Heegaard Floer Homology
Lecture 3: Structure of bordered HF homology

Dylan Thurston

Joint with Robert Lipshitz, Peter Ozsváth

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http://www.math.columbia.edu/~dpt/speaking

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Outline

- Surfaces
  - Modules
  - Bimodules
  - Computability
  - 4-manifolds
## Overview

<table>
<thead>
<tr>
<th>Geometry</th>
<th>Algebra</th>
</tr>
</thead>
<tbody>
<tr>
<td>Closed 4-manifold $W^4$</td>
<td>Invariant $HF(W)$</td>
</tr>
<tr>
<td>3-manifold $Y^3$</td>
<td>Homology $HF(Y)$</td>
</tr>
<tr>
<td>Surface $F$</td>
<td>Algebra $A(F)$</td>
</tr>
<tr>
<td>3-manifold w/ $\partial Y = F$</td>
<td>$A_\infty$ module $\widehat{CFA}(Y)$</td>
</tr>
<tr>
<td>Gluing $Y = Y_1 \cup_F Y_2$</td>
<td>Projective module $\widehat{CFD}(Y)$</td>
</tr>
<tr>
<td>3-manifold w/ $\partial Y^3 = F_1 \cup F_2$</td>
<td>$\widehat{CF}(Y) \simeq \widehat{CFA}(Y_1) \widehat{\otimes}_{A(F)} \widehat{CFD}(Y_2)$</td>
</tr>
<tr>
<td>Cobordism $\partial W^4 = -Y_1 \cup Y_2$</td>
<td>Bimodules $\widehat{CFDA}(Y)$, ...</td>
</tr>
<tr>
<td>Map $\widehat{HF}(W) : \widehat{HF}(Y_1) \to \widehat{HF}(Y_2)$</td>
<td></td>
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*Overview*
A handle decomposition of a surface $F$ is a way of writing $F$ as

- a disk $D_0$,
- some handles attached to $\partial D_0$, and
- a disk $D_2$ attached to remaining boundary.

We also fix a basepoint on $\partial D_0$.

Definition

A pointed matched circle $Z$ of genus $g$ is a pairing of the points $\{1, \ldots, 4g\} \subset [0, 4g + 1]$ so that surgery on pairs of matched points yields a connected interval.
Strands algebra $\mathcal{A}(n, k)$

Recall strands algebra $\mathcal{A}(n, k)$ (really a category).

- Idempotents (objects): $k$-element subsets $S \subset \{1, \ldots, n\}$
- Elements (morphisms): $\text{Mor}(S, T)$ spanned by $\phi : S \xrightarrow{\sim} T$, $\phi(i) \geq i$
- Product: composition
- Differential: sum over smoothings of crossings
- Set double-crossings to 0.
For a pointed matched circle \( \mathcal{Z} \) of genus \( g \), we define \( \mathcal{A}(\mathcal{Z}) \subset \mathcal{A}(4g) = \bigoplus_k \mathcal{A}(4g, k) \).

\( \mathcal{A}(\mathcal{Z}) \) is the subalgebra of sums of diagrams in which, if a diagram with a horizontal strand appears, the diagram with the horizontal strand at the matching position appears with equal weight.

The idempotents of \( \mathcal{A}(\mathcal{Z}) \) correspond to subsets of matched pairs \( (2^{2g} \text{ in all rather than } 2^{4g}) \).

\( \mathcal{A}(\mathcal{Z}, i) \) is the subalgebra with \( g + i \) strands.
Properties of $\mathcal{A}(\mathcal{Z})$

- $\mathcal{A}(\mathcal{Z}, -g) \cong \mathbb{F}_2$.

- $\mathcal{A}(\mathcal{Z}, -g + 1)$ has no differential, and is a quiver algebra.

- For $\mathcal{Z}$, $\mathcal{Z}'$ of same genus, $\mathcal{A}(\mathcal{Z}) \not\cong \mathcal{A}(\mathcal{Z}')$. Derived categories are isomorphic.

- $\mathcal{A}(\mathcal{Z})$ is not $\mathbb{Z}$ graded. It is $G$-graded for a non-commutative group $G$.

- $\mathcal{A}(\mathcal{Z}, i) \cong \mathcal{A}(-\mathcal{Z}, i)^{\text{op}}$.

- $\mathcal{A}(\mathcal{Z}, i)$ is Koszul dual to $\mathcal{A}(\mathcal{Z}, -i)$.

- $\mathcal{A}(\mathcal{Z}, i) \cong \mathcal{A}(\mathcal{Z}^*, -i)$. ($\mathcal{Z}^*$ is the dual pointed matched circle.)

