Heegaard Floer homology

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Joint with/work of Sarkar, Lipshitz, Manolescu, Ozsváth, Szabó

http://www.math.columbia.edu/~dpt/speaking

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Outline

► Introduction

Computing $HFK$

Variants of Heegaard Floer homology

Making Heegaard Floer homology more computable

   Method 1: Nice diagrams
   Method 2: Surgery
   Method 3: Cut the 3-manifold

Further questions
There are multiple smooth structures on the same topological 4-manifold.

Several theories give 4-manifold invariants to detect this:\textsuperscript{1}

Donaldson theory
\[ \leftrightarrow \text{(conj. Seiberg–Witten '94)} \]
Monopole Floer homology (Seiberg-Witten)
\[ \cong \text{(Taubes '08)} \]
Embedded contact homology
\[ \cong \text{(Cutluhan-Lee-Taubes, Colin-Ghiggini-Honda '10)} \]
Heegaard Floer homology

We topologists only have one trick in 4 dimensions!

Monopoles: More computable version of Donaldson invariants
HF homology: More computable version of monopoles

\textsuperscript{1}Some theories/equivalences currently only work in 3 dimensions.
By contrast, there are many 3-manifold invariants. Focus on a knot $K$ in $\mathbb{R}^3$.

One of oldest: Alexander polynomial $\Delta(K)$.

Algebraic topology: Look at $H_1(\mathbb{R}^3 \setminus K)$ under deck transforms.

Skein theory: $\Delta(L_+) - \Delta(L_-) = (t^{1/2} - t^{-1/2})\Delta(L_0)$

More recent: Jones polynomial, HOMFLYPT polynomial, ...

How are 3- and 4-dimensional theories related?
Many knot invariants are one- or two-variable Laurent polynomials. Can often find a doubly- or triply-graded homology theory whose Euler characteristic is the polynomial invariant.

<table>
<thead>
<tr>
<th>Knot polynomial</th>
<th>Knot homology</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Instanton Floer (2008)</td>
</tr>
</tbody>
</table>

Passage polynomial $\Rightarrow$ homology called *categorification*. 
Heegaard Floer homology

\[
dim(\hat{\text{HFK}}(K; s)) = (K = 10_{132})
\]

Characteristics of \(\hat{\text{HFK}}\):
- Bigraded;
- Euler characteristic is Conway-Alexander polynomial \(\Delta\);
- Max grading is knot genus (so detects unknot); (Ozsváth-Szabó 2001)
- Determines knot fibration; (Ghiggini, Ni 2006)
- Defined via pseudo-holomorphic curves.

Today we will see a simple algorithm for computing \(\hat{\text{HFK}}\)...

...and one of the world’s simplest algorithms for detecting unknot!
Heegaard Floer homology

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Knot genus

Definition
A *Seifert surface* for a knot is an oriented surface embedded in space whose boundary is the knot. The *genus* of a knot is the minimal genus of any Seifert surface.

Seifert surfaces always exist. The genus of a knot is 0 iff it is the unknot.

Theorem (Neuwirth 1960)

Genus of $K \geq$ degree of Alexander-Conway polynomial

Theorem (Ozsváth-Szabó 2001)

Genus of $K = \max s$ so that $\widehat{HFK}_*(K; s) \neq 0$
Heegaard Floer homology

\[ \dim(\hat{HFK};(K;s)) : \]
\[ (K = 10_{132}) \]

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Fibered knots

Definition

A knot is *fibered* if complement is a fiber bundle of a surface over the circle.
Or: Seifert surface can be swept around to cover complement.

Theorem (Neuwirth 1960)

\( K \) is fibered \( \Rightarrow \) Alexander-Conway polynomial is monic

Theorem (Ghiggini-Ni 2006)

\( K \) is fibered \( \Leftrightarrow \) \( \widehat{HFK}_i(K; s) \) is monic w.r.t. \( s \)
\( \Leftrightarrow \) for max \( s \) so that \( \widehat{HFK}_*(K; s) \neq 0 \), \( \dim(\widehat{HFK}_*(K; s)) = 1 \)
Heegaard Floer homology

\[
\dim(\hat{HFK};(K; s)): \quad (K = 10_{132})
\]

\begin{align*}
\hspace{1cm} & \uparrow i \\
\hspace{1cm} & \downarrow \text{Maslov} \\
1 & \hspace{1cm} \downarrow 1 \hspace{1cm} 1 \hspace{1cm} 2 \hspace{1cm} 1 \hspace{1cm} 2 \\
1 & \hspace{1cm} \downarrow 1 \hspace{1cm} 1 \hspace{1cm} 2 \\
\hspace{1cm} & \downarrow \text{Alexander} \\
1 & \hspace{1cm} \downarrow 1 \hspace{1cm} 1 \hspace{1cm} 1 \hspace{1cm} -1 \hspace{1cm} 1 \\
\hspace{1cm} & \text{genus}
\end{align*}

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Setting: Grid diagrams

Grid diagram: square diagram with one X and one O per row and column.

