N_e)_{L_e} & \xrightarrow{id \otimes \xi} & L_e(M_e \otimes K_e \otimes N_e)_{L_e} \\
\downarrow \mu(e) & & \downarrow \mu(e) \\
L_e(L_e)_{L_e} \otimes_k L_e(L_e)_{L_e} & \xrightarrow{} & L_e(L_e)_{L_e} \\
\end{array}
\]

and remaining inner product elements are obtained from \( \xi' \) as described above.

The following theorem gives a classification of \( \text{Alg}^2 \)-valued oriented E-HFTs with target \( X \).

Theorem 3.5. Any oriented 2-dimensional E-HFT determines a triple \((A, B, \zeta)\) where \( A \) and \( B \) are quasi-biangular \( G \)-algebras and \( \zeta \) is a compatible \( G \)-graded Morita context between \( A \) and the opposite algebra \( B^{op} \). Moreover, any such triple \((A, B, \zeta)\) is realized by an oriented 2-dimensional E-HFT.

Proof. Let \( Z : X \text{Bord}_2 \to \text{Alg}^2 \) be an oriented 2-dimensional E-HFT and \( Z' \) be an object of \( XP(\text{Alg}^2) \) corresponding to \( Z \) under the equivalence of bicategories. For each generating object of \( XP \) there is a \( k \)-algebra in \( Z'_0(X \mathcal{G}_0) \) as

\[
Z'(\emptyset) = A_e, \\
Z'(\star) = B_e.
\]

There are four types of generating 1-morphisms and each is indexed by elements of \( G \). For every \( g \in G \) they give following bimodules in \( Z'_1(X \mathcal{G}_1) \)

\[
\begin{align*}
Z'(\emptyset \to g) &= M_g \quad (B_e \otimes_k A_e, k)\text{-bimodule} \\
Z'(\emptyset \to g) &= N_g \quad (k, A_e \otimes_k B_e)\text{-bimodule} \\
Z'(\star \to g) &= A_g \quad (A_e, A_e)\text{-bimodule} \\
Z'(\star \to g) &= B_g \quad (B_e, B_e)\text{-bimodule}.
\end{align*}
\]
The first 2-morphism in Figure 19 defines a $G$-graded product on $(A_e, A_e)$-bimodules $\{A_g\}_{g \in G}$. Associativity of this product is the obvious relation in Figure 18. Denote the corresponding $G$-algebra by $A = \bigoplus_{g \in G} A_g$. The first relation in Figure 18 shows that the bimodule map

$$A_e(A_{g'}) \otimes A_e(A_g) \xrightarrow{\cong} A_e(A_{gg'})$$

is invertible for all $g, g' \in G$. In particular, choosing $g'$ as $g^{-1}$ in the morphism (3) gives that $A_g$ and $A_{g^{-1}}$ are rank one $(A_e, A_e)$-bimodules. Similar arguments for $(B_e, B_e)$-bimodules $\{B_g\}_{g \in G}$ yield another $G$-algebra $B = \bigoplus_{g \in G} B_g$ whose components are rank one $(B_e, B_e)$-bimodules.

A key observation is that the algebra action on bimodules can be turned around using the opposite algebra. More precisely, a left $B_e$ action on $A_e \otimes_{B_e} M_g$ can be turned into a right $B_e^{\text{op}}$ action on $A_e(M_g)B_e^{\text{op}}$ and similarly the right $B_e$ module action on $(N_g)_{A_e \otimes B_e}$ can be turned into a left $B_e^{\text{op}}$ action on $B_e^{\text{op}}(N_g)_{A_e}$. The second 2-morphism in Figure 19 can be obtained by composing generators and is invertible by the relations in the same figure. In $Z'_1(XG_2)$ this 2-morphism gives the bimodule map

$$B_e \otimes_{A_e} (B_g^{-1} \otimes_k A_{g'}) \otimes_{B_e} B_e \otimes_{A_e} M_{h} \xrightarrow{\cong} B_e \otimes_{A_e} M_{g^{-1}h'g}.$$ 

Turning $B_e$ actions on $B_g$ around gives $A_e(M_{g'g})B_e^{\text{op}}$ and collection of all such bimodule maps turns $(A_e(M_g)B_e^{\text{op}})_{g \in G}$ into a $G$-graded $(A, B_e^{\text{op}})$-bimodule $M = \bigoplus_{g \in G} M_g$. Similarly, reflections of this 2-morphism and relations show that $N = \bigoplus_{g \in G} N_g$ is a $G$-graded $(B_e^{\text{op}}, A)$-bimodule.

There are four types of cusp generators and each is index by two elements of $G$. For every $g, g' \in G$ they give following bimodule maps in $Z'_2(XG_2)$

$$f_{1}^{gg'}: A_e(A_{gg'})_A \rightarrow A_e(M_g \otimes B_e^{\text{op}}(N_g')_A),$$

$$f_{2}^{gg'}: B_e^{\text{op}}N_g \otimes_{A_e}(M_g)^{B_e^{\text{op}}} \rightarrow B_e^{\text{op}}(B_{gg})^{B_e^{\text{op}}},$$

$$f_{3}^{gg'}: B_e^{\text{op}}(B_{gg})^{B_e^{\text{op}}} \rightarrow B_e^{\text{op}}N_g \otimes_{A_e}(M_g)^{B_e^{\text{op}}},$$

$$f_{4}^{gg'}: A_e(M_g \otimes B_e^{\text{op}}(N_g')_A) \rightarrow A_e(A_{gg'})_A,$$

given in the order of cusp generators in Figure 17. These bimodule maps are required to satisfy relations $X'R$ among 2-morphisms. Relations containing cusp generators indicate that these bimodule maps are both sided inverses i.e. $f_{1}^{gg'} = (f_{3}^{gg'})^{-1}$ and $f_{2}^{gg'} = (f_{3}^{gg'})^{-1}$. It is not hard to see that for each $i$ the collection $\{f_{i}^{gg'}\}_{g, g' \in G}$ of bimodule maps form a $G$-graded bimodule map $f_i$. The collection of swallowtail morphisms corresponds to following compositions of graded bimodule maps

$$B^{\text{op}}N_A \rightarrow B^{\text{op}}N \otimes_A A_A \xrightarrow{id \otimes f_1} B^{\text{op}}N \otimes_A M \otimes B^{\text{op}}N_A \xrightarrow{f_{2} \otimes id} B^{\text{op}}B^{\text{op}} \otimes B^{\text{op}}N_A \rightarrow B^{\text{op}}N_A,$$

$$A_M B^{\text{op}} \rightarrow A_M A \otimes A_M B^{\text{op}} \xrightarrow{f_{1} \otimes id} A_M \otimes B^{\text{op}}N \otimes_A M \otimes B^{\text{op}} \xrightarrow{id \otimes f_2} A_M \otimes B^{\text{op}}B^{\text{op}}_B \rightarrow A_M B^{\text{op}}.$$ 

The opposite $G$-algebra of $B$ is defined as $B^{\text{op}} = \bigoplus_{g \in G} B_{g^{-1}}$ where the order of multiplication is reversed.
Swallowtail relations (last relation in Figure 18 and its reflection) imply that both are identity bimodule maps of $M$ and $N$ respectively. In other words, $f_1$ and $f_2$ are unit and counit of the the $G$-graded Morita context $\xi = (B^{op}N_A, A^{op}M_{B^{op}}, f_1, f_2)$ respectively.

The $G$-graded Morita context $\xi$ between $G$-algebras $A$ and $B^{op}$ can be used to replace $B^{op}$-module actions with $A$-module actions. More precisely, a bimodule with left (right) $B^{op}$ action can be turned into a left (right) $A$-module by tensoring with the bimodule $A^{op}M_{B^{op}} (B^{op}(N)_A)$. For example, tensoring $A^{op}M_{B^{op}}$ with $B^{op}(N)_A$ yields $A^{op}M \otimes B^{op} N_A$ which is isomorphic to $A^{op}A$ via $f_4$ and turning the right $A$-module action on $A^{op}A$ around, $A^{op}B^{op}M$ can be replaced by $A^{op}A^{op}A$. In the same way, $B^{op}N \otimes A^{op}M_{B^{op}}$ is isomorphic to $B^{op}B^{op}_{B^{op}}$ via $f_2$ and using $\xi$ bimodule $B^{op}B^{op}_{B^{op}}$ can be replaced by $A^{op}A$.

Remaining 2-generators are Morse generators consisting of saddles, cup and cap. The collection of bimodule maps in $Z^2(XG_2)$ for the first saddle morphisms in Figure 17 yields a graded bimodule map of the form

$$A^{op}B^{op}M \otimes_k N_{A^{op}B} \to A^{op}B(A \otimes_k B)A^{op}B$$

by turning the $B$-module actions around we obtain

$$A^{op}B^{op}(M \otimes_k N)_{A^{op}B^{op}} \to A^{op}B^{op}(A \otimes_k B)A^{op}B^{op}$$

where the left $B^{op}$-action is over $N$ and the right $B^{op}$-action is over $M$. As pointed out above $B^{op}$-module actions can be replaced by $A$-module actions and we get a graded $(A_1 \otimes A_3, A_2 \otimes A_4)$ bimodule map of the form

$$\xi : A_1^{op}A_2 \otimes_k A_3 A_4 \to A_1^{op}A_4 \otimes_k A_3 A_2$$

where numbers indicate module actions i.e. $A_i = A$ for $i = 1, 2, 3, 4$. The graded bimodule map $\xi$ is determined at $1 \otimes 1 \in A_e \otimes A_e$ which we denote by

$$\xi(1 \otimes 1) = \sum_i p_i^e \otimes q_i^e \in A_e \otimes_k A_e$$

where sum is finite and it satisfies $\sum a p_i^e \otimes q_i^e = \sum p_i^e \otimes q_i^e a$ for all $a \in A$. Similarly, we denote the image of $1 \otimes 1$ under the first 2-morphism in Figure 20 by $\eta_1^a = \sum_i p_i^e \otimes q_i^g a$ for all $g \in G$ (compare with equation 2).