- $\mathcal{A}(\mathcal{Z})$ is derived equivalent to the “partially wrapped Fukaya category” of $F$ [Auroux].
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$\mathcal{A}(\mathcal{Z})$ is not $\mathbb{Z}$-graded

\begin{align*}
x &= \begin{tikzpicture}[baseline] 
\begin{scope}[scale=0.5]
\draw[dashed] (0,0) .. controls (1,1) and (1,-1) .. (0,0);
\draw (0,0) .. controls (-1,1) and (-1,-1) .. (0,0);
\end{scope}
\end{tikzpicture} \\
\end{align*}

\begin{align*}
y &= \begin{tikzpicture}[baseline] 
\begin{scope}[scale=0.5]
\draw[dashed] (0,0) .. controls (1,1) and (1,-1) .. (0,0);
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\begin{align*}
x \cdot y &= \begin{tikzpicture}[baseline] 
\begin{scope}[scale=0.5]
\draw[dashed] (0,0) .. controls (1,1) and (1,-1) .. (0,0);
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\begin{align*}
\partial y &= \begin{tikzpicture}[baseline] 
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\begin{align*}
(\partial y) \cdot x &= \begin{tikzpicture}[baseline] 
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\end{align*}

\begin{align*}
x \cdot y &= \partial((\partial y) \cdot x)
\end{align*}
Aside: Non-commutative gradings

Let $G$ be a group, possibly non-commutative.

An algebra $A$ is $G$-graded if we have a decomposition

$$A = \bigoplus_{g \in G} A_g$$

so that

$$A_g \cdot A_h \subseteq A_{gh}.$$ 

For $\lambda \in G$ a fixed central element, a differential algebra is $G$-graded if in addition

$$\partial(A_g) \subseteq A_{\lambda^{-1}g}.$$ 

$\mathcal{A}(\mathcal{Z})$ is graded by

$$\begin{cases} 
\text{a canonical } \mathbb{Z} \text{ central extension of } H_1(F) \\
\text{vector fields on } F \times [0, 1]/\text{isotopy rel } \partial.
\end{cases}$$
Properties of $\mathcal{A}(\mathcal{Z})$

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Surfaces

- Modules

  Bimodules

Computability

4-manifolds
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<table>
<thead>
<tr>
<th>Geometry</th>
<th>Algebra</th>
</tr>
</thead>
<tbody>
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Type $D$ modules

Pointed matched circle $\mathcal{Z} \rightsquigarrow$ Surface $F(\mathcal{Z})$ with marked disk $D_0$

A bordered 3-manifold is a 3-manifold $Y$ with a homeomorphism $\phi : F(\mathcal{Z}) \to \partial Y$, defined up to isotopy rel $D_0$.

Corresponding notion of bordered Heegaard diagrams $\mathcal{H}$, with boundary $\partial \mathcal{H}$ a pointed matched circle.

For $\mathcal{H}$ a bordered Heegaard diagram, $\widehat{\text{CFD}}(\mathcal{H})$ is a left, projective module over $A(-\partial \mathcal{H}, 0)$.

**Theorem**

If $\mathcal{H}_1$ and $\mathcal{H}_2$ represent the same bordered 3-manifold,

$$\widehat{\text{CFD}}(\mathcal{H}_1) \simeq \widehat{\text{CFD}}(\mathcal{H}_2).$$

Can therefore write $\widehat{\text{CFD}}(Y)$ for $Y$ a bordered 3-manifold.
Type A modules

For $\mathcal{H}$ a bordered Heegaard diagram, $\widehat{\text{CFA}}(\mathcal{H})$ is a right, $A_\infty$ module over $A(\partial \mathcal{H}, 0)$.

**Theorem**

If $\mathcal{H}_1$ and $\mathcal{H}_2$ represent the same bordered 3-manifold,

$$\widehat{\text{CFA}}(\mathcal{H}_1) \simeq \widehat{\text{CFA}}(\mathcal{H}_2).$$

A nice (based) Heegaard diagram is one in which every region (except for the one containing the basepoint) is a square or bigon.

**Theorem (Sarkar-Wang)**

Any 3-manifold has a nice diagram.

