1 Correlated Equilibria

Sometimes players can form binding contracts to achieve an outcome strictly preferred by each player to any Nash equilibria; however, when such contracts are not binding, players may have an incentive to deviate from the contractually prescribed strategies. In such situations, players might be able to communicate and coordinate with one another to achieve superior outcomes than Nash equilibria.

We say that a game is with communications if, in addition to the strategy options explicitly specified in the structure of the game, the players have a very wide range of implicit options to communicate with each other.

Example 1. Look at the game in Table 1. Without communication, there are three equilibria of this game: \((x_1, x_2)\), \((y_1, y_2)\), and a randomized equilibrium with payoff \((2.5, 2.5)\).

\[
\begin{array}{c|cc}
& C_1 & C_2 \\
\hline
x_1 & 5,1 & 0,0 \\
y_1 & 4,4 & 1,5 \\
\end{array}
\]

With communication, the players can achieve a better outcome, say, by tossing a coin together and play \((x_1, x_2)\) and \((y_1, y_2)\) with equal probability (and achieve a payoff of \((3, 3)\). Even though the coin toss has no binding force on the players, such a plan is self-enforcing in the sense that neither player could gain by unilaterally deviating from this plan.
With the help of a mediator (a person or machine that can help the players communicate and share information) better self-enforcing outcomes can be achieved. Suppose the mediator recommends \((x_1, x_2), (y_1, x_2),\) and \((y_1, y_2)\) each with probability \(1/3\); suppose also that each player learns only her own recommended strategy from the mediator. Even though the mediators recommendation has no binding force, there is still an equilibrium of the transformed game in which players are incentivized to follow the mediation.

Note that the implementation of this correlated strategy \(1/3[x_1, x_2] + 1/3[y_1, x_2] + 1/3[y_1, y_2]\) without contracts requires that each player get different partial information about the outcome of the mediator’s randomization.

Remark 1. With only direct unmediated communication in which all players observe anyone’s statements or the outcomes of any randomizations, the only self-enforcing plans that the players could implement without contracts would be randomizations among the Nash equilibria of the original game (without communication).

Let \(\Gamma = (N, (C_i)_{i \in N}, (u_i)_{i \in N})\) be a finite strategic-form game. A mediator would at least need to let each player \(i\) know which strategy in \(C_i\) (the set of pure strategies) was recommended for him. If no player would be told the recommendations for any other players, without contracts, after hearing the recommendation, player \(i\) would be able to choose any strategy in \(C_i\). That is, the set of strategies with mediation of player \(i\) includes all mappings \(\delta_i : C_i \rightarrow C_i\).

Let \(C = \times_{i \in N} C_i\). Suppose it is common knowledge that the mediator will recommend according to the probability \(\mu \in \Delta(C)\) (the set of all probability distributions with support in \(C\).) The condition for \(\mu\) to be a correlated equilibrium is for all players to willingly obey the mediation:

**Definition 1** (correlated equilibrium). Let \(\Gamma = (N, (C_i)_{i \in N}, (u_i)_{i \in N})\) be a finite strategic-form game. \(\mu\) is a correlated equilibrium of \(\Gamma\) iff \(\mu \in \Delta(C)\) and

\[
U_i(\mu) \geq \sum_{c \in C} \mu(c)u_i(c_i, \delta_i(c_i)), \quad \forall i \in N, \forall \delta_i : C_i \rightarrow C_i
\]

where \(U_i(\mu) = \sum_{c \in C} \mu(c)u_i(c)\).

**Proposition 1.** \(\mu \in \Delta(C)\) is a correlated equilibrium of \(\Gamma\) iff

\[
\sum_{c_{-i} \in C_{-i}} \mu(c_i)(u_i(c_i) - u_i(c_{-i}, e_i)) \geq 0, \quad \forall i \in N, \forall c_i \in C_i, \forall e_i \in C_i
\]

**Remark 2.** Conditions (1) and (2) are called the strategic incentive constraints. They guarantee that the mediator’s recommendations could be rationally obeyed by all players.

