§41 The Measure of an Open Set

In this and the following sections we describe an improvement on the Riemann integral which was devised by Lebesgue about 1900. The Lebesgue integral is based on the concept of the (Lebesgue) measure of a measurable subset of \( \mathbb{R} \), two terms whose definition will come later. Before that we will define the measure of a bounded open set.

**THEOREM** Each bounded open subset of \( \mathbb{R} \) is a finite or countable union of disjoint open intervals.

**Proof** Let \( U \subseteq \mathbb{R} \) be open and bounded (i.e., contained in some finite interval \((a, b)\)). For \( x, y \in U \), put

\[
x \sim y \iff [x, y] \subseteq U.
\]

This is an equivalence relation on \( U \). In fact,

- \( x \in U \Rightarrow x \sim x \) since \([x, x] = \{x\} \subseteq U\).
- \( x, y \in U \), \( x \sim y \Rightarrow y \sim x \) since \([x, y] = [y, x]\)
- \( x, y, z \in U \), \( x \sim y \), \( y \sim z \) \( \Rightarrow [x, y], [y, z] \subseteq U \Rightarrow [x, z] \subseteq U \Rightarrow x \sim z \).

Thus \( U \) is the disjoint union of its equivalence classes.

We show next that each equivalence class \( I \) is an open interval.
Correction to 41.2. Starting at the 5th line:

Clearly \( I \subset [a_0, b_0] \). Actually \( I \subset (a_0, b_0) \) since \( a \neq U \), \( b \neq U \).

In fact, if \( b_0 \in U \) then some \( \epsilon \)-neighborhood of \( b_0 \) could be in \( U \), since \( U \) is open. Since \( b_0 \in (a_0, b_0) \) \( \forall I \), there a \( y \in (b_0 - \epsilon, b_0 + \epsilon) \) \( \forall I \) and since \( b_0 \) is an 'a.b. for \( I \), each \( y \in (b_0 - \epsilon, b_0 + \epsilon) \) belongs to \( I \).

\[
\begin{array}{c}
\theta_i \quad \theta_2 \\
\theta \quad b_0 \\
\theta - \epsilon \quad b_0 + \epsilon
\end{array}
\]

\( \therefore \) \( [y_1, y_2] \subset (b_0 - \epsilon, b_0 + \epsilon) \subset U \), \( \therefore y_1 \sim y_2 \) so since \( y_1 \in I \) we conclude that \( y_2 \in I \) a contradiction.

Similarly we can show that \( a_0 \notin U \).

Now pick any \( x \in L \), etc.
In fact, we will show that \( I = (a, b) \) where

\[
\begin{align*}
  b_0 &= \text{least upper bound of } I \\
  a_0 &= \text{greatest lower bound of } I.
\end{align*}
\]

(Since \( I \) is nonempty and bounded (as a subset of \( U \)), \( a_0 \) and \( b_0 \) exist.) Clearly \( I \subset (a_0, b_0) \). Now pick any \( x \in I \). For \( x \leq y < b_0 \) there exists \( z \in (y, b_0) \) with \( z \in I \) (since \( b_0 \) is the least upper bound for \( I \)), so \( [x, z] \subset U \), so \( [x, y] \subset U \) so \( y \in I \).

Thus \( [x, b_0] \subset I \). Similarly \( (a_0, x] \subset I \). Thus \( (a_0, b_0) \subset I \), giving \( U \).

So far we have shown that each bounded open subset \( U \) of \( \mathbb{R} \)

is a disjoint union of open intervals. It remains to show that

the set of all these disjoint open intervals making up \( U \) is finite or countable.

In each such interval we may pick a rational number.

Since the set of all rationals is countable this shows that the set

of open intervals in one decomposition of \( U \) is finite or countable.

**NOTATION.** In what follows, we write

\[(*) \quad U = \sum_n I_n \subset I\]

for an open set \( U \) contained in an interval \( I = (a, b) \),

expressed as a finite or countable disjoint union of open intervals

\( I_n = (a_n, b_n) \).

**LEMMA.** If \( (*) \), then

\[
\sum_n (b_n - a_n) \leq b - a.
\]
Proof. If there are only finitely many summands in \( U \), i.e.,
\[
U = \sum_{n=1}^{N} \Sigma_n,
\]
we may suppose, after a possible reordering, that
\[
a_1 < a_2 < \ldots < a_N,
\]
and then \( b_1 \leq a_2, \ldots, b_{n-1} \leq a_n \) by disjointness. Thus
\[
a < a_1 < b_1 < a_2 < b_2 \leq \ldots \leq a_n < b_n \leq b.
\]

\[ \alpha \quad a_1 \quad b_1 \quad a_2 \quad b_2 \quad a_n \quad b_n \quad \beta \]

From which clearly
\[
\sum_{n=1}^{N} (b_n - a_n) \leq b - a.
\]

In the case of infinitely many summands in \( U \),
\[
U = \sum_{n=1}^{\infty} \Sigma_n,
\]
the preceding argument shows that each partial sum of \( \sum (b_n - a_n) \)
is bounded by \( b - a \), so the full series converges to a number \( \leq b - a \).

**Remark** In general, if there are infinitely many summands in \( U \), it may be impossible to list them all "from left to right."