**Lemma**

If $\mathcal{H}$ is nice, then $\widehat{\text{CFA}}(\mathcal{H})$ is a differential module over $A(\mathcal{Z})$. 
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Pairing theorems

Theorem

If $Y_1$ is bordered by $F(Z)$ and $Y_2$ is bordered by $-F(Z)$, then

$$\hat{CF}(Y_1 \cup \partial Y_2) \simeq \hat{CFA}(Y_1) \hat{\otimes}_{A(Z)} \hat{CFD}(Y_2).$$

Theorem

With $Y_1$, $Y_2$ as above,

$$\hat{CF}(Y_1 \cup \partial Y_2) \simeq \text{Mor}_{A(Z)}(\hat{CFA}(-Y_2), \hat{CFA}(Y_1))$$
$$\simeq \text{Mor}_{A(-Z)}(\hat{CFD}(-Y_2), \hat{CFD}(Y_1))$$

($\text{Mor}(M, N)$ is a chain complex whose homology is $\text{Ext}(M, N)$.)
Dualities

The two pairing theorems are related by dualities between $\widehat{CFA}$ and $\widehat{CFD}$.

**Theorem**

For $Y$ bordered by $F(Z)$,

\[
\text{Mor}_{A(-Z)}(\widehat{CFD}(Y), A(-Z)) \simeq \widehat{CFA}(-Y)
\]
\[
\text{Mor}_{A(Z)}(\widehat{CFA}(Y), A(Z)) \simeq \widehat{CFD}(-Y).
\]

**Theorem (Suggested by Auroux)**

For $Y$ bordered by $F(Z)$,

\[
\widehat{CFA}(Y, \overline{s}) \simeq \widehat{CFD}(Y, \overline{s}).
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Bimodules of various types

An arced, bordered 3-manifold is a 3-manifold $Y$ with two boundary components, parametrized by $F(Z_1)$ and $F(Z_2)$, together with a framed arc connecting the base disks in the two boundary components.

For a Heegaard diagram $\mathcal{H}$ representing an arced, bordered 3-manifold, there are bimodules of various types:

$$\widehat{\text{CFAA}}(\mathcal{H}) A(Z_1,i), A(Z_2,-i),$$

$$A(-Z_1,i) \widehat{\text{CFDA}}(\mathcal{H}) A(Z_2,i),$$

$$A(-Z_1,i), A(-Z_2,-i) \widehat{\text{CFDD}}(\mathcal{H}).$$

Theorem

For $\mathcal{H}$ representing an arced, bordered 3-manifold $Y$, the bimodules $\widehat{\text{CFAA}}(\mathcal{H})$, $\widehat{\text{CFDA}}(\mathcal{H})$, $\widehat{\text{CFDD}}(\mathcal{H})$ are invariants up to quasi-isomorphism of $Y$. 
Pairing theorems

Theorem

We can glue (arced) bordered 3-manifold (bi)modules in any way that matches an A with a D.
For instance, if $\partial Y_1 = F(Z_1) \cup F(Z_2)$, $\partial Y_2 = -F(Z_2)$,

$$\widehat{CFD}(Y_1 \cup_{F(Z_2)} Y_2) = \widehat{CFDA}(Y_1) \otimes_{A(Z_2)} \widehat{CFD}(Y_2).$$

If $\partial Y_3 = -F(Z_2) \cup F(Z_3)$,

$$\widehat{CFDA}(Y_1 \cup_{F(Z_2)} Y_3) = \widehat{CFDA}(Y_1) \otimes_{A(Z_2)} \widehat{CFDA}(Y_3).$$

There are also Hom-pairing and duality theorems for bimodules. Some of these involve a boundary Dehn twist $\tau_\partial$, which is the Serre functor in $\mathcal{A}(Z)$-Mod.

Theorem

$$\text{Mor}_{\mathcal{A}(Z)}(N, M \otimes \widehat{CFDA}(\tau_\partial)) \simeq \text{Mor}_{\mathcal{A}(Z)}(M, N)^*. $$
Mapping class group

As a special case, we can consider \( Y = [0, 1] \times F \), with the two boundaries possibly parametrized differently.

**Theorem**

\[ \text{CFDA}([0, 1] \times F(\mathcal{Z})) \cong \mathcal{A}(\mathcal{Z}), \text{ with both boundaries parametrized by identity.} \]

**Corollary**

*There is a compositional map from the strongly based mapping class group of \( F(\mathcal{Z}) \) to \( \mathcal{A}(\mathcal{Z}) \) bimodules, so \( \text{MCG}_0(F(\mathcal{Z})) \) acts (weakly) on \( \mathcal{A}(\mathcal{Z})\)-Mod.*

**Corollary**

*If \( \mathcal{Z} \) and \( \mathcal{Z}' \) have the same genus, \( \mathcal{A}(\mathcal{Z}) \) and \( \mathcal{A}(\mathcal{Z}') \) are derived equivalent.*
Theorem

The mapping class group action of a surface of genus $g$ on $A(\mathcal{Z}, -g + 1)$-$\text{Mod}$ is faithful.