**Proof of Proposition 1.** We would like to show that conditions (1) and (2) are equivalent.
For particular \(i, c_i, e_i\), let \(\delta_i : C_i \rightarrow C_i\) be

\[
\delta_i(f_i) = \begin{cases} 
  e_i & \text{if } f_i = c_i \\
  f_i & \text{otherwise}
\end{cases}.
\]  

(3)

Then (1) becomes

\[
\sum_{f_{-i} \in C_{-i}, f_i = c_i} \mu(f)u_i(f) + \sum_{f_{-i} \in C_{-i}, f_i \neq c_i} \mu(f)u_i(f) \geq \sum_{f_{-i} \in C_{-i}, f_i = e_i} \mu(f)u_i(f_{-i}, e_i) + \sum_{f_{-i} \in C_{-i}, f_i \neq e_i} \mu(f)u_i(f)
\]  

or

\[
\sum_{f_{-i} \in C_{-i}, f_i = e_i} \mu(f)u_i(f_{-i}, e_i) \geq \sum_{f_{-i} \in C_{-i}, f_i = c_i} \mu(f)u_i(f_{-i}, c_i)
\]  

or

\[
\sum_{f_{-i} \in C_{-i}} \mu(f_{-i}, c_i)(u_i(f_{-i}, c_i) - u_i(f_{-i}, e_i)) \geq 0.
\]  

(6)

Now replace \(f_{-i}\) with \(c_{-i}\).

(1) \(\Rightarrow\) (2). Note that for \(e_i = \delta_i(c_i)\), (2) implies

\[
\sum_{c_{-i} \in C_{-i}} \mu(c)(u_i(c) - u_i(c_{-i}, \delta_i(c_i))) \geq 0
\]  

or

\[
\sum_{c_{-i} \in C_{-i}} \mu(c)u_i(c) \geq \sum_{c_{-i} \in C_{-i}} \mu(c)u_i(c_{-i}, \delta_i(c_i)).
\]  

(8)

Sum over \(c_i \in C_i\) on both sides to get (1).

Remark 3. \(\mu \in \Delta(C)\) iff it satisfies the **probability constraints**

\[
\sum_{e \in C} \mu(e) = 1 \text{ and } \mu(e) \geq 0, \quad \forall e \in C.
\]  

(9)

A vector \(\mu \in R^N\) is a correlated equilibrium iff both (1) and (9) hold. This makes the set of correlated equilibria a compact and convex set.

**Proposition 2.** The set of correlated equilibria of a finite strategic-form game is a compact and convex set.
Proof. Verify with elementary analysis.

If we’d like to find the correlated equilibria that maximize the players’ combined payoffs \( \sum_{i \in N} U_i(\mu) \) (which always exist by the continuity of \( \sum_{i \in N} U_i(\mu) \) in \( \mu \) and Weierstrass’ theorem), we would just need to subject it to the linear constraints discussed above and solve the linear program.

Example 2. Let’s revisit the example above. The problem of finding the correlated equilibria that maximize the expected sum of the players’ payoffs is

\[
\max \quad 6\mu(x_1, x_2) + 0\mu(x_1, y_2) + 8\mu(y_1, x_2) + 6\mu(y_1, y_2)
\]

s.t.

\[
(5 - 4)\mu(x_1, x_2) + (0 - 1)\mu(x_1, y_2) \geq 0 \\
(4 - 5)\mu(y_1, x_2) + (1 - 0)\mu(y_1, y_2) \geq 0 \\
(1 - 0)\mu(x_1, x_2) + (4 - 5)\mu(y_1, x_2) \geq 0 \\
(0 - 1)\mu(x_1, y_2) + (5 - 4)\mu(y_1, y_2) \geq 0
\]

\[
\mu(x_1, x_2) + \mu(x_1, y_2) + \mu(y_1, x_2) + \mu(y_1, y_2) = 1
\]

\[
\mu(x_1, x_2) \geq 0, \quad \mu(x_1, y_2) \geq 0, \quad \mu(y_1, x_2) \geq 0, \quad \mu(y_1, y_2) \geq 0
\]

Using a Lagrangean analysis (or any standard solution method for linear programs), we can find the unique solution to be

\[
\mu(x_1, x_2) = \mu(y_1, x_2) = \mu(y_1, y_2) = 1/3, \quad \mu(x_1, y_2) = 0.
\]

with payoffs \((10/3, 10/3)\).

2 The Revelation Principle

The reason for focusing our discussion on centralized mediation among other possible communication schemes is that such mediation can simulate any equilibrium of any game that can be generated from any given finite strategic-form game by adding any preplay communication system.