![Figure 20. Morphism giving inner product elements and cusp flip relation](image)

In the same way the collection of bimodule maps in $Z^2(XG_2)$ for the second saddle morphism gives a graded $(A_1 \otimes A_3, A_2 \otimes A_4)$ bimodule map of the form

$$\eta : A_1^{op}A_2 \otimes_k A_3 A_4 \to A_1^{op}A_4 \otimes_k A_3 A_2^{op}.$$  

Cusp flip relation in Figure 2 implies that $\xi = \eta$. Note that cusp flip relation explains replacing algebra actions using the graded Morita context. For the cup and cap generators we describe
Thus, and Morse relation implies that it is equivalent to \( \text{id} \). Morse relations involving cup morphism indicates the nondegeneracy of \( k \). For any \( \eta \) the collection of \( g \) components cup morphism is given by multiplication followed by \( g \). \( g \) morphism in Figure 22 corresponds to the following composition

\[
\Lambda : \oplus_{g \in G} A_g \otimes A_e A_{g-1} \rightarrow k
\]

\[u : k \rightarrow \oplus_{g \in G} A_g \otimes A_e A_{g-1} \]

respectively. Figure 21 implies that cup and cap morphisms are determined on the principal component \( (g = e) \). Since \( A_e \otimes A_e A_{g-1} A_e = A_e /[A_e, A_e] \), cup morphism on principal component can be considered as symmetric linear map \( \Lambda : A_e \rightarrow k \). Additionally, Figure 21 shows that on nonprincipal components cup morphism is given by multiplication followed by \( \Lambda \) and it is symmetric. This defines \( k \)-bilinear map

\[
\eta_g : A_g \otimes A_{g-1} \rightarrow A_e \rightarrow A_e /[A_e, A_e] \rightarrow k.
\]

Morse relations involving cup morphism indicates the nondegeneracy of \( \eta_g \) as follows. Assuming \( \beta_g \) is a basis for \( A_g \) for any \( a \in A_g \) the first 2-morphism in Figure 22 corresponds to following compositions

\[
a \rightarrow 1 \otimes 1 \otimes a \rightarrow 1 \otimes (\beta_g \otimes 1 \otimes \beta_{g-1}) \otimes a \rightarrow \beta_g \otimes \left( \sum_i p_i^e \otimes q_i^g \right) \otimes \beta_{g-1} \otimes a \rightarrow \sum_i p_i^a \otimes q_i^g \otimes a \rightarrow \sum_i p_i^a \eta_i(q_i^g, a)
\]

and Morse relation implies that it is equivalent to \( \text{id} A_g \). Similarly, reflection of this morphism with \( g^{-1} \) label gives \( b = \sum_j \eta_j(b, p_i^g)q_i^g \) for any \( b \in A_{g^{-1}} \), which shows that \( \eta_g \) is nondegenerate. Thus, \( (A, \eta_A) \) is a Frobenius \( G \)-algebra where \( (\eta_A)|_{A_g \otimes A_{g-1}} = \eta_g \) for all \( g \in G \) and zero otherwise. Remaining Morse relations contain cap morphisms. As noted above cap morphisms are determined on the principal component. For any \( c \in A_c \), assuming \( u(1)|_{A_c \otimes A_c} = \sum_j a_j \otimes b_j \) the second 2-morphism in Figure 22 corresponds to the following composition

\[
c \otimes \sum_j a_j \otimes b_j \rightarrow \sum_{i,j} c p_i^c a_j^g \otimes b_j \rightarrow \sum_{i,j} c p_i^c b_j a_j^e \rightarrow c \sum_i p_i^e z q_i^c
\]

where \( z = \sum_j b_j a_j \in A_c \). Morse relation implies that \( \sum_i p_i^e z q_i^c = 1 \) and consequently \( \sum_i p_i^e \otimes z q_i^c \) is a separability idempotent of the algebra \( A_e \). Thus, \( (A_e, \eta_e) \) is a separable symmetric Frobenius algebra as shown in [5]. Similarly, we have \( \sum_i p_i^e z q_i^c = 1 \) using the saddle which gives \( \eta_g^A \). Until now we used \( \zeta \) to replace \( B^{op} \) actions by \( A \) actions. By changing the roles of \( A \) and \( B \) we obtain a quasi-biangular \( G \)-algebra \( B \) and in this case \( \zeta \) is clearly a compatible graded Morita context.

We showed that any oriented 2-dimensional E-HFT determines a triple \((A, B, \zeta)\) by analyzing the corresponding object in \( X \mathbb{P}(\mathbb{G}_2) \). For any such triple there exists an oriented 2-dimensional E-HFT by forming a strict 2-functor \( Z' : X \mathbb{B} \rightarrow \mathbb{G}_2 \) using the given triple and precomposing \( Z' \) with the equivalence \( X \mathbb{B} \mathcal{O}d_2 \xrightarrow{\sim} X \mathbb{P} \mathcal{O}d_2 \).
Definition 3.10. [[Tu2]] A Frobenius $G$-algebra $(K = \bigoplus_{g \in G} K_g, \eta)$ is crossed if $K$ is endowed with a group homomorphism $\varphi : G \to \text{Aut}(K)$ satisfying the following conditions

(i) $\varphi$ is conjugation type i.e. $\varphi_h(K_g) = K_{gh^{-1}}$ and $\varphi_h|K_h = id_{K_h}$ for every $g, h \in G$,

(ii) $ba = \varphi(a)(b)$ for any $a \in K$ and $b \in K_h$,

(iii) $\text{Tr}(\mu_c \varphi_h : K_g \to K_g) = \text{Tr}(\varphi_g \mu_c : K_h \to K_h)$ for all $g, h \in G$ and $c \in K_{gh^{-1}h^{-1}}$ where $\mu_c : K \to K$ is a left multiplication by $c$ and $\text{Tr}$ is the trace of a map,

(iv) $\eta$ is invariant under $\varphi$.

Theorem 3.6. (Turaev, [[Tu2]]) Every oriented 2-dimensional HFT with target $X = K(G, 1)$ determines a crossed Frobenius $G$-algebra. This induces a bijection between the isomorphism classes of 2-dimensional oriented HFTs with target $X$ and the isomorphism classes of crossed Frobenius $G$-algebras.

In the following we generalize Turaev’s $G$-center construction for biangular $G$-algebras (see [[Tu1]]) to quasi-biangular $G$-algebras. In general, $G$-center is not commutative and it differs from the usual center of the algebra. However, it has a crossed Frobenius $G$-algebra structure.

Lemma 3.5. Let $(K, \eta)$ be a quasi-biangular $G$-algebra with collection of inner product elements $\{\sum_i p_i^g \otimes q_i^g\}_{g \in G}$ and a central element $z \in K_e$. The $G$-center $Z_G(K)$ of $K$ is defined using the map $\Psi(a) = \sum_i p_i^g a q_i^g$ for all $a \in K_g$ as $Z_G(K) = \bigoplus_{g \in G} \Psi(K_g)$. Then $Z_G(K)$ is a $G$-subalgebra and the triple $(Z_G(K), \eta, \{\varphi|_{Z_G(K)}\}_{g \in G})$ is a crossed Frobenius $G$-algebra where $\varphi_g(a) = \sum_i p_i g a z q_i^g$ for all $a \in K$ and all $g \in G$.

Proof. Since $\sum_i p_i^g \otimes z q_i^g$ is a separability idempotent we have $\Psi(z) = 1 \in Z_G(K)_e$. By Lemma 3.3 we have the equality

$$\sum_i p_i^g b z q_i^h c = c \sum_i p_i^g h b q_i^g h$$

for all $c \in K_{g^{-1}}$, $b \in K$ and $g, h \in G$. Taking $z = 1$ and $g = h = e$ gives

$$\Psi(a \Psi(b)) = \sum_{i,j} p_i^g a p_j^g b q_j^g q_i^g = \sum_{i,j} p_i^g a q_i^g p_j^g b q_j^g = \Psi(a) \Psi(b)$$

which implies that $Z_G(K)$ is a $G$-subalgebra of $K$. Restriction of $\eta$ to $Z_G(K)$ is an inner product and hence $(Z_G(K), \eta, Z_G(K))$ is a Frobenius $G$-algebra. For any $b \in K$ and for all $h \in G$ we have