Turn it into a knot: connect X to O in each column; O to X in each row.

Cross vertical strands over horizontal.

Grid diagrams exist: take any diagram, rotate crossings so vertical crosses over horizontal.

The knot is unchanged under cyclic rotations:
Move top segment to bottom.
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Computing the Alexander polynomial

We categorify the following formula:

\[
\begin{vmatrix}
1 & 1 & 1 & t & t & t \\
1 & 1 & t^{-1} & 1 & t & t \\
1 & t & 1 & 1 & t & t \\
1 & t & t & t & t^2 & t \\
1 & t & t & t & t & 1 \\
1 & 1 & 1 & 1 & 1 & 1
\end{vmatrix}
= \pm t^*(1 - t)^{n-1} \Delta(K; t)
\]

- Make matrix of $t^{-\text{winding \#}}$
  (with extra row/column of 1's);

- det determines the Conway-Alexander polynomial $\Delta$
  ($n =$ size of diagram; here 6)
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- Method 1: Nice diagrams
- Method 2: Surgery
- Method 3: Cut the 3-manifold

Further questions
Define a chain complex $\tilde{C}_K$ over $\mathbb{F}_2$.

- $n!$ generators: matchings between horizontal and vertical grid circles (as counted in det for Alexander).
- Boundary $\partial$ switches corners on empty rectangles:

Sum over all ways to switch SW-NE corners of an empty rectangle to NW-SE corners. (*Empty* means: no X’s, O’s, or other points in generator.)
Computing $HFK$: Chain complex $\tilde{CK}$

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Computing $HFK$: $\partial^2 = 0$

Each term in $\partial^2$ must have a mate:

- If rectangles are disjoint, take rectangles in either order.
- If rectangles share a corner, decompose the union in another way.
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Computing $HFK$: Gradings on $\tilde{\mathbb{C}K}$

In the plane,

removes one inversion.

For $A, B, C \subset \mathbb{R}^2$, \[ I(A, B) := \# \{ a \boxtimes b \mid a \in A, b \in B \} \]

\[ I(A - B, C) := I(A, C) - I(B, C) \]

For $x$ a generator, $\mathbb{X} =$ set of $X$'s, $\mathbb{O} =$ set of $O$'s, gradings are:

- **Maslov:** $M(x) := I(x - \mathbb{O}, x - \mathbb{O}) + 1$.

- **Alexander:** Sum of winding numbers around generator pts, or \[ A(x) := \frac{1}{2} (I(x - \mathbb{O}, x - \mathbb{O}) - I(x - \mathbb{X}, x - \mathbb{X}) - (n - 1)) \].
Computing $HFK$: The answer

Theorem (Manolescu-Ozsváth-Sarkar ’06)

For $G$ a grid diagram for $K$,

$$H_*(\widetilde{CK}(G)) \cong \widehat{HFK}(K) \otimes V^{\otimes n-1}$$

where $V := (\mathbb{F}_2)_{0,0} \oplus (\mathbb{F}_2)_{-1,-1}$.

(Remember the factor of $(1 - t)^{n-1}$ in determinant formula for $\Delta$.)

Gillam and Baldwin used this to compute $\widehat{HFK}$ for all knots with $\leq 11$ crossings, including new values of knot genus.
Exercises

- Find a grid diagram for the trefoil.
- Compute $\hat{HFK}$ of the unknot from the diagram below.

![Grid Diagram](attachment:grid_diagram.png)

- Show that the figure 8 knot is fibered by computing $\hat{HFK}$ of the figure 8 knot in the highest Alexander grading.
- Show that the Alexander polynomial is invariant under some moves that preserve the knot.
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Further questions
The Heegaard Floer homology package

\[ \dim \widehat{HFK}_i(K; s): \]

\[
\begin{array}{cc}
  & 1 & 1 \\
 1 & 1 & 2 \\
 2 & 1 & 2 \\
 1 & 1 & 1 \\
\end{array}
\]

HF comes in many variants. So far, we have seen the simplest, $\widehat{HF}$. Also $HF^+$, $HF^-$, and $HF^\infty$.

Thinking about variants will allow removing factors of $V \otimes n^{-1}$.

Also allows computing 4-manifold invariants.

