To receive full credit, you must explain your answers.
No calculators of any type are allowed.

(1) Let $P_1, P_2, P_3$ be the planes defined by the equations below.

$P_1 : x + y = 1$
$P_2 : y - z = 2$
$P_3 : y + z = -1$

(a) Find the line of intersection between $P_1$ and $P_2$.

**Answer.** The line of intersection is parallel to the vector $(1, 1, 0) \times (0, 1, -1) = (-1, 1, 1)$. One point on the line is at $x = 0, y = 1, z = -1$, so the line is $(x, y, z) = (0, 1, -1) + t(-1, 1, 1)$ or

\[
\begin{align*}
x &= -t \\
y &= 1 + t \\
z &= -1 + t
\end{align*}
\]

(b) Find the point of intersection between the line you found in part (a) and $P_3$.

**Answer.** Probably the easiest way to do this is to find the value of $t$ that also satisfies the third equation:

\[
-1 = y + z \\
= (1 + t) + (-1 + t) \\
t = -1/2 \\
x = -t = 1/2 \\
y = 1 + t = 1/2 \\
z = -1 + t = -3/2.
\]

*(Check: This point (1/2, 1/2, -3/2) satisfies all of the original equations.)*

(c) Find the $z$ coordinate of the same point (the point of intersection of $P_1, P_2,$ and $P_3$) using Cramer’s rule.

**Answer.** We are trying to solve the simultaneous equations

\[
\begin{align*}
x + y &= 1 \\
y - z &= 2 \\
y + z &= -1
\end{align*}
\]
Cramer’s rule tells us that the $x$-coordinate is

\[
x = \frac{\begin{vmatrix} 1 & 1 & 0 \\ 2 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 0 \\ 2 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{vmatrix}} = \frac{1}{2}
\]

(2) Let $\mathbf{v} = \langle 1, 1, 0 \rangle$, and let $\mathbf{w}$ be a vector of length 3 at 60° counterclockwise from $\mathbf{v}$, also in the $xy$ plane.

(a) What is $\mathbf{v} \cdot \mathbf{w}$?

**Answer.** $\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}||\mathbf{w}| \cos \theta = \sqrt{2} \cdot 3 \cdot \frac{1}{2} = 3\sqrt{2}/2$

(b) What is $\mathbf{v} \times \mathbf{w}$?

**Answer.** The length of $\mathbf{v} \times \mathbf{w}$ is given by $|\mathbf{v} \times \mathbf{w}| = |\mathbf{v}||\mathbf{w}| \sin \theta = \sqrt{2} \cdot 3 \cdot \frac{\sqrt{3}}{2} = 3\sqrt{6}/2$. The direction of the cross product is always perpendicular to the two vectors. Since both vectors are in the $xy$ plane, the cross product is along the $z$ axis, either positive or negative. The direction of the cross product is given by the right hand rule, which in this case means it points along the positive $z$ axis. Combining all this, we have:

$\mathbf{v} \times \mathbf{w} = \langle 0, 0, 3\sqrt{6}/2 \rangle$.

(c) What is $\mathbf{w}$?

**Answer.** This is most easily solved in polar coordinates (ignoring the $z$ axis, which is 0 for everything). $\mathbf{v}$ has a length of $\sqrt{2}$ and an angle of 45°. From the information given, $\mathbf{w}$ has a length of 3 and an angle of $45° + 60° = 105°$. Thus $\mathbf{w}$ is given by

$\mathbf{w} = \langle r \cos \theta, r \sin \theta, 0 \rangle = \langle 3 \cos 105°, 3 \sin 105°, 0 \rangle$.

(3) Let $f(x,y) = xe^{x+y}$.

(a) Let $\mathbf{u} = \langle 1/\sqrt{2}, 1/\sqrt{2} \rangle$. Find the directional derivative $D_{\mathbf{u}}f(1,-1)$.

(b) Find $\nabla f(1,-1)$.

**Answer.** We first find the gradient:

$\nabla f(1,-1) = \langle \frac{\partial f}{\partial x}(1,-1), \frac{\partial f}{\partial y}(1,-1) \rangle$

$= \langle (1 + x)e^{x+y}, xe^{x+y} \rangle|_{x=1,y=-1}$

$= \langle 2, 1 \rangle$.

Then the directional derivative is given by

$D_{\mathbf{u}}f(1,-1) = \mathbf{u} \cdot \nabla f(1,-1)$

$= \frac{3}{\sqrt{2}}$. 

(c) Find the tangent plane to the graph of \( f \) at \((x, y) = (1, -1)\).

**Answer.** First note that \( z_0 = f(x_0, y_0) = f(1, -1) = 1 \). The tangent plane is given by

\[
z - z_0 = f_x(x_0, y_0) \cdot (x - x_0) + f_y(x_0, y_0) \cdot (y - y_0)
\]

\[
z - 1 = 2(x - 1) + 1(y + 1).
\]

(d) Use the linear approximation to \( f \) near \((1, -1)\) to approximate \( f(1.1, -1.2) \).

