1. We’ve seen that $S^1$ is a $K(\mathbb{Z}, 1)$, and so homotopy classes of maps from a CW complex $X$ to $S^1$ are in bijection with elements of the singular cohomology $H^1(X, \mathbb{Z})$. Give a more direct proof of this fact.

2. Give an example of a space $X$ for which the above bijection is not true. You may find it useful to think about the topologist’s sine curve, the graph of the function $\sin(1/x)$ for $x \in [0, \infty)$, and build from there.

3. Show that $\pi_1(SL(n)) \cong \pi_1(SO(n)) \cong \mathbb{Z}/2$ for $n \geq 3$. (Use the fact that homotopy groups of $SO(n)$ stabilize.)

4. For an $n$-dimensional oriented real vector bundle $\eta$ over a space $X$, consider the space of oriented orthonormal frames $OV(\eta)$ as a fiber bundle over $X$, with fiber $SO(n)$. Show that in the Serre spectral sequence of this fibration with $\mathbb{Z}/2$ coefficients, the map from $H_2(X; \mathbb{Z}/2)$ to $H_1(SO(n); \mathbb{Z}/2)$ is dual to the second Stiefel-Whitney class $w_2$.

5. The double cover of $SO(n)$ is called $Spin(n)$, and an oriented vector bundle $\eta$ is said to have a spin structure if the frame bundle $OV(\eta)$ has a double cover which is non-trivial on the fiber (so that the fiber is $Spin(n)$). Show that $\eta$ has a spin structure iff $w_2(\eta) = 0$. (Compare to the earlier result that $\eta$ is orientable iff $w_1(\eta) = 0$.)

6. Recall that a Moore space $M(G, n)$ is a space whose only non-trivial reduced homology group is $G$ in dimension $n$. Define $\mu_n(X; G)$ to be based homotopy classes of maps from $M(G, n)$ to $X$. (These are used in the construction of homotopy groups with coefficients.) Show that $\mu_n(X; G)$ has a natural group structure for $n > 1$, and is Abelian for $n > 2$. 