(Recall $A(\mathcal{Z}, -g) = \mathbb{F}_2$, and $A(\mathcal{Z}, -g + 1)$ has no differential.)

Inspired by Seidel-Thomas ’00: Rank of homology of bimodule counts intersections.

Conjecture

There is a faithful linear representation of the mapping class group.

Unfortunately, our representation presumably decategorifies to a relative for surfaces of Burau representation, which is not faithful.
## Overview

<table>
<thead>
<tr>
<th>Geometry</th>
<th>Algebra</th>
</tr>
</thead>
<tbody>
<tr>
<td>Closed 4-manifold $W^4$</td>
<td>Invariant $HF(W)$</td>
</tr>
<tr>
<td>3-manifold $Y^3$</td>
<td>Homology $HF(Y)$</td>
</tr>
<tr>
<td>$F$</td>
<td>Algebra $\mathcal{A}(F)$</td>
</tr>
<tr>
<td>3-manifold w/ $\partial Y = F$</td>
<td>$\mathcal{A}_\infty$ module $\widehat{CFA}(Y)$</td>
</tr>
<tr>
<td>Gluing $Y = Y_1 \cup_F Y_2$</td>
<td>Projective module $\widehat{CFD}(Y)$</td>
</tr>
<tr>
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</tr>
<tr>
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Hochschild (co)homology

Hochschild homology in a TQFT is typically related to self-gluing. Idea:

\[ HH_* (A M_A) = H_* (\text{Tor}_{A \otimes A^{op}} (AA_A, A M_A)) \]

For \( Y \) an arced, bordered 3-manifold param. by \( \mathcal{Z} \) and \(-\mathcal{Z}\). The open book decomposition \((Y^\circ, K)\) is obtained by gluing the two boundary components, doing surgery on framed knot coming from the framed arc in \( Y \), retaining the core \( K \) of the surgery.

Theorem

For \( Y \) an arced, bordered 3-mfld as above,

\[ HH_* (\widehat{\text{CFDA}}(Y)) \simeq \widehat{\text{CFK}}(Y^\circ, K). \]
Outline

Surfaces

Modules

Bimodules

- Computability

4-manifolds
Everything is entirely computable! Especially the $\mathcal{A}_\infty$ modules.

$$\sum_i \dim(H_*(\mathcal{A}(\mathcal{Z}, i))) =$$

$$\begin{cases} T^{-2} + 32 T^{-1} + 98 + 32 T + T^2 & \mathcal{Z} \text{ split, genus 2} \\ T^{-2} + 32 T^{-1} + 70 + 32 T + T^2 & \mathcal{Z} \text{ antipodal, genus 2} \end{cases}$$

In a genus 3 example, get dimension 1224 in middle dimension.

Computing pairings by computing Hom-spaces multiplies by dimension of the algebra.

Computing pairing $\widehat{\text{CFA}} \otimes \widehat{\text{CFD}}$ has a model which does not increase rank at all.

For $\mathcal{A}_\infty$ modules allows passing to homology.
Computing in practice


Enough to find invariants of

- handlebodies and
  - generators of mapping class group(oid).

This can be done.

Computations can be done in practice for genus 2.
Outline

Surfaces

Modules

Bimodules

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» 4-manifolds
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Computing 4-manifold invariants

One approach to computing 4-manifold invariants: for $T_0$ and $T_\infty$ the 0-framed and $\infty$-framed solid torus, respectively, compute a cobordism

$$\widehat{HF}(D^4) : \widehat{CFD}(T_0) \to \widehat{CFD}(T_\infty).$$

This works.

Easier approach: Use composition map

$$\text{Mor}(\widehat{CFD}(Y_2), \widehat{CFD}(Y_3)) \otimes \text{Mor}(\widehat{CFD}(Y_1), \widehat{CFD}(Y_2)) \to \text{Mor}(\widehat{CFD}(Y_1), \widehat{CFD}(Y_3))$$

where $Y_1$, $Y_2$, $Y_3$ all parametrized by $\mathcal{Z}$. This is a map

$$\widehat{CF}(-Y_2 \cup \partial Y_3) \otimes \widehat{CF}(-Y_1 \cup \partial Y_2) \to \widehat{CF}(-Y_1 \cup \partial Y_3).$$

**Theorem**

This is the cobordism map for a pair-of-pants cobordism.