Heegaard Floer Homology
Lecture 2: Bordered HF homology, a toy model

Dylan Thurston
Joint with Robert Lipshitz, Peter Ozsváth

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http://www.math.columbia.edu/~dpt/speaking

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Outline

- **TQFT setup**
  
  Splitting diagrams
  
  The algebra
  
  Engineering $\mathcal{A}$
Extending down a dimension

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<th>Geometry</th>
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<td>4-manifold w/ $\partial W^4 = Y$</td>
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<td>Surface $F$</td>
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<td>3-manifold w/ $\partial Y = F$</td>
<td>Module $CF(Y)$ over $A(F)$</td>
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<td>$Y = Y_1 \cup_F Y_2$</td>
<td>$CF(Y) \simeq CF(Y_1) \otimes_{A(F)} CF(Y_2)$</td>
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Benefits:
- Computability (theoretical)
- Computability (practical)
- Axioms
Extending down a dimension

Have: Extend $HF$ as a TQFT down a dimension

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$\widehat{CF}(Y) \simeq \widehat{CFA}(Y_1) \widehat{\otimes}_{A(F)} \widehat{CFD}(Y_2)$

Notes:
- Two versions of modules, left and right actions
- Disconnected surfaces (etc.) behave differently
- Only $\widehat{HF}$ so far
- Everything graded
Decategorifying

What happens when we decategorify?

Invariants $HF(W^4)$ $\rightarrow$ $\emptyset$

Homology $HF(Y^3)$ $\rightarrow$ Invariant

Algebra $\mathcal{A}(F^2)$ $\rightarrow$ Abelian group

Module $CF(Y^3)$ $\rightarrow$ Elt. of Grothendieck group

$CF(Y_1) \otimes_{\mathcal{A}(F)} CF(Y_2)$ $\rightarrow$ Pairing of Grothendieck groups

Right hand side looks like a 3D TQFT. (Presumably this is like a Reshetikhin-Turaev TQFT, but this has not been worked out.)

Slogan

*Categorification is going up a dimension.*
Comparison: Reshetikhin-Turaev categorification

- Higher-dimensional invariants are better understood.
- Still looking for convincing algebraic/geometric background. Don’t have anything like Grassmannians.
- We need differentials in the algebra. Homological direction is built in.
- Best developed for surfaces/3-manifolds (rather than tangles).
- Related to $GL(1|1)$ supergroup. This may cause some of the difficulties above.
- Tensor product of representations would be another level of structure: connect sum of surfaces.
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- TQFT setup
  - Splitting diagrams
- The algebra
- Engineering $\mathcal{A}$
Splitting Heegaard diagrams

Recall that $HF$ homology is defined based on Heegaard diagrams.

Split the diagram into two by stretching along a neck intersecting only $\alpha$ circles.

Need to keep track of differentials that cross the boundary.
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Need to keep track of differentials that cross the boundary.
A grid diagram represents a knot. A planar diagram $P$ is a square grid with blocks. Do not identify sides. Chain complex $CF(P)$:
- Generators given by permutations
- Differential counts empty rectangles (not wrapping)
- Not an invariant of anything
**Toy model: Planar diagrams**

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Want to split planar diagrams in two: \( P = P_1 \cup P_2 \).

Associate modules \( CPA(P_1) \), \( CPD(P_2) \)

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CP(P) = CPA(P_1) \otimes CPD(P_2)
\]

\( CPA(P_1) \) is a right differential module. Interactions on boundary encoded in algebra action.

\( CPD(P_2) \) is a left, projective module. Interactions on boundary encoded in differential.

Tensor product is nice since \( CPD(P_2) \) is projective.
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TQFT setup

Splitting diagrams

- The algebra

Engineering $\mathcal{A}$
Defining a (naive) version of strands algebra $\tilde{A}(n, k)$ (really a category).

- **Objects (idempotents):**
  - $k$-element subsets $S \subseteq \{1, \ldots, n\}$
- **Elements (morphisms):**
  - $\text{Mor}(S, T)$ spanned by $\phi : S \to T$, $\phi(i) \geq i$
- **Product:** composition
- **Differential:** sum over smoothings of crossings

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Properties of $\mathcal{A}$

$\mathcal{A}(n, k)$ is ...

- finite-dimensional;
- Koszul dual to $\mathcal{A}(n, n - k)$ (in suitable sense); and
- very simple when $k = 1$.

The actual algebra $\mathcal{A}(F)$ (for non-toy model) is ...

- an idempotent restriction of $\mathcal{A}(n, k)$,
- incorporates structure of the surface,
- a quiver algebra when $k = 1$, and
- not formal in general (not equivalent to its homology).
Outline

- TQFT setup
- Splitting diagrams
- The algebra
  - Engineering $A$
A generator of a partial planar diagram is a subset $x$ of the intersections with
- one element per vertical strand;
- at most one element per horizontal strand.
To form a complete generator, the set of occupied horizontal strands must be complementary on left and right.
For $x$ a generator, define
- $I_A(x) =$ set of horizontal strands occupied in $x$
- $I_D(x) =$ complement of $I_A(x)$
So idempotents in $\mathcal{A}$ correspond to subsets of $\{1, \ldots, n\}$, and $I_A$ and $I_D$ are the idempotents of generators in $CPA$ and $CPD$. 
From rectangles to chords

Must arrange that $CPA(P_1) \otimes CPD(P_2)$ counts all empty rectangles in $P$.

For rects on left or right, include a term in differential on appropriate side. For rects crossing boundary, need the algebra.

We have an algebra element associated to possible intersections of rectangles with the boundary. $CPD(P_2)$ is a projective module generated over $\mathcal{A}$ by generators. Differential counts half-rectangles. $CPA(P_1)$ is a module generated over $\mathbb{F}_2$ by generators. Algebra action counts half-rectangles. Combination counts rectangles across boundary in tensor product.
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- always true for $CPA(P_1)$;
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Relation 1: commutativity

From disjoint rectangles, get commutativity of disjoint chords.
From L-shaped region, get composition of strands.
From L-shaped region facing other way, get a differential.
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