Relation:
- $CFK^-$ is a complex over $\mathbb{F}_2[U]$
- $U$ has degree $(-1, -2)$
- Related to $\widehat{HFK}$ by Universal Coefficient Theorem (set $U$ to 0 on chains).
The Heegaard Floer homology package

\[ \dim HFK^-_i(K; s): \]

\[ \begin{array}{c}
\begin{array}{c}
1 \\
1 \\
1 \\
1 \\
1
\end{array}
\end{array} \]

\[ s \]

\[ i \]

\[ 2 \]

\[ 1 \]

\[ 1 \]

\[ 1 \]

\[ 1 \]

\[ \ldots \]

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\[ ✔️ U \text{ has degree } (-1, -2) \]

\[ ✔️ \text{Related to } HFK \text{ by Universal Coefficient Theorem (set } U \text{ to 0 on chains).} \]
Computing $HFK^-$

To compute $HFK^-$ for grid diagrams:

- Add one $U_i$ for each $O$.
- Complex $CK^-(G)$ generated by same generators over $\mathbb{F}_2[U_1, \ldots, U_n]$
- $\partial$ counts rects. that contain only $O$’s, weighted by corresponding $U_i$.

**Theorem (Manolescu-Ozsváth-Sarkar)**

$H_* (CK^-(G)) \simeq HFK^-(K)$.
Each $U_i$ acts by $U$ on the homology.

Other variants are similar, but treat $O$’s in different ways.
E.g., $CFK^\infty$ is similar, but over $\mathbb{F}_2[U, U^{-1}]$ instead of $\mathbb{F}_2[U]$. 
Further variants

Can also:

- Allow rectangles to cross $X$’s as well to get a filtered complex with trivial total homology.

- Add signs (in essentially unique way) to work over $\mathbb{Z}[U]$.

<table>
<thead>
<tr>
<th>Cross $X$’s?</th>
<th>Cross $O$’s? (count with $U_i$ variables)</th>
<th>No</th>
<th>Yes</th>
</tr>
</thead>
<tbody>
<tr>
<td>No</td>
<td>$\hat{HFK} \otimes V^{n-1}$</td>
<td></td>
<td>Filtered version</td>
</tr>
<tr>
<td>All but 1</td>
<td>$\hat{HFK}$</td>
<td></td>
<td>”</td>
</tr>
<tr>
<td>Yes</td>
<td>$\hat{HFK}^-$</td>
<td></td>
<td>”</td>
</tr>
</tbody>
</table>
Exercises

▶ Check that $\partial^2$ is still 0 in the version where rectangles can cross O’s.

▶ Compute $HFK^-$ of the unknot from the same diagram as before.
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► Making Heegaard Floer homology more computable
  
  Method 1: Nice diagrams
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Further questions
Rest of the theory?

Grid diagrams can compute Heegaard Floer homology for knots.

Problem

_Inefficient: basis of chain complex grows like n!_

But what about 3- and 4-manifolds? How to compute for those?
Heegaard diagrams

The grid diagram technique can be generalized.

Definition

A Heegaard diagram $\mathcal{H}$ is a surface $\Sigma$, with two sets of $k$ curves $\{\alpha_i\}$ and $\{\beta_i\}$, with each set of curves not intersecting itself and homologically independent. Heegaard diagrams represent 3-manifolds.

Heegaard Floer homology is defined by counting pseudo-holomorphic curves related to $\mathcal{H}$. Either:

- Count holomorphic disks in $\text{Sym}^k(\Sigma)$ or
- Count holomorphic curves in $\Sigma \times [0, 1] \times \mathbb{R}$.

Boundary conditions given by the $\{\alpha_i\}$ and $\{\beta_i\}$. 
Method 1: Nice diagrams

Definition

A *nice Heegaard diagram* is one in which all regions, except for one, have 2 or 4 sides.

Theorem (Sarkar-Wang ’06)

*Nice Heegaard diagrams exist for any 3-manifold.*

In nice diagrams, holomorphic curve counts are easy: Count empty bigons, rectangles.

Problem

*Nice diagrams are huge! Cannot handle 4-manifolds.*
Method 2: Surgery approach

Every 3-manifold is \textit{surgery} on a link: cut out a neighborhood of the link and reglue another way.

Manolescu-Ozsváth, Manolescu-Ozsváth-T: the effect of surgery can be computed using grid diagram techniques. Also compute 4-manifold invariants this way.