$$\Psi(\varphi_h(bz)) = \sum_j p_j^g \left( \sum_i p_i^g b z q_i^h \right) q_j^g = \sum_{i,j} p_i^g q_j^g b q_i^h = \sum_i p_i^g b z q_i^h = \varphi_h(b)$$
which shows that \( \varphi_h(K) \subset Z_G(K) \). Similarly for any \( \sum_i p_i^a q_i^b \in Z_G(K) \) we have
\[
\varphi_c \left( \sum_i p_i^a q_i^b \right) = \sum_j p_j^c \left( \sum_i p_i^a q_i^b \right) z_j^q = \sum_i p_i^c z_i^q p_i^a q_i^b = \sum_i p_i^0 a q_i^b
\]
showing \( \varphi_c |_{Z_G(K)} = id_{Z_G(K)} \). Note that for any \( g \in G \) and \( a \in K \) we have
\[
\varphi_g(\Psi(a)) = \sum_j p_j^g \left( \sum_i p_i^a q_i^e \right) z_j^q = \sum_j p_j^g z_j^q \sum_i p_i^0 a q_i^b = \sum_i p_i^0 a q_i^b
\]
and using this we have the following equality for all \( a, b \in K \) and \( g \in G \)
\[
\varphi_g(\Psi(a) \Psi(b)) = \sum_k p_k^g \left( \sum_{i,j} p_i^a q_i^e p_j^b q_j^e \right) z_k^q = \left( \sum_{i,k} p_k^g p_i^a q_i^e z_k^q \right) \left( \sum_j p_j^b q_j^e \right)
\]
showing \( \varphi_g \) is an algebra homomorphism. Using equation (6) we have
\[
\varphi_g(a \varphi_h(b)) = \sum_k p_k^g a p_k^g b \sum_j q_j^g z_j^q = \sum_i \eta(a z_i^q b, p_i^0) = \eta(a, \varphi_{g^{-1}}(b))
\]
Taking \( a = 1 \) gives \( \varphi_g \circ \varphi_h = \varphi_{gh} \) for all \( g, h \in G \), which also implies that \( \varphi_{g^{-1}} \) is the inverse of \( \varphi_g \) for all \( g \in G \). For all \( a, b \in K \) and \( g \in G \) using the cyclic symmetry of \( \eta \) we have
\[
\eta(\varphi_g(a), b) = \eta \left( \sum_i p_i^a z_i^q, b \right) = \sum_i \eta(a z_i^q b, p_i^0) = \eta \left( a, \sum_i q_i^0 b z_i^q \right) = \eta(a, \varphi_{g^{-1}}(b))
\]
showing the inner product \( \eta \) is invariant under \( \varphi : G \to \text{Aut}(Z_G(K)) \). Lemma 3.3 implies that for any \( b \in K_{h^{-1}} \) and \( g, h \in G \)
\[
\varphi_{g^{-1}}(b) = \sum_i q_i^g b z_i^q = \sum_i \eta(b g^{-1} p_i^0, q_i^g) = \varphi_{h^{-1}}(b) = \varphi_{g^{-1} h^{-1}}(b)
\]
and taking \( g^{-1} = e \) we get \( \varphi_{h^{-1}}(b) = \varphi_e(b) = b \). This implies that \( \varphi_g \) acts by identity on \( Z_G(K) \) for all \( g \in G \). Equation (5) gives \( \varphi_g(a)b = b \varphi_{h^{-1}}(a) \) for \( a \in K, b \in K_h \) and \( g, h \in G \). Taking \( g = h \) we have \( \varphi_h(a)b = ba \). Let \( \mu_c : K \to K \) be a multiplication by \( c \) in \( K \) for any \( a, h \in G \) and \( c \in K_{gh^{-1}} \). We have
\[
\text{Tr}(\mu_c \varphi_h : K_g \to K_g) = \sum_{i,j} \eta(c \varphi_h(p_i^0), q_j^0) = \sum_{i,j} \eta(c p_j^b z_i^q, q_j^0) = \sum_{i,j} \eta(q_i^0 c p_j^b z_i^q, q_j^0)
\]
Lastly, we want to show that the \( G \)-center of a quasi-biangular \( G \)-algebra is well-defined. By definition of inner product elements \( Z_G(K) = \oplus_{g \in G} \Psi(K_g) \) is independent of the choice of inner product elements. Let \( \{ \sum_i p_i^0 \otimes q_i^0 \}_{g \in G}, \{ r_i^g z_i^q s_i^q \}_{g \in G}, z' \) be tuples of inner product and central elements for quasi-biangular \( G \)-algebra \( (K, \eta) \). By definition of a quasi-biangular \( G \)-algebra we have \( \sum_i p_i^0 z_i^q = 1 \) and \( \sum_i r_i^g z_i^q s_i^q = 1 \) for all \( g \in G \). Thus, \( z = \left( \sum_i p_i^0 q_i^0 \right)^{-1} \) and \( z' = \left( \sum_i r_i^g s_i^g \right)^{-1} \) for any \( g \in G \) for any \( \ell \in K_z \). Lemma 1.4 in [Tu1] gives the following equalities
\[
\eta \left( \ell, \sum_i p_i^0 q_i^0 \right) = \eta \left( \ell p_i^0, q_i^0 \right) = \text{Tr}(\mu_{\ell} |_{K_g} : K_g \to K_g).
\]
We have the same equality using inner product elements \( \{ r_i^g \otimes s_i^g \}_{g \in G} \) for all \( g \in G \), which implies that \( z = z' \).
The following corollary describes the relation between nonextended oriented 2-dimensional HFTs obtained from E-HFTs and Turaev’s classification of oriented 2-dimensional HFTs by crossed Frobenius G-algebras.

**Corollary 3.6.1.** The nonextended oriented HFT associated to the G-center of quasi-biangular G-algebra \((A, \eta_A)\) is the nonextended oriented HFT obtained from the oriented E-HFT by restricting it to oriented X-circles and oriented X-cobordisms between them.

**Proof.** Using the notation of previous proof the image of a \(g\)-labeled circle is given by

\[ A_e \otimes A_e \otimes A_e^\circ \ A_g = \{ b \in A_g \mid a \cdot b = b \cdot a \text{ for all } a \in A_e \}. \]

The G-center \(Z_G(A)\) of a quasi-biangular G-algebra \((A, \eta)\) with a collection of inner product elements \(\{ \sum_i p_i^g \otimes q_i^g \}_{g \in G} \) and a central element \(z \in A_e\) is defined as \(Z_G(A) = \oplus_{g \in G} \Psi(A_g)\) where \(\Psi(a) = \sum_i p_i^g a z q_i^g\). By Lemma 3.3, the G-center is a crossed Frobenius G-algebra \((Z_G(A), \eta |_{Z_G(A)}, \{ \varphi_g |_{Z_G(A)} \}_{g \in G})\) where \(\varphi_g(a) = \sum_i p_i^g a z q_i^g\). For any \(a \in A_e \otimes A_e \otimes A_e^\circ \ A_g\) we have

\[ a = 1.a = \left( \sum_i p_i^g z q_i^g \right) a = \sum_i p_i^g a z q_i^g = \Psi(az) \in \Psi(A_g) \]

and for any \(\sum_i p_i^g a q_i^g \in \Psi(A_g)\) and \(b \in A_e\) we have

\[ \left( \sum_i p_i^e a q_i^e \right) b = \left( \sum_i p_i^e a q_i^e \right) b = \sum_i b p_i^e a q_i^e = b \left( \sum_i p_i^e a q_i^e \right) \]

where the middle equality is the result of Lemma 3.3. Thus, \(A_e \otimes A_e \otimes A_e^\circ \ A_g = \Psi(A_g)\) for all \(g \in G\). The third morphism in the Figure 22 gives the crossed structure on the restricted nonextended HFT and for any \(a \in A_g\) this 2-morphism corresponds to following sequence of compositions

\[ 1 \otimes a \rightarrow 1 \otimes \sum_j a_j b_j \otimes a \rightarrow 1 \otimes \beta_h z \beta_h^{-1} \otimes a \rightarrow \]

\[ 1 \otimes \beta_h \left( \sum_i p_i^h \otimes z q_i^h \right) \beta_h^{-1} \otimes a \rightarrow 1 \otimes \sum_i p_i^h \otimes z q_i^h \otimes a \rightarrow 1 \otimes \sum_i p_i^h a z q_i^h \]

which coincides with the crossed structure of \(Z_G(A)\). \(\blacksquare\)

**Example 3.1.** Let \(k\) be an algebraically closed field. Then separable \(k\)-algebras are the same as semisimple \(k\)-algebras. By the Artin-Wedderburn structure theorem any separable algebra is isomorphic to a product of finitely many matrix algebras over \(k\). Consider the \(G\)-algebra \(A = \oplus_{g \in G} A_g\) whose principal component is a product \(A_e = \prod_{i=1}^n M_{k_i}(k)\) of \((k_i \times k_i)\)-matrix algebras over \(k\) such that each \(k_i\) is invertible in \(k\) and each component is given by \(A_g = \ell_g A_e\) where \(\ell_g\) is a basis. Define an inner product \(\eta\) on \(A\) as

\[ \eta(a, b) = \begin{cases} r \text{Tr}(L_{ab} : A_e \rightarrow A_e) & \text{when } ab \in A_e \\ 0 & \text{otherwise} \end{cases} \]

where \(r \in k\) is invertible and \(\text{Tr}(L_{ab})\) is the trace of left multiplication by \(ab\) map. We can express inner product concretely as \(\eta(\ell_g A_e, \ell_{g'} A_e) = \sum_{i=1}^n k_i \text{Tr}(A_i B_i)\) where \(\text{Tr}(A_i B_i)\) is the trace of the matrix \(A_i B_i\). For each \(g \in G\) inner product element can be chosen as \(\eta_g = r^{-1} \prod_{i=1}^n k_i^{-1} k_{i+1} \sum_{\alpha, \beta=1}^n \ell_g E_{\alpha, \beta} \otimes \ell_{g^{-1}} E_{\beta, \alpha} \in A_g \otimes A_{g^{-1}}\) where \(E_{\alpha, \beta}\) is the \((\alpha, \beta)\)-elementary matrix. In this case the central element \(z \in A_e\) is given by \((rI_{k_1}, \ldots, rI_{k_n})\) where \(I_{k_i}\) denote \((k_i \times k_i)\) identity matrix. Note that \(\prod_{i=1}^n k_i^{-1} k_{i+1} \sum_{\alpha, \beta=1}^n E_{\alpha, \beta} \otimes E_{\beta, \alpha}\) is a separability idempotent of \(A_e\). Thus, the map
\( \Psi : A_g \to A_g \) is given by
\[
\Psi \left( \ell_g \prod_{i=1}^{n} A_i \right) = r^{-1} \prod_{i=1}^{n} k_i^{-1} \sum_{\alpha, \beta = 1}^{\ell_{g}} E_{\alpha, \beta}(\ell_{g} A_i)E_{\beta, \alpha} = r^{-1} \prod_{i=1}^{n} k_i^{-1} \ell_g \text{Tr}(A_i)I_{k_i}
\]
which is a projection onto its center \( \ell_g k^n \).