**Answer.** The linear approximation is the value given by the tangent plane above: for \( x, y \) near \((1, -1)\),

\[
f(x, y) \approx z = 1 + 2(x - 1) + (y + 1)
\]

\[
f(1.1, -1.2) \approx 1 + 2(1.1 - 1) + (-1.2 + 1) = 1 + 0.2 - 0.2 = 1.
\]

(4) (a) Find the tangent plane to the surface \( z^2 = x^2 + 2y^2 + 3 \) at the point \((2, 1, 3)\).

**Answer.** The surface is a level surface of the function \( F(x, y, z) = -x^2 - 2y^2 + z^2 \). The normal to the surface is given by the gradient of \( F \):

\[
\mathbf{n} = \nabla F(2, 1, 3) = \langle -2x, -4y, 2z \rangle|_{(2,1,3)} = \langle -4, -4, 6 \rangle
\]

Thus the plane is given by

\[
\langle -4, -4, 6 \rangle \cdot \langle x - 2, y - 1, z - 3 \rangle = 0.
\]

**Note.** An alternative way to solve this problem is to solve for \( z \) in terms of \( x \) and \( y \) and proceed as in problem 3. You are responsible for both methods.

(b) If \( z = xy^3 + x^4 + xy \) and \( x = t + 1, y = t^2 \), find \( dz/dt \) at \( t = 1 \).

**Answer.** Note that at \( t = 1, x = 2 \) and \( y = 1 \). We compute using the chain rule:

\[
\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}
\]

\[
= (y^3 + 4x^3 + y)(1) + (3xy^2 + x)(2t)
\]

\[
= (1 + 4 \cdot 8 + 1)(1) + (3 \cdot 2 \cdot 1 + 2)(2)
\]

\[
= 34 + 16 = 50.
\]

(5) Find the critical points of the function

\[
f(x, y) = (x^3 - 3x)e^{-y^2}
\]

and classify whether they are local minimima, maxima, or saddle points.
**Answer.** First we find the partial derivatives:

\[
\begin{align*}
    f_x &= (3x^2 - 3)e^{-y^2} \\
    f_y &= (x^3 - 3x)e^{-y^2}(-2y) \\
    f_{xx} &= 6xe^{-y^2} \\
    f_{xy} &= (3x^2 - 3)e^{-y^2}(-2y) \\
    f_{yy} &= (x^3 - 3x)e^{-y^2}(4y^2 - 2).
\end{align*}
\]

The critical points are the values where \( f_x = f_y = 0 \):

\[
\begin{align*}
    f_x &= (3x^2 - 3)e^{-y^2} = 0 \\
    f_y &= -2y(x^3 - 3x)e^{-y^2} = 0.
\end{align*}
\]

From the first equation we deduce that \( 3x^2 - 3 = 0 \) so \( x = \pm 1 \) (since \( e^{-y^2} \) is never \( 0 \)). In the second equation, we then see that \( e^{-y^2} \) and \( x^3 - 3x \) are not \( 0 \), so \( y \) must be zero. Thus the two critical points are

\[
\begin{align*}
P_1 : (x, y) &= (1, 0) \\
P_2 : (x, y) &= (-1, 0).
\end{align*}
\]

To classify the critical points, we need to evaluate \( D = f_{xx}f_{yy} - f_{xy}^2 \). Fortunately, \( e^{-y^2} = 1 \) and \( y = 0 \), simplifying the calculations. We calculate:

\[
\begin{align*}
P_1 : D &= (6)(4) - 0 > 0 \\
P_2 : D &= (-6)(-4) - 0 > 0
\end{align*}
\]

Thus both are either maxima or minima. Looking at the signs of \( f_{xx} \), we see that \( P_1 \) is a local minimum and \( P_2 \) is a local maximum.

(6) Find the largest box with one vertex at \((0, 0, 0)\), the opposite vertex on the ellipsoid \( E = \{(x, y, z) | x^2 + 4y^2 + 9z^2 = 1\} \) and with sides aligned with the \( x \), \( y \), and \( z \) axes. Use Lagrange multipliers.

**Answer.** We want to maximize the objective function \( f(x, y, z) = xyz \) subject to the constraint that \( g(x, y, z) = x^2 + 4y^2 + 9z^2 = 1 \). By Lagrange multipliers, we need to solve

\[
\begin{align*}
    \nabla f &= \lambda \nabla g \\
    (yz, xz, xy) &= \lambda(2x, 8y, 18z) \\
    x^2 + 4y^2 + 9z^2 &= 1.
\end{align*}
\]

Solving the first 3 equations for \( \lambda \) and setting the results equal, we find

\[
\begin{align*}
    \frac{yz}{2x} &= \frac{xz}{8y} = \frac{xy}{18z}
\end{align*}
\]

which yields (after multiplying by 2, dividing by \( xyz \) and inverting each equation)

\[
\begin{align*}
x^2 &= 4y^2 = 9z^2.
\end{align*}
\]
From the constraint equation, we then find

\[ 3x^2 = 1 \]
\[ x = \frac{1}{\sqrt{3}} \]
\[ y = \frac{1}{2\sqrt{3}} \]
\[ z = \frac{1}{3\sqrt{3}} \]

(7) (a) Compute \((-1 + i)^4\).