Problem

\textit{Complicated, inefficient.}
Method 3: Bordered Floer homology

Another approach: Cut 3-manifold into simpler pieces along surfaces. (Lipshitz-Ozsváth-T, ongoing)

Want: Extend $HF$ as a TQFT down a dimension

<table>
<thead>
<tr>
<th>Geometry</th>
<th>Algebra</th>
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<tbody>
<tr>
<td>Closed 4-manifold $W^4$</td>
<td>Invariant $HF(W, s)$</td>
</tr>
<tr>
<td>3-manifold $Y^3$</td>
<td>Homology $HF(Y, s)$</td>
</tr>
<tr>
<td>4-manifold w/ $\partial W^4 = Y$</td>
<td>$HF(W) \in HF(Y)$</td>
</tr>
<tr>
<td>Surface $F$</td>
<td>Algebra $A(F)$</td>
</tr>
<tr>
<td>3-manifold w/ $\partial Y = F$</td>
<td>Module $CF(Y)$ over $A(F)$</td>
</tr>
<tr>
<td>$Y = Y_1 \cup_F Y_2$</td>
<td>$CF(Y) \cong CF(Y_1) \otimes_{A(F)} CF(Y_2)$</td>
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</table>

Benefits:

- Computability (theoretical)
- Computability (practical)
- Axioms
A *grid diagram* represents a knot.

A planar diagram $P$ is a square grid with blocks. Do not identify sides.

Chain complex $CF(P)$:

- Generators given by permutations
- Differential counts empty rectangles (*not* wrapping)
- Not an invariant of anything
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Splitting planar diagrams

Want to split planar diagrams in two: $P = P_1 \cup P_2$.

Associate modules $CPA(P_1), CPD(P_2)$

$$CP(P) = CPA(P_1) \otimes CPD(P_2)$$

$CPA(P_1)$ is a right differential module. Interactions on boundary encoded in algebra action.

$CPD(P_2)$ is a left, projective module. Interactions on boundary encoded in differential.

Tensor product is nice since $CPD(P_2)$ is projective.
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Naive algebra $\tilde{A}(n, k)$

Defining a (naive) version of strands algebra $\tilde{A}(n, k)$ (really a category).

- Idempotents (objects):
  - $k$-element subsets $S \subset \{1, \ldots, n\}$
- Elements (morphisms):
  - $\text{Mor}(S, T)$ spanned by $\phi : S \xrightarrow{\sim} T$, $\phi(i) \geq i$
- Product: composition
- Differential: sum over smoothings of crossings
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- **Differential:** sum over smoothings of crossings
Strands algebra $\mathcal{A}(n, k)$

$\tilde{\mathcal{A}}(n, k)$ has a filtration by number of crossings.
$\mathcal{A}(n, k)$ is the associated graded algebra.

- Product is 0 if it introduces a double-crossing.
- Differential: likewise.
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- Differential: likewise.
Strands algebra $\mathcal{A}(n, k)$

$\partial \left[ \begin{array}{c} \bullet \bullet \bullet \\ \bullet \bullet \bullet \\ \bullet \bullet \bullet \\ \bullet \bullet \bullet \\ \bullet \bullet \bullet \end{array} \right] = \left[ \begin{array}{c} \bullet \bullet \bullet \\ \bullet \bullet \bullet \\ \bullet \bullet \bullet \\ \bullet \bullet \bullet \\ \bullet \bullet \bullet \end{array} \right] + \left[ \begin{array}{c} \bullet \bullet \bullet \\ \bullet \bullet \bullet \\ \bullet \bullet \bullet \\ \bullet \bullet \bullet \\ \bullet \bullet \bullet \end{array} \right]$

$\tilde{\mathcal{A}}(n, k)$ has a filtration by number of crossings. $\mathcal{A}(n, k)$ is the associated graded algebra.

- Product is 0 if it introduces a double-crossing.
- Differential: likewise.

▶ Product is 0 if it introduces a double-crossing.
▶ Differential: likewise.
Outline

Introduction

Computing $HFK$

Variants of Heegaard Floer homology

Making Heegaard Floer homology more computable

  Method 1: Nice diagrams
  Method 2: Surgery
  Method 3: Cut the 3-manifold

▶ Further questions
Further questions

There are many questions left!

- Combinatorial proofs of genus equality and other nice properties?
- Construct a topological space with this homology?
- Manifold version of other link homologies?
- Make entire theory computable?
- Do computations!
Appendix: Crossing number vs. Grid number

Knots are usually ordered by *crossing number*: Minimum number of crossings in a planar diagram.

For grid diagrams, natural to consider *grid number* (or *arc index*): Minimum size of a grid diagram.

**Theorem (Bae–Park, Morton–Beltrami)**

*Grid number of an alternating knot is equal to crossing number + 2. For non-alternating knots, grid number strictly less.*