In order to extend Theorem 3.5 to an equivalence of bicategories we need to study symmetric monoidal transformations between E-HFTs and symmetric monoidal modifications between such transformations. Cofibrancy theorem implies that up to an equivalence it is enough to study 1- and 2-morphisms of the bicategory \( XP(\text{Alg}_G^2) \).

Assume that two oriented extended homotopy field theories \( Z_0 \) and \( Z_1 \) give triples\(^20\) \((A, B, \zeta)\) and \((A', B', \zeta')\) respectively. A 1-morphism \( \alpha \) in \( XP(\text{Alg}_G^2) \) from \((A, B, \zeta)\) to \((A', B', \zeta')\) gives 1-morphisms
\[
\alpha_0(\cdot) = A_e R_{A_e} \\
\alpha_0(\cdot) = B_e S_{B_e}
\]
and 2-morphisms
\[
\alpha_1(\cdot, \cdot \cdot) : A'_g A'_g \otimes A'_g R_{A_e} \to A'_g R \otimes A_e (A_g)_{A_e} \\
\alpha_1(\cdot, \cdot \cdot) : B'_g B'_g \otimes B'_g S_{B_e} \to B'_g S \otimes B_e (B_g)_{B_e} \\
\alpha_1(\cdot, \cdot \cdot) : A'_g B'_e (M'_g)^{\ell} \to A'_g B'_e (R \otimes S) \otimes A_e B_e (M_g)^{k} \\
\alpha_1(\cdot, \cdot \cdot) : k N'_g \otimes B'_e A'_e (S \otimes R) B_e A_e \to k (N_g) B_e A_e
\]
which are isomorphisms for all \( g \in G \) and \( G \)-graded bimodules \( M, M', N \) and \( N' \) are the components of given graded Morita contexts. These morphisms are natural with respect to generating 2-morphisms. Naturality with respect to 2-morphism corresponding to graded multiplication leads to the commutativity of the diagram
\[
\begin{array}{ccc}
A'_g A'_g \otimes A'_g R_{A_e} & \xrightarrow{\alpha_1(\cdot, \cdot \cdot)} & A'_g R \otimes A_e (A_g)_{A_e} \\
\downarrow Z_1(\cdot, \cdot \cdot) & & \downarrow Z_0(\cdot, \cdot \cdot) \\
A'_g \otimes A'_g R_{A_e} & \xrightarrow{\alpha_1(\cdot, \cdot \cdot)} & A'_g R \otimes A_e (A_g)_{A_e}
\end{array}
\]
for all \( g, g' \in G \). We denote bimodules \( A'_g A'_g \otimes A'_g R_{A_e} \) and \( A'_g R \otimes A_e (A_g)_{A_e} \) by \( A'_g (R'_g)_{A_e} \) and \( A'_g (R''_g)_{A_e} \) respectively. Commutativity of the above diagram implies that they are naturally isomorphic under \( G \)-grading. Thus, using these isomorphisms we can use one of them and denote it by \( R_g \). Similarly, \( S_g \) denotes rank one \((B'_e, B_e)\)-bimodule. These assignments and naturality with respect to graded multiplication generator turn \( R = \bigoplus_{g \in G} R_g \) and \( S = \bigoplus_{g \in G} S_g \) into \( G \)-graded \((A', A)\) and \((B', B)\)-bimodules respectively. Similarly, naturality with respect to \( G \)-module generators turns collections \( \{\alpha_1(\cdot, \cdot \cdot)\} \) into \( G \)-graded \((A' \otimes B', k)\) and \((k, B \otimes A)\)-bimodule maps respectively.

Using \( \alpha_0(\cdot) \) we define a 1-morphism \( \alpha_0' \) of \( A'_e R'_{A'_e} \) as follows
\[
\alpha_0'(\cdot) = [\alpha(\cdot) \otimes \sigma_{Z_0}(\cdot), Z_1(\cdot)] \circ \nu(\cdot) \circ \sigma_{Z_0}(\cdot), Z_1(\cdot)] \\
\alpha_0'(\cdot) = \nu(\cdot) \circ \sigma_{Z_0}(\cdot), Z_1(\cdot)]
\]
\(^20\)Quasi-biangular \( G \)-algebras and a compatible Morita context between them.
where $\sigma$ is the symmetric braiding of $\text{Alg}_G^2$. In the same way, using $\alpha'_0(\star)$ we define a 2-morphism
\[
\alpha'_1(- \star -) = A_e A_e \otimes A_e R'_{\Lambda'_e} \to A_e R' \otimes A_e (A'_g)_{\Lambda'_e},
\]
\[
\alpha'_1(- \star -) = Z_0(- \star -) \circ \alpha'_0(\star) \to \alpha'_0(\star) \circ Z_1(- \star -).
\]

Using the naturality $R'$ is turned into a $G$-graded $(A, A')$-bimodule $R' = \oplus_{g \in G} R'_g$. The 1-morphism $\alpha_0(\star)$ can be obtained from $\alpha'_0(\star)$ by applying $Z_1(\overline{\gamma}) \circ \text{id}_{\alpha_0(\star)} \circ Z_0(\overline{\gamma})$ to the 1-morphism $[Z_1(\overline{\gamma}) \otimes Z_0(\overline{\gamma}) \otimes \text{id}_{Z_1(\overline{\gamma})}] \circ [\alpha_0(\star) \otimes \sigma_{Z_0(\overline{\gamma})} Z_1(\overline{\gamma}) \otimes \sigma_{Z_0(\overline{\gamma})} Z_0(\overline{\gamma})] \circ [Z_0(\overline{\gamma}) \otimes Z_1(\overline{\gamma}) \otimes \text{id}_{Z_0(\overline{\gamma})}]$ and similarly $\alpha_1(- \star -)$ can be obtained from $\alpha'_1(- \star -)$. Likewise, using $\alpha'_0(\star)$ in the images of cusps generators under $Z_0$ the 2-morphisms $\alpha'_1(\star \star -)$ and $\alpha'_1(- \star \star)$ are defined and $\alpha_1(\star \star -)$ and $\alpha_1(- \star \star)$ can be obtained from these 2-morphisms.

As in the proof of Theorem 3.5 using $G$-graded Morita contexts $\zeta$ and $\zeta'$ graded bimodules $M$ and $M'$ can be replaced by $A_{\otimes A}'$ and $A'_{\otimes A'}$. We can also replace graded bimodule $S$ by $R'$ using $\alpha'_0(\star)$. Thus, naturality with respect to $G$-module generators turn the collection $\{\alpha'_1(\star \star -)\}_{g \in G}$ into $A'_{\otimes A'} \to A' R \otimes A' R'_{\Lambda'_e}$. Similarly, the collection $\{\alpha'_1(- \star \star)\}_{g \in G}$ is turned into $A' R' \otimes A' R'_{\Lambda'_e} \to A' R'_{\Lambda'_e}$.

Naturality with respect to cusps generators indicate that compositions
\[
\begin{align*}
A' R A &\to A' A'_{\otimes A'} \otimes A' R \overset{\alpha'_1(\star \star -) \otimes \text{id}}{\longrightarrow} A' R \otimes A' A_{\otimes A'} A A \to A' R A \\
A' R'_{\Lambda'_e} &\to A' R' \otimes A' A'_{\otimes A'} \overset{\text{id} \otimes \alpha'_1(- \star \star)}{\longrightarrow} A' R' \otimes A' R'_{\Lambda'_e} \overset{\alpha'_1(- \star \star) \otimes \text{id}}{\longrightarrow} A A_{\otimes A'} A' R'_{\Lambda'_e} \to A' R'_{\Lambda'_e}
\end{align*}
\]
are $\text{id}_R$ and $\text{id}_{R'}$ respectively. In other words, the 1-morphism $\alpha$ gives a $G$-graded Morita context between $A$ and $A'$. Note that one can define $\alpha'_0(- \star -)$ and similar arguments give a $G$-graded Morita context between $B$ and $B'$. Naturality with respect to Morse generators indicates that $G$-graded Morita contexts are compatible. Hence, $\alpha$ leads to two compatible $G$-graded Morita contexts. In the theory of bicategories this means that both $\alpha'_0(\star)$ and $\alpha_0(\star)$ are parts of two adjoint equivalences. Since an adjoint equivalence is the same as equivalence (see Proposition A.27 in [Sc]), $Z_0$ and $Z_1$ are equivalent.

Let $\alpha'_1, \alpha'_2 : Z_0 \to Z_1$ be symmetric monoidal transformations. By the cofibrancy theorem any symmetric monoidal modification $\theta : \alpha'_1 \to \alpha'_2$ is determined by 2-morphisms $\theta_0(\star)$ and $\theta_0(\star')$ which are natural with respect to image of generating 1-morphisms in $XP(\text{Alg}_G^2)$. To see the naturality with respect to $\star \star$ assume that $Z_0$ and $Z_1$ give triples $(A, B, \zeta)$ and $(A', B', \zeta')$ as before and transformations give $\alpha'_1(\star) = A_e R_{\Lambda_e}$ and $\alpha'_0(\star') = A_e P_{\Lambda_e}$. In this case $\theta_0(\star) = A_e R_{\Lambda_e} \to A_e P_{\Lambda_e}$ and the naturality of $\theta_0(\star)$ with respect to $\star \star$ is the commutativity of the following diagram
\[
\begin{array}{ccc}
A_e A_e' \otimes A_e P_{\Lambda_e} & \overset{\alpha'_1(\star \star -)}{\longrightarrow} & A_e A_e' \otimes A_e A_e (A'_e) A_e \\
\theta_0(\star) \downarrow & & \theta_0(\star) \\
A_e A_e' \otimes A_e P_{\Lambda_e} & \overset{\alpha'_0(\star \star -)}{\longrightarrow} & A_e A_e' \otimes A_e A_e (A'_e) A_e
\end{array}
\]
which shows that $\theta_0(\star')$ is a $G$-graded bimodule map. Assuming $(\alpha'_0)^1(\star) = A_e R_{\Lambda_e}$, and $(\alpha'_0)^2(\star) = A_e P_{\Lambda_e}$, we have similarly a graded bimodule map $\theta'_0(\star') : A_e R'_{\Lambda_e} \to A_e P'_{\Lambda_e}$ using $\theta_0(\star)$ along with $(\alpha'_0)^1(\star')$ for $i = 1, 2$. Naturality with respect to $\star \star$ and $\star \star$ corresponds commutativity of these bimodule maps with the unit and counit of the adjunctions. In other words, $\theta_0(\star')$ and $\theta'_0(\star')$ is an equivalence of graded Morita contexts (see Definition 3.8). In the same way, using $B$ and $B'$ one gets an equivalence of graded Morita contexts between $B$ and $B'$. Hence, $\theta$ leads to two equivalences of compatible $G$-graded Morita contexts.