Answer.

\[ (-1 + i)^2 = 1 - 2i - 1 = -2i \]
\[ (-1 + i)^4 = ((-1 + i)^2)^2 = (-2i)^2 = -4. \]

(b) Find all 4th roots of \(-4\).

Answer. Note that we found one 4th root of \(-4\) in the first part. We can find the others by taking that answer and rotating repeatedly by \(360^\circ / 4 = 90^\circ\). Alternatively, we use polar coordinates: with \(z = (r, \theta)^P\) in polar coordinates, we solve:

\[ z^4 = -4 \]
\[ (r, \theta)^P = ((4, 180^\circ)^P)^{1/4} \]
\[ = (\sqrt{2}, (180^\circ + n360^\circ)/4) \]

where \(n\) ranges from 0 to 3 (to give the four different fourth roots). These points are

\[ z_1 = (\sqrt{2}, 45^\circ)^P = 1 + i \]
\[ z_2 = (\sqrt{2}, 45^\circ + 90^\circ)^P = -1 + i \]
\[ z_3 = (\sqrt{2}, 45^\circ + 180^\circ)^P = -1 - i \]
\[ z_4 = (\sqrt{2}, 45^\circ + 270^\circ)^P = 1 - i. \]

(8) Show that absolute value is multiplicative on complex numbers:

\[ |z_1 z_2| = |z_1||z_2| \]

for \(z_1\) and \(z_2\) complex numbers.
Answer. Remember that if $z = x + iy$, then $|z| = \sqrt{x^2 + y^2}$. With $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, we then have

$$\begin{align*}
|z_1z_2| &= |(x_1 + iy_1)(x_2 + iy_2)| \\
&= |x_1x_2 - y_1y_2 + i(x_1y_2 + y_1x_2)| \\
&= \sqrt{(x_1x_2 - y_1y_2)^2 + (x_1y_2 + y_1x_2)^2} \\
&= \sqrt{x_1^2x_2^2 + y_1^2y_2^2 + x_1^2y_2^2 + y_1^2x_2^2}.
\end{align*}$$

(In the last step, the middle term in both squares, involving $x_1x_2y_1y_2$, cancels out.)

On the other side, we have

$$\begin{align*}
|z_1||z_2| &= \sqrt{x_1^2 + y_1^2}\sqrt{x_2^2 + y_2^2} \\
&= \sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)} \\
&= \sqrt{x_1^2x_2^2 + y_1^2y_2^2 + x_1^2y_2^2 + y_1^2x_2^2}.
\end{align*}$$

The last lines match from the two sides, so the two sides are equal.

(9) (a) Find the general solution to $6y' + 9y = 0$.

Answer. The characteristic equation is

$$6r + 9 = 0$$

with solution $r = -3/2$, so the general solution is

$$y(x) = k_1e^{-3x/2}.$$ 

(b) Find the general solution to $y'' + 6y' + 9y = 0$.

Answer. The characteristic equation is

$$r^2 + 6r + 9 = 0$$

$$(r + 3)^2 = 0$$

with a double root at $r = -3$. Thus the general solution is

$$y(x) = k_1e^{-3x} + k_2xe^{-3x}.$$ 

(c) Find the solution to $y'' + 6y' + 9y = 0$ where $y(0) = 0$ and $y'(0) = 1$.

Answer. This is the same equation as above, which we solve for $k_1$ and $k_2$. We find

$$y(0) = k_1 = 0$$

$$y'(x) = -3k_1e^{-3x} + k_2(1 - 3x)(e^{-3x})$$

$$= -3k_1 + k_2 = 1$$

$$k_2 = 1$$

$$y(x) = xe^{-3x}.$$
(d) Find the general solution to $y'' + 6y' + 9y = x^2 + e^x$.

**Answer.** The complementary equation is just the one we found above, so we just need to find a particular solution. We break it into two parts: one particular solution with RHS $x^2$, and another with RHS $e^x$. For the first, we guess that $y$ might be a polynomial in $x$ of degree 2:

$$y_1(x) = Ax^2 + Bx + C$$
$$y'_1(x) = 2Ax + B$$
$$y''_1(x) = 2A$$

$$y''_1 + 6y'_1 + 9y_1 = 2A + (12Ax + 6B) + (9Ax^2 + 9Bx + 9C)$$
$$= 9Ax^2 + (12A + 9B)x + (2A + 6B + 9C)$$
$$= x^2$$
$$9A = 1$$
$$12A + 9B = 0$$
$$2A + 6B + 9C = 0$$

which has a solution $A = 1/9$, $B = -4/27$, $C = -2/27$, so

$$y_1(x) = x^2/9 - 4x/27 - 2/27.$$  

For the second, we guess that $y$ might be a multiple of $e^x$:

$$y_2 = y'_2 = y''_2 De^x$$
$$y''_2 + 6y'_2 + 9y_2 = 16De^x$$
$$= e^x$$
$$D = 1/16$$
$$y_2(x) = e^x/16.$$  

Combining all these, the general solution is

$$y(x) = k_1e^{-3x} + k_2xe^{-3x} + x^2/9 - 4x/27 - 2/27 + e^x/16.$$