Given a finite strategic-game \( \Gamma \) as before, Let \( R_i \) denote the set of all possible reports that player \( i \) can send out to the communication system, and \( M_i \) all possible messages that player \( i \) may receive from the system. Note that any “report” \( r_i \in R_i \) can represent a long-winded and complex communication strategy so this is a general formulation. Given \( r \in R = \times_{i \in N} R_i \) and \( m \in M = \times_{i \in N} M_i \), let \( \nu : R \rightarrow \Delta(M) \) be the function such that \( \nu(m|r) \) is the probability that \( m \) would be the messages received by the players when \( r \) is their collective reports.
Given such a communication system, the set of pure strategies is

$$B_i = \{(r_i, \delta_i) | r_i \in R_i, \delta_i : M_i \rightarrow C_i\}$$  \hspace{1cm} (18)

and player $i$’s payoff $\bar{u}$ is given by

$$\bar{u}_i((r_j, \delta_j)_{j \in N}) = \sum_{m \in M} \nu(m| r) u_i((\delta_j(m_j))_{j \in N}).$$  \hspace{1cm} (19)

**Definition 2** (communication game). Given a finite strategic-form game $\Gamma$ and a communication system $\nu : R \rightarrow \Delta(M)$, the communication game $\Gamma_\nu$ is given by

$$\Gamma_\nu = (N, (B_i)_{i \in N}, (\bar{u}_i)_{i \in N})$$  \hspace{1cm} (20)

It is clear that a correlated equilibrium $\mu$ of $\Gamma$ can be represented by an equilibrium of the augmented communication game $\Gamma_\nu$, since the type of mediation we described is a kind of preplay communication that can be describe by setting $M$ to be $C$ and $\nu(c|r) = \mu(c), \forall r \in R$. The players will simply adopt the strategies $\delta_i(m_i) = m_i, \forall m_i \in M_i$. The revelation principle states that the converse is also true.

**Proposition 3** (revelation principle). Let $\sigma = (\sigma_i)_{i \in N}$, where $\sigma_i \in \Delta(B_i), \forall i \in N$, be an equilibrium of the communication game $\Gamma_\nu$ generated from a finite strategic-form game $\Gamma = (N, (C_i)_{i \in N}, (u_i)_{i \in N})$ by adding a preplay communication system $\nu$, then $\mu \in \Delta(C)$ defined by

$$\mu(c) = \sum_{(r, \delta) \in B} \left(\prod_{i \in N} \sigma_i(r_i, \delta_i)\right) \sum_{m \in \delta^{-1}(c)} \nu(m|r), \quad \forall c \in C$$  \hspace{1cm} (21)

where $B = \times_{i \in N} B_i$, is a correlated equilibrium of $\Gamma$.

**Remark 4.** The correlated strategy profile $\mu$ defined as such assigns probability to each $c \in C$ equal to that assigned by $\sigma$ in the communication game, and consequently the two equilibria gives the same expected payoff to each player. The two equilibria are “equivalent” in these senses.

**Remark 5.** The proof is algebraically involved, but the reasoning behind is intuitively clear: If $\mu$ is not a correlated equilibrium of $\Gamma$, and there exists $i \in N$ and $e_i \in C_i$ such that condition (2) is violated, then by changing player $i$’s strategy in $\sigma$ to choosing $e_i$ whenever her own strategy $\sigma_i$ prescribes $c_i$, player $i$ can attain a higher payoff. Hence $\sigma$ cannot be an equilibrium of the communication game $\Gamma_\nu$.

For any Nash equilibrium $\bar{\sigma}$ of the original game $\Gamma$, there exists an equilibrium $\sigma$ of the communication game $\Gamma_\nu$, where each player disregards the report that she sends or the message that she receives from the preplay communication, and chooses a strategy according to $\bar{\sigma}$. Therefore the equilibria of the communication game $\Gamma_\nu$ (or, equivalently,
the correlated equilibria of \( \Gamma \) include all equilibria of \( \Gamma \). Such equilibria of \( \Gamma_\nu \), where all communication are treated as meaningless and ignored, are called **babbling equilibria**.

The set of correlated equilibria, closed, convex, and characterizable by a system of linear inequalities, is usually more analytically tractable than the equilibria of a specific communication game. This demonstrates the power the revelation principle, and of abstracting from the communication structure of games and incorporate the communication into solution concepts, as opposed to modeling all possible communication explicitly. (There is a series of revelation principles addressing similar problems in different settings, particularly in the mechanism design literature, some of which you will encounter later in this course or in future studies.)

**References**