Let Frob$^G$ be a bicategory whose objects are quasi-biangular $G$-algebras, 1-morphisms are compatible $G$-graded Morita contexts, and 2-morphisms are equivalences of $G$-graded Morita contexts.
The forgetting 2-functor $F : \mathcal{EHFT}(X, \text{Alg}_k^2) \to \text{Frob}^G$ is defined as follows. On E-HFTs it is defined as $(A, B, \zeta) \to A$, i.e. by forgetting the second $G$-algebra and the $G$-graded Morita context. On the transformation level it gives a compatible $G$-graded Morita context between quasi-biangular $G$-algebras whose principal components are values of positive points under E-HFTs. On the modification level it gives an equivalence of the given compatible $G$-graded Morita contexts.

**Theorem 3.7.** The forgetting 2-functor $F$ is an equivalence of bicategories

$$\mathcal{EHFT}(X, \text{Alg}_k^2) \simeq \text{Frob}^G.$$ 

**Proof.** By the Whitehead theorem for bicategories (see Theorem A.16 in [Se]) it is enough to show that $F$ is essentially surjective on objects, essentially full on 1-morphisms and fully faithful on 2-morphisms. For a given quasi-biangular $G$-algebra $A$ the triple $(A, A^\text{op}, \text{id})$ gives an extended homotopy field theory $Z : \text{XBord}_2 \to \text{Alg}_k^2$ such that $F(Z) = A$. Let $\alpha$ be a compatible $G$-graded Morita context between quasi-biangular $G$-algebras $A$ and $A'$. Then triples $(A, A^\text{op}, \alpha)$ and $(A', A'^\text{op}, \alpha)$ give extended homotopy field theories $Z_0, Z_1 : \text{XBord}_2 \to \text{Alg}_k^2$ such that $F(\alpha') = \alpha$ where $\alpha' : Z_0 \to Z_1$.

For any two transformations $\alpha^1, \alpha^2 : Z_0 \to Z_1$, we claim that

$$F(\alpha^1, \alpha^2) : \text{Hom}(\alpha^1, \alpha^2) \to \text{Hom}(F(\alpha^1), F(\alpha^2))$$

is an injection. Assume that different modifications $\theta^1, \theta^2 : \alpha^1 \to \alpha^2$ give the same equivalence of $G$-graded Morita contexts. This means that tuples $(\theta^1_0(\bullet), (\theta^1_0)^1(\bullet))$ and $(\theta^2_0(\bullet), (\theta^2_0)^2(\bullet))$ give different graded bimodule maps while tuples, images of $\theta^1, \theta^2$ under $F$, $((\theta^1_0(\bullet), (\theta^1_0)^1(\bullet))$ and $((\theta^2_0(\bullet), (\theta^2_0)^2(\bullet))$ give the same graded bimodules maps. However, this is a contradiction because each $(\theta^i_0)^i(\bullet)$ is obtained from $(\theta_0)^i(\bullet)$ and each $(\theta^i_0)^i(\bullet)$ is obtained from $(\theta_0)^i(\bullet)$ for $i = 1, 2$. For the surjectivity let $\theta : F(\alpha^1) \to F(\alpha^2)$ be an equivalence of graded Morita contexts. Then the equivalence of graded Morita contexts $(\theta_0(\bullet), (\theta_0)^1(\bullet))$ can be obtained from $\theta_0(\bullet)$, $\theta_0^1(\bullet)$ and $(\alpha_0^1)^i(\bullet)$ for $i = 1, 2$. \hfill \blacksquare

Another approach to categorical classification of (fully-)extended oriented HFTs is by the structured cobordism hypothesis due to J. Lurie [Lu]. Cobordism hypothesis is conjectured by J. Baez and J. Dolan in their seminal paper [BD]. Lurie (Lu) reformulated the cobordism hypothesis using $(\infty, n)$-categories and generalized it to a structured cobordism hypothesis using homotopy fixed points as follows.

**Theorem 3.8.** ($G$-structured Cobordism Hypothesis, [Lu]). Let $\mathcal{C}$ be a symmetric monoidal $(\infty, n)$-category (see [CS]) and $\text{Bord}_n^G$ be the $G$-equivariant extended bordism $(\infty, n)$-category for a group $G$. Then there is a canonical equivalence of $(\infty, n)$-categories

$$\text{Fun}^\otimes(\text{Bord}_n^G, \mathcal{C}) \simeq ((\text{C}^\text{fd})^\sim)^{hG}$$

where $\text{Fun}^\otimes$ is the $(\infty, n)$-category of symmetric monoidal functors between symmetric monoidal $(\infty, n)$-categories, $\text{C}^\text{fd}$ is the sub-$(\infty, n)$-category of fully dualizable objects with duality data, $(\text{C}^\text{fd})^\sim$ is the underlying $\infty$-groupoid and $((\text{C}^\text{fd})^\sim)^{hG}$ is the $\infty$-groupoid of homotopy $G$-fixed points given by

$$((\text{C}^\text{fd})^\sim)^{hG} = \text{Hom}_G(\text{EG}, (\text{C}^\text{fd})^\sim)$$

where $\text{EG}$ is a weakly contractible $\infty$-groupoid equipped with a free $G$-action.

In particular, $\text{Fun}^\otimes(\text{Bord}_n^G, \mathcal{C})$ is an $\infty$-groupoid.

**Remark.** An oriented 2-dimensional E-HFT with target $X \simeq K(G, 1)$ i.e. a classifying space $BG$, is a $(G \times SO(2))$-structured (fully-)extended TFT by pulling back universal bundle along characteristic maps of oriented $X$-manifolds. The fact that characteristic maps are pointed homotopy classes instead of pointed maps does not lead to a problem because in the context of structured E-TFTs we would like to glue equivalent bundles not the same bundles.

**Corollary 3.8.1.** For any algebraically closed field $k$ of characteristic zero the $(G \times SO(2))$-structured cobordism hypothesis for $\text{Alg}_k^2$-valued $(G \times SO(2))$-structured E-TFTs holds true.
Proof. O. Davidovich [Da] showed that for a finite group $G$ and such field $k$ isomorphism classes of homotopy $(G \times SO(2))$-fixed points in $\text{Alg}_k^2$ are given by $G$-equivariant algebras. A $G$-equivariant algebra is a Frobenius $G$-algebra with semisimple principal component such that each component is left and right rank 1 module over the principal component. Her methods do not particularly require group $G$ to be finite and can be extended to discrete groups directly. Thus, using Davidovich’s result Theorem 3.7 verifies the $(G \times SO(2))$-structured cobordism hypothesis for $\text{Alg}_k^2$-valued $(G \times SO(2))$-structured fully-extended TFTs. □

3.4. Classification of unoriented 2-dimensional E-HFTs. In this section we define and classify unoriented 2-dimensional E-HFTs with target $X \simeq K(G, 1)$ where every element of $G$ except identity has order two.

For a symmetric monoidal bicategory $\mathcal{C}$, we define a $\mathcal{C}$-valued unoriented 2-dimensional E-HFT with target $X$ as a symmetric monoidal 2-functor from $X\text{Bord}_2^{un}$ to $\mathcal{C}$. The bicategory $\mathcal{E}\text{-HFT}^{un}(X, \mathcal{C}) = \text{SymBicat}(X\text{Bord}_2^{un}, \mathcal{C})$ has $\mathcal{C}$-valued unoriented E-HFTs as objects, symmetric monoidal transformations as 1-morphisms, and symmetric monoidal modifications as 2-morphisms. Proposition 3.1 implies that $X\text{Bord}_2^{un}$ and $X\text{Bord}^{PD, un}_2$ are symmetric monoidally equivalent.

The presentation $XP^{un}$ of freely generated unbiased semistrict symmetric monoidal 2-category $X\text{Bord}^{PD,un}_2$ is given by Theorem 3.2. Single unoriented point $\{\cdot\}$ forms the set of generating objects and the set $\{g, \overline{g}, \frac{g}{g} \}$ for $g \in G$ gives the generating 1-morphisms. Generating 2-morphisms are shown in Figures 17 and 23 and relations among them are shown in Figures 18 and 24. Using the cofibrancy theorem (see Theorem 3.3) we state the classification of unoriented 2-dimensional E-HFTs as the equivalence of bicategories

$$\mathcal{E}\text{-HFT}^{un}(X, \mathcal{C}) \simeq XP^{un}(\mathcal{C}).$$

Remark. There is a symmetric monoidal 2-functor $\text{Forget} : X\text{Bord}_2 \to X\text{Bord}_2^{un}$ given by forgetting the orientation. In the same way, any oriented or unoriented 2-dimensional E-TFT leads to an oriented or unoriented E-HFT respectively by forgetting the $X$-manifold data. The following diagram indicates the universality of unoriented 2-dimensional E-TFTs in this context

$$\begin{array}{ccc}
\mathcal{E}\text{-TFT}^{un}(\mathcal{C}) & \longrightarrow & \mathcal{E}\text{-TFT}(\mathcal{C}) \\
\downarrow & & \downarrow \\
\mathcal{E}\text{-HFT}^{un}(X, \mathcal{C}) & \longrightarrow & \mathcal{E}\text{-HFT}(X, \mathcal{C})
\end{array}$$

where bicategories $\mathcal{E}\text{-TFT}^{un}(\mathcal{C})$ and $\mathcal{E}\text{-TFT}(\mathcal{C})$ are defined similarly using $\text{Bord}_2^{un}$ and $\text{Bord}_2$ respectively.

3.4.1. Classification of $\text{Alg}_k^2$-valued unoriented 2-dimensional E-HFTs. K. Tagami [Ta] classified nonextended unoriented 2-dimensional HFTs by extended crossed Frobenius $G$-algebras. Similar to oriented case our goal is to understand the relation between his classification and the restriction of $\text{Alg}_k^2$-valued unoriented 2-dimensional E-HFTs to circles and cobordisms between them.
Firstly, we introduce necessary algebraic notions. Let $K$ be $G$-algebra and $N$ be a $(K, K^{\text{op}})$ bimodule. Conjugate of $N$ is the $(K, K^{\text{op}})$ bimodule $N_k$ obtained by turning actions around. Similarly, the conjugate of a graded Morita context $\zeta = (K^{\text{op}} M_K, K N_{K^{\text{op}}}, \tau, \mu)$ is the graded Morita context given by $\tilde{\zeta} = (K^{\text{op}} M_K, K N_{K^{\text{op}}}, \tau, \mu)$. Note that conjugate of a Morita context between algebras $K$ and $L$ gives a Morita context between their opposite algebras. From now on we assume that $G$ is a discrete group whose nonidentity elements have order 2. We generalize stellar algebras introduced in [Sc] to stellar $G$-algebras as follows.

**Definition 3.11.** A stellar $G$-algebra is a $G$-algebra $K = \oplus_{g \in G} K_g$, equipped with a $G$-graded Morita context $\zeta = (K^{\text{op}} M_K, K N_{K^{\text{op}}}, \tau, \mu)$ together with an isomorphism of $G$-graded Morita contexts $\sigma : \zeta \cong \tilde{\zeta}$ such that $\sigma \circ \bar{\sigma}$ is the identity isomorphism where $\bar{\sigma}$ is the induced isomorphism between $\zeta$ and $\tilde{\zeta}$.

Stellar structure on a $G$-algebra can be transferred along a graded Morita context as follows. Let $\rho = (X L, \eta L, K, \kappa, \nu)$ be a $G$-graded Morita contexts between $G$-algebras $K$ and $L$ and let $(K, \zeta, \sigma)$ be a $G$-algebra on $K$. Then $(L, \rho, \zeta, \sigma)$ is a stellar algebra on $L$ where $\rho_{\text{op}} \zeta = (L^{\text{op}} X \otimes K^{\text{op}} M_K \otimes K X L, \eta \otimes \kappa, \nu \otimes \mu)$ and $\rho_{\text{op}} \zeta \cong \rho_{\text{op}} \tilde{\zeta}$ is given by $\sigma$.

**Definition 3.12.** Let $(K, \zeta, \sigma)$ be a stellar $G$-algebra with $\zeta = (K^{\text{op}} M_K, K N_{K^{\text{op}}}, \tau, \mu)$ and let $(K, \eta)$ be a quasi-biangular $G$-algebra. We call that stellar structure is compatible with quasi-biangular $G$-algebra if there exists an element $\sum_j a_j \otimes b_j \in K \otimes K$ giving the central element $z = \sum_j b_j a_j$ and the following diagrams commute.

\[
\begin{array}{ccccccccc}
K \otimes K & \xrightarrow{\tau \otimes \text{id}} & (N \otimes M) \otimes K & \xrightarrow{\sigma \otimes \text{id}} & (N \otimes M) \otimes K & \xrightarrow{\tau^{-1} \otimes \text{id}} & K \otimes K \\
\downarrow \text{id} \otimes \tau & & \downarrow \text{id} \otimes \sigma & & \downarrow \text{id} \otimes \tau & & \downarrow \text{id} \otimes \sigma \\
K \otimes (N \otimes M) & \xrightarrow{\text{id} \otimes \sigma} & K \otimes (N \otimes M) & \xrightarrow{\text{id} \otimes \tau^{-1}} & K \otimes K & \xrightarrow{\text{id} \otimes \tau^{-1}} & K \otimes K \\
\xi & \xrightarrow{i} & K \otimes K & \xrightarrow{\tau \otimes \text{id}} & (N \otimes M) \otimes K & \xrightarrow{\sigma \otimes \text{id}} & (N \otimes M) \otimes K \\
\xi & \xrightarrow{\text{id} \otimes \tau} & K \otimes (N \otimes M) & \xrightarrow{\text{id} \otimes \sigma} & K \otimes (N \otimes M) & \xrightarrow{\text{id} \otimes \tau^{-1}} & K \otimes K \\
K \otimes K & \xrightarrow{\tau \otimes \text{id}} & (N \otimes M) \otimes K & \xrightarrow{\sigma \otimes \text{id}} & (N \otimes M) \otimes K & \xrightarrow{\tau^{-1} \otimes \text{id}} & K \otimes K \\
\downarrow \text{id} \otimes \tau & & \downarrow \text{id} \otimes \sigma & & \downarrow \text{id} \otimes \tau & & \downarrow \text{id} \otimes \sigma \\
K \otimes (N \otimes M) & \xrightarrow{\text{id} \otimes \sigma} & K \otimes (N \otimes M) & \xrightarrow{\text{id} \otimes \tau^{-1}} & K \otimes K & \xrightarrow{\text{id} \otimes \tau^{-1}} & K \otimes K \\
\end{array}
\]

where $i(1)|_{K_i \otimes K_i} = \sum_j a_j \otimes b_j$ and $\xi : K_i K_{i2} \otimes K_3 K_{4} \to K_i K_{4} \otimes K_3 K_{K_2}$ is a graded bimodule map with each $K_i$ is $K$ and $\xi(1) = \sum_j p_j \otimes q_j^t$ inner product element of the principal component. We call such a compatible quadruple $(K, \eta, \zeta, \sigma)$ a quasi-biangular stellar $G$-algebra.

**Definition 3.13.** A morphism of quasi-biangular stellar $G$-algebras $(K, \eta^k, \zeta^K, \sigma^K)$ and $(L, \eta^L, \zeta^L, \sigma^L)$ is a compatible $G$-graded Morita context $\rho = (K M_L, L N_K, \tau, \mu)$ together with an equivalence of $G$-graded Morita contexts $\phi : \zeta^L \to \rho_{\text{op}} \zeta^K$ such that $\rho_{\text{op}} \sigma^K \circ \phi = \phi \circ \sigma^L$ where $\phi : \zeta^L \to \rho_{\text{op}} \zeta^K$ and $\sigma^i : \xi^i \to \xi^i$ for $i = K, L$. Two morphisms $(\rho, \phi)$ and $(\rho', \phi')$ of quasi-biangular stellar $G$-algebras are isomorphic if there exists an equivalence of $G$-graded Morita contexts $\alpha : \rho \to \rho'$ such that $\phi' = \alpha \circ \phi$ and $\phi' = \alpha \circ \phi$ for $\alpha : \rho \to \rho'$.

---

21 Tensors in diagrams are taken over $K, K^{\text{op}}$ or $K \otimes K^{\text{op}}$. 
Now we state the classification of $\text{Alg}_{2}^{\text{un}}$-valued unoriented E-HFTs with target $X$.

**Theorem 3.9.** Any $\text{Alg}_{2}^{\text{un}}$-valued unoriented 2-dimensional E-HFT determines a quasi-biangular stellar $G$-algebra $(A, \eta, \zeta, \sigma)$. Moreover, any quasi-biangular stellar $G$-algebra $(A, \eta, \zeta, \sigma)$ is determined by an unoriented 2-dimensional E-HFT.

**Proof.** Let $Z : X\text{Bord}_{2}^{\text{un}} \to \text{Alg}_{2}^{\text{un}}$ be an unoriented 2-dimensional E-HFT and $Z'$ be the corresponding object of $X\text{P}^{\text{un}}(\text{Alg}_{2}^{\text{un}})$ under the equivalence of bicategories $X\text{Bord}_{2}^{\text{un}} \simeq X\text{B}^{PD,\text{un}}$.

Following the proof of Theorem 3.5 we have a $G$-algebra $A = \bigoplus_{g \in G} A_{g}$ where $Z'(\bullet) = A_{e}$ and each $A_{g}$ is both left and right rank one $A_{e}$-module. We also have $G$-graded $(A \otimes A, k)$ and $(k, A \otimes A)$-bimodules $M = \bigoplus_{g \in G} M_{g}$, $N = \bigoplus_{g \in G} N_{g}$ respectively. By turning actions around we obtain $(A, A^{\text{op}})$-bimodule $M$ and $(A^{\text{op}}, A)$-bimodule $N$.

Bimodule maps in $Z_{2}^{\text{un}}(X\text{un}G_{2})$ corresponding to cusp generators (subject to relations) yield a $G$-graded Morita context $\zeta = (A^{\text{op}}N_{A}, A^{\text{op}}M_{A^{\text{op}}}, f_{1}, f_{2})$ between $A$ and $A^{\text{op}}$ where $f_{1} : A^{\text{op}}A_{A} \to A^{\text{op}}M \otimes A^{\text{op}}N_{A}$ and $f_{2} : A^{\text{op}}N \otimes A^{\text{op}}M_{A^{\text{op}}} \to A^{\text{op}}A^{\text{op}}M_{A^{\text{op}}}$ are invertible $G$-graded bimodule maps. Bimodule maps in $Z_{2}^{\text{un}}(X\text{un}G_{2})$ for the Morse generators satisfying relations imply that $(A, \eta)$ is a quasi-biangular $G$-algebra. Generators in Figure 23 give the following graded bimodule maps in $Z_{2}^{\text{un}}(X\text{un}G_{2})$

$$\sigma_{1} : A^{\text{op}}M_{A^{\text{op}}} \to A^{\text{op}}M_{A^{\text{op}}} \quad \sigma_{2} : A^{\text{op}}N_{A} \to A^{\text{op}}N_{A}$$

$$\sigma_{1}' : A^{\text{op}}M_{A^{\text{op}}} \to A^{\text{op}}M_{A^{\text{op}}} \quad \sigma_{2}' : A^{\text{op}}N_{A} \to A^{\text{op}}N_{A}.$$  

These graded bimodule maps are subject to relations in Figure 24. Thereby, we have $\sigma_{1}' \circ \sigma_{1} = \text{id}_{M}$, $\sigma_{1} \circ \sigma_{1}' = \text{id}_{M}$, $\sigma_{2}' \circ \sigma_{2} = \text{id}_{N}$ and $\sigma_{2} \circ \sigma_{2}' = \text{id}_{N}$. These isomorphisms of bimodules lead to an isomorphism $\sigma : \zeta \cong \zeta$. Applying $\sigma$ to $\zeta$ gives another isomorphism $\sigma : \zeta \to \zeta$ and composition with $\sigma$ gives $\sigma \circ \sigma : \zeta \cong \zeta$. Third relation on the first row of Figure 24 and its reflections indicate that compositions of bimodule maps $M \to M \to M$ and $N \to N \to N$ are identity maps.

Thus, additional generators and relations among them lead to a stellar structure $(\zeta, \sigma)$ on the quasi-biangular $G$-algebra $A$. Remaining relations imply that they are compatible giving quasi-biangular stellar $G$-algebra $(A, \eta, \zeta, \sigma)$. For any quasi-biangular stellar $G$-algebra there exists an unoriented 2-dimensional E-HFT by forming a strict 2-functor $Z' : X\text{B}^{PD,\text{un}} \to \text{Alg}_{2}^{\text{un}}$ using the quadruple and precomposing with the equivalence $X\text{Bord}_{2}^{\text{un}} \simeq X\text{B}^{PD,\text{un}}$.

\[\text{Figure 24. Additional relations of } X^{\text{un}} R\]
Definition 3.14. (Tagami) Let \((K, \eta, \varphi)\) be a crossed Frobenius \(G\)-algebra over \(\mathbb{k}\). An extended structure on \(K\) consist of a \(k\)-module homomorphism \(\Phi : K \to K\) and a family of elements \(\{\theta_g \in K_e\}_{g \in G}\) satisfying the following conditions

1. \(\Phi\) is an involution commuting with \(\varphi\) and \(\Phi\) maps each component to itself i.e. \(\Phi(K_g) \subset K_g\), such that \(\Phi(\theta_g) = \theta_g\) for all \(g \in G\).
2. for any \(v, w \in K, \Phi(vw) = \Phi(w)\Phi(v)\) and \(\Phi(1_K) = 1_K\).
3. \(\Phi^2 = id\) and \(\eta \circ (\Phi \otimes \Phi) = \eta\).
4. for any \(g, h, l \in G\) and \(v \in K_{gh}\), we have

   \[ m \circ (\Phi \circ \varphi_l) \circ \Delta_{g,h}(v) = \varphi_l(\theta_g \theta_iv), \]
   \[ m \circ (\varphi_l \circ \Phi) \circ \Delta_{g,h}(v) = \varphi_l(\theta_h \theta_lv), \]

   where \(\Delta_{g,h} : K_{gh} \to K_g \otimes K_h\) is defined by the equation \((id_g \otimes \eta) \circ (\Delta_{g,h} \otimes id_h) = m\). Since \(\eta\) is nondegenerate and each \(K_g\) is finitely generated, such a map \(\Delta_{g,h}\) is uniquely determined.

5. for any \(g, h \in G\) and \(v \in K_g\), we have \(\Phi(\theta_h \theta_l) = \theta_{gh}\).

6. for any \(g, h, l \in G\), we have \(\varphi_l(\theta_g) = \theta_{gh}\).

7. for any \(g, h, l \in G\), we have \(\theta_g \theta_h \theta_l = q(1)\theta_{ghl}\), where \(q : \mathbb{k} \to K_e\) is defined as follows; let \(\{a_i \in K_{gh}\}_{i=1}^n\) and \(\{b_i \in K_{ghl}\}_{i=1}^n\) be families of elements of \(K_{gh}\) satisfying the equation \(\sum_i \eta(b_i \otimes v)a_i = \varphi_l(h)v\) for any \(v \in K_{gh}\). As in (3), such \(a_i\) and \(b_i\) are uniquely determined and \(q(1) = \sum_i a_i b_i\).

Theorem 3.10. (Tagami, Tagami) Every unoriented 2-dimensional HFT with target \(X = K(G, 1)\) determines an extended crossed Frobenius \(G\)-algebra. This induces a bijection between the isomorphism classes of unoriented 2-dimensional HFTs with target \(X\) and the isomorphism classes of extended crossed Frobenius \(G\)-algebras.

The following corollary describes the relation between unoriented 2-dimensional HFTs obtained from unoriented E-HFTs and Tagami’s classification.

Corollary 3.10.1. Assume that \(Z : \text{Bord}_2^m \to \text{Alg}_G^\mathbb{k}\) determines a quasi-biangular stellar \(G\)-algebra \((A, \eta, \zeta, \sigma)\). Then the stellar structure \((\zeta, \sigma)\) gives an extended structure on the crossed Frobenius \(G\)-algebra \(ZG(A)\). Moreover, the corresponding 2-dimensional HFT is the unoriented HFT obtained by restricting \(Z\) to unoriented \(X\)-circles and unoriented \(X\)-cobordisms.

Proof. We have a crossed Frobenius \(G\)-algebra \((ZG(A) = \oplus_{g \in G} ZG(A)_g, \eta, ZG(A), \{\varphi|_{ZG(A)}\}_{g \in G}\) By Tagami’s classification, the unoriented 2-dimensional HFT given by the restriction of \(Z\) to unoriented circles and cobordism between them induces an extended structure on \(ZG(A)\). We claim that homomorphism \(\Phi\) and elements \(\{\theta_g \in ZG(A)_e\}_{g \in G}\) come from stellar structure \((\zeta, \sigma)\) on \(A\).

In Figure 23, for each \(g \in G\) the restriction \(\Phi|_{ZG(A)_g} : ZG(A)_g \to ZG(A)_g\) is the involution induced by an orientation reversing homeomorphism of the \(g\)-labeled circle. In our case this morphism is given by additional 2-morphisms (Figure 23). More precisely, \(\Phi|_{ZG(A)_g} : A_e \otimes A_e \otimes A_e \otimes A_e \to A_e \otimes A_e \otimes A_e \otimes A_e\) is defined by \(\Phi(a \otimes b) = a \otimes \Phi_g(b)\) where \(\Phi_g\) is defined so that the following diagram

\[ A_e M_g \otimes A_e \otimes A_e \overset{\Phi_g}{\longrightarrow} A_e (A_g) A_e \]

commutes. It is not hard to see that \(\Phi\) reverses the orientation of the to oriented (input) circle. In Figure 23, for every \(g \in G\) the element \(\theta_g\) is the image of HFT under the Möbius strip whose boundary is labeled by \(g^2 = e\) where the Möbius strip is considered as the cobordism from the empty 1-manifold to the boundary circle. In our case, \(\theta_g \in A_e \otimes A_e \otimes A_e\) is the image of \(1 \in \mathbb{k}\) under the following composition; \(\{g, g\}\)-labeled cap morphism followed by new generators (see the second equality of
the second row in Figure 24 which is composed with module actions turning boundary labels into \{e,e\} (see Figure 21).

We see that the involution \(\Phi\) and elements \(\{\theta_{g}\}_{g \in G}\) are defined according to their topological description given in \(\text{Ta}\). Hence, \((Z_{G}(A), \eta|_{Z_{G}(A)}, \{\varphi_{g}|_{Z_{G}(A)}\}_{g \in G}, \Phi, \{\theta_{g}\}_{g \in G}\) is an extended crossed Frobenius \(G\)-algebra which by definition corresponds to the restriction of \(Z : \text{X Bord}^{2}_{\text{un}} \rightarrow \text{Alg}^{2}_{E}\) to \(X\)-circles and unoriented \(X\)-cobordisms between them.

In order to upgrade Theorem 3.11 to an equivalence of bicategories we study morphisms in the bicategory \(XP^{an}(\text{Alg}^{2}_{E})\) as in the oriented case. Let \(\alpha : Z_{0} \rightarrow Z_{1}\) be a symmetric monoidal modification between unoriented \(E\)-HFTs giving quasi-biangular stellar \(G\)-algebras \((A, \eta, \zeta, \sigma)\) and \((A', \eta', \zeta', \sigma')\) respectively. We know from the oriented case that \(\alpha\) gives two compatible \(G\)-graded Morita contexts between \(A\) and \(A'\). Both Morita contexts are equal, we denote by \(\xi\) as there is no orientation on points. Assuming \(\alpha_{0}(\bullet) = A_{\xi}R_{A_{\xi}}\) and \(\xi = (A_{\xi}R_{A_{\xi}}', A_{\xi}R_{A_{\xi}}', \tau, \mu)\) naturality with respect to the first generator in Figure 23 is the commutativity of the following diagram

\[
\begin{array}{c}
\alpha'_{2}(\zeta_{\geq \varnothing}) \downarrow \\
A'_{\xi}(M'_{g})_{(A'_{\xi})}^{op} \xrightarrow{\alpha_{1}(\zeta_{\leq \varnothing})} A'_{\xi}R'_{\xi} \otimes_{A_{\xi}} M_{g} \otimes_{A_{\xi}^{op}} R_{(A'_{\xi})}^{op} \\
\end{array}
\]

where \(M\) and \(M'\) are components of the graded Morita contexts \(\zeta\) and \(\zeta'\) respectively. There are similar commutative diagrams for the remaining three generators. These diagrams indicate that the \(G\)-graded Morita context \(\xi\) gives an equivalence of \(G\)-graded Morita contexts \(\zeta'\) and \(\xi \otimes \zeta\) with \(\alpha \circ \sigma' = \xi_{\ast} \sigma \circ \alpha\). In other words, \(\alpha\) induces a morphism of stellar \(G\)-algebras (see Definition 3.13).

The bicategory \(\text{Frob}^{G}_{E}\) has quasi-biangular \(G\)-algebras with stellar structures as objects, morphisms of stellar \(G\)-algebras as 1-morphisms and isomorphisms of stellar \(G\)-algebra morphisms as 2-morphisms. Above arguments imply that there is a 2-functor \(\mathcal{F} : \mathcal{E}\text{-HFT}^{an}(X, \text{Alg}^{2}_{E}) \rightarrow \text{Frob}^{G}_{E}\).

**Theorem 3.11.** The 2-functor \(\mathcal{F} : \mathcal{E}\text{-HFT}^{an}(X, \text{Alg}^{2}_{E}) \rightarrow \text{Frob}^{G}_{E}\) is an equivalence of bicategories.

**Proof.** Proof follows from above arguments and the Whitehead theorem for bicategories. \(\blacksquare\)
Parallel to oriented case we want to compare Theorem 3.11 with the classification given by the \((G \times O(2))\)-structured cobordism hypothesis. To do this we need to understand homotopy \((G \times O(2))\)-fixed points in \(\text{Alg}_k\) which are given by

\[
\left( \left( \text{Alg}_k \right)^{fd} \right)^{h(G \times O(2))} = \text{Hom}_G\left( \text{EG}, \text{Hom}_{O(2)}(EO(2), \mathcal{X}) \right)
\]

where \(G\)-acts on a Hom space trivially and \(\mathcal{X}\) is the 2-type corresponding to the \(\infty\)-groupoid \(\left( \text{Alg}_k^{fd} \right)^{\sim}\). Recall that unoriented Grassmannian \(\text{Gr}(2, \mathbb{R}^\infty)\) is a model for \(BO(2)\) and Stiefel manifold \(V(2, \mathbb{R}^\infty)\) for \(EO(2)\). Universal principal \(O(2)\)-bundle \(p : V(2, \mathbb{R}^\infty) \to \text{Gr}(2, \mathbb{R}^\infty)\) is given by \(p((e_1, e_2)) = \langle e_1, e_2 \rangle\) i.e. the plane generated by the orthonormal 2-frame \((e_1, e_2)\).

**Lemma 3.6.** Reflection invariant maps in \(\text{Hom}_{O(2)}(V(2, \mathbb{R}^\infty), \chi)\) equip algebras with stellar structure.

**Proof.** A reflection \(\omega\) in \(O(2)\) acts on \(\chi\) by sending a \(k\)-algebra \(A\) to its opposite algebra \(A^{op}\). Let \(f\) be a reflection invariant map with \(f((e_1, e_2)) = A\). Since oriented Grassmanian \(\widetilde{\text{Gr}}(2, \mathbb{R}^\infty)\) is a 2-fold cover of \(\text{Gr}(2, \mathbb{R}^\infty)\) by orientation forgetting map and (simply connected) \(CP^\infty\) is another model for \(BSO(2)\) we have \(\pi_1(\text{Gr}(2, \mathbb{R}^\infty), (e_1, e_2)) \cong \mathbb{Z}/2\mathbb{Z}\). Let \(\gamma\) be a representative of the nontrivial element. Lift the \(\gamma\) to \(\tilde{\gamma}\) starting at \((e_1, e_2)\) and ending at \(\omega((e_1, e_2))\) (see Figure 25). Then \(f(\tilde{\gamma})\) is a \((A^{op}, A)\)-bimodule \(M\) and invariance under \(\omega\) means \(f(\omega(\tilde{\gamma})) = A^{op}M_A = \omega(M)\). Lifting \(\gamma\) to \(\tilde{\gamma}'\) starting at \(\omega((e_1, e_2))\) gives a path ending at \((e_1, e_2)\). Similarly, \(f(\tilde{\gamma}')\) is a \((A, A^{op})\)-bimodule \(N\) and we have \(f(\omega(\tilde{\gamma}')) = A^{op}N_A = \omega(N)\).

**Figure 25.** A reflection invariant map

Loops \(\tilde{\gamma}' * \tilde{\gamma}\) and \(\tilde{\gamma} * \tilde{\gamma}'\) bound disks since \(V(2, \mathbb{R}^\infty)\) is contractible. This implies that bimodules \(M\) and \(N\) are part of a Morita context \(\zeta = (A^{op}M_A, A^{op}N_A, \tau, \mu)\). Similarly, loops \(\tilde{\gamma} * \omega(\tilde{\gamma})\) and \(\tilde{\gamma}' * \omega(\tilde{\gamma}')\) bound which implies that there is an equivalence of Morita contexts \(\sigma : \zeta \cong \tilde{\zeta}\). Since the reflection is order 2 we have \(\sigma \circ \sigma = \text{id}\). Thus, any reflection invariant map leads to stellar algebra structures on algebras.

**Lemma 3.7.** For an algebraically closed field \(k\) of characteristic zero, homotopy \((G \times O(2))\)-fixed points of \(\text{Alg}_k\) are quasi-biangular stellar \(G\)-algebras.

**Proof.** Serre automorphism trivializes homotopy \(SO(2)\)-action (see [Da]), which turns the space of homotopy \((G \times SO(2))\)-fixed points into \(\text{Hom}_G(EG, \text{Hom}(\widetilde{\text{Gr}}(2, \mathbb{R}^\infty), \chi))\). Davidovich [Da] showed that homotopy \(SO(2)\)-fixed points are finite dimensional semisimple symmetric Frobenius \(k\)-algebras.

\[22\] We write \(\text{Alg}_k\) instead of \(\text{Alg}_k^2\) for convenience.
Then understanding homotopy $O(2)$-fixed points is amount to understanding invariance under reflections. Then using Lemma 3.6 we conclude that homotopy $O(2)$-fixed points are finite dimensional semisimple symmetric Frobenius $k$-algebra with a stellar structure.

Stellar structure is compatible with the Frobenius form as follows. Frobenius from on a $k$-algebra $A$ is determined by a central element which is the image of 1 under a bimodule map $z : A A_A \to A A_A$. Geometrically, $z(1)$ is an element of $\pi_2(\text{Hom}(BSO(2),\chi^r), f) = (k^\times)^r$ where an algebra $A \in \chi^r \subset \chi = \Pi_{r=1}^\infty \chi_r$ is isomorphic to $\text{End}(V_1) \times \text{End}(V_2) \times \cdots \times \text{End}(V_r)$ under Artin-Wedderburn isomorphism for finite dimensional $k$-vector spaces $V_1, \ldots, V_r$. Compatibility means that (horizontal) composition of $z$ with $\zeta$ yields $z$ again. Geometrically, this corresponds to conjugating the representing sphere based at $\bar{f}$ with loops in $\chi_r$ given by bimodules of $\zeta$. Since this loop is contractible conjugation does not change $z(1)$ in the second homotopy group. Thus, we have a compatible stellar structure and following Davidovich’s methods we obtain that for a discrete group $G$ homotopy $(G \times O(2))$-fixed points are quasi-biangular stellar $G$-algebras.

Corollary 3.11.1. Any algebraically closed field $k$ of characteristic zero the $(G \times O(2))$-structured (fully-)extended cobordism hypothesis for 2-dimensional $(G \times O(2))$-structured $E$-TFTs with values in $\text{Alg}_k^2$ holds true.

Proof. Follows from Theorem 3.11 and Lemma 3.7

Let $G$ be any discrete group, $(Y, y) \simeq (K(H, 1), y)$ be a pointed CW-complex for a subgroup $H \leq G$ and $p : (Y, y) \to (X, x)$ be a covering. We consider the relationship between oriented $E$-HFTs with target $Y$ and with target $X$. Any $Y$-manifold ($Y$-cobordism) can be turned into an $X$-manifold ($X$-cobordism) by postcomposing the representative of characteristic map with $p$. This gives a symmetric monoidal 2-functor $\iota_H : Y \text{Bord}_2 \to X \text{Bord}_2$ and precomposing any $E$-HFT with $\iota_H$ yields an oriented extended homotopy field theory $Z' := Z \circ \iota_H$ with target $Y$. In the case of $\text{Alg}_k^2$-valued oriented $E$-HFTs, we have the following commutative diagram of 2-functors

$$
\begin{array}{ccc}
\mathcal{E}\mathcal{H}F\mathcal{T}(X, \text{Alg}_k^2) & \xrightarrow{\sim} & \text{Frob}^G \\
\downarrow & & \downarrow \\
\mathcal{E}\mathcal{H}F\mathcal{T}(Y, \text{Alg}_k^2) & \xrightarrow{\sim} & \text{Frob}^H
\end{array}
$$

where vertical 2-functors are given by forgetting the $G\backslash H$ components of quasi-biangular $G$-algebras, compatible $G$-graded Morita contexts and equivalences of $G$-graded Morita contexts. In other words, a $G$-graded Morita context can be considered as collection of Morita contexts indexed by the subgroups of $G$ (see [Bo]). More generally, for any symmetric monoidal bicategory $\mathcal{C}$ there is a 2-functor $XP(\mathcal{C}) \to YP(\mathcal{C})$ where $XP$ and $YP$ are presentations of $X\text{Bord}_2$ and $Y\text{Bord}_2$ respectively. There are similar 2-functors in the unoriented case.

References


Kürşat Sözer, Department of Mathematics, Indiana University, Bloomington, Indiana 47405

E-mail address: ksozer@indiana.edu