**Example** The open-middle-third Cantor set is
\[
U = \left( \frac{1}{3}, \frac{2}{3} \right) + \left( \frac{7}{9}, \frac{8}{9} \right) + \left( \frac{25}{27}, \frac{26}{27} \right) + \ldots
\]
Here we have listed the component intervals by size, starting with the longest. The sum of the lengths in this example is

\[
\frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \ldots = \frac{1}{3} \left(1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \ldots \right)
\]

\[
= \frac{1}{3} \cdot \frac{1}{1 - \frac{1}{3}} = 1.
\]

**DEF.** For each bounded open set \( U \) given by (\(*\)), the measure \( m(U) \) is defined by

\[
(**) \quad m(U) = \sum m(I_n)
\]

where \( m(I_n) = b_n - a_n \). It doesn't matter in what way we enumerate the component intervals of \( U \) since the sum of a series of positive terms is independent of the order of summation, by the theorem of Riemann.

**EXAMPLE.** The open-middle-third Cantor Set has measure 1.

The LEMMA on 41.2 may now be written as

**LEMMA.** If \( U \subseteq I \), then \( m(U) \leq m(I) \).

**EXERCISE.** If \( U \) satisfies (\(*\)) and \( I \) is any open interval \( U \), then \( I \) is some \( I_n \).
THEOREM. For open sets $U, V, \ldots$ all contained in some open interval,

1. If $U \subseteq V$, then $m(U) \leq m(V)$.

2. If $U = \bigcup U_k$, then $m(U) = \sum m(U_k)$

3. If $U = U \cap V$, then $m(U) \leq \sum m(U_k)$

4. $m(U \cup V) + m(U \cap V) = m(U) + m(V)$.

(In (2) and (3) the index set is finite or countable.)

Proof (1) Let

$$U = \bigcup I_n, \quad V = \bigcup J_m.$$ 

Since $U \subseteq V$, each $I_n$ is contained in some $J_m$, by the EXERCISE.

By the LEMMA, in each partial sum $\sum I_n$, the contribution by the terms with $I_n$ in any particular $J_m$ is $\leq m(J_m)$, so the partial sum is $\leq \sum m(J_m) = m(V)$, from which $m(U) \leq m(V)$.

(2) Let $U_k = \bigcup I_{k,n}$. Then $U = \bigcup \bigcup I_{k,n}$, so

$$m(U) = \sum \sum m(I_{k,n}) = \sum m(U_k).$$

More precisely, we must list the $I_{k,n}$ in some linear order and sum their lengths as a single series to get $m(U)$. That this sum is the same as the sum of the double series above follows from a general fact about absolutely convergent double series:
THEOREM. If the double series \( \sum_{k} \sum_{n} c_{k,n} \) is absolutely convergent and its terms are listed in some order as \( c_{1}, c_{2}, c_{3}, \ldots \), then

\[
\sum_{k} \sum_{n} c_{k,n} = \sum_{l} c_{l}.
\]

Proof. (for the case in which all \( c_{k,n} \geq 0 \)). Let

\[
S_{k} = \sum_{n} c_{k,n} , \quad S = \sum_{k} S_{k}.
\]

assumed finite. For each \( L \), each term \( c_{k} \) with \( L \leq k \) occurs in some \( S_{k} \) so there is a \( K_{L} \) so that each \( c_{k} \) with \( L \leq k \) occurs in some \( S_{k} \) with \( L \leq K_{L} \). It follows that

\[
\sum_{L} c_{L} = \sum_{L \leq K_{L}} \sum_{n} c_{k,n} = \sum_{L \leq K_{L}} S_{k} \leq S,
\]

and therefore \( \sum_{L} c_{L} = \tilde{S} \leq S \). In the other direction, each \( c_{k,n} \) with \( k \leq K \) and \( n \leq N \) is of the form \( c_{L} \) for some \( L \leq \min K, N \), so

\[
\sum_{k \leq K} \sum_{n \leq N} c_{k,n} \geq \sum_{L \leq \min K, N} c_{L} = \tilde{S}.
\]

Letting first \( N \to \infty \), then \( K \to \infty \) gives first

\[
\sum_{k \leq K} \sum_{n} c_{k,n} \leq \tilde{S}
\]

then

\[
S = \sum_{k} \sum_{n} c_{k,n} \leq \tilde{S}.
\]

Thus \( S = \tilde{S} \), as claimed.
Proof of (3). We deal first with the case in which \( U \) is an interval.

**Lemma:** If \( J \) is an open interval, and \( J = \bigcup_k I_k \), then
\[
    m(J) = \sum_k m(I_k) \quad \text{(no convergence asserted)}
\]

**Proof:** Let \( I_k = \sum I_{k,n} \). Then \( J = \bigcup_k \sum I_{k,n} \). Let \( J = [\alpha, \beta] \) and put \( \tilde{\alpha} = \alpha + \varepsilon, \tilde{\beta} = \beta - \varepsilon \) for any positive \( \varepsilon < \frac{1}{2} (\beta - \alpha) \). Then
\[
    [\tilde{\alpha}, \tilde{\beta}] \subset \bigcup_k \sum I_{k,n} = \bigcup_{k=1}^L I_L
\]

where the \( I_L \) are the \( I_{k,n} \) listed in some linear order. Since \( [\alpha, \beta] \) is compact, we have
\[
    [\tilde{\alpha}, \tilde{\beta}] \subset \bigcup_{k=1}^L I_L
\]

for some \( L \in \mathbb{N} \).

Since \( \tilde{\alpha} \) is covered by \( I_1, \ldots, I_L \) we may suppose, after possibly reindexing, that \( \tilde{\alpha} \in I_1 = (a_1, b_1) \). If \( b_1 > \tilde{\beta} \), stop. If \( b_1 < \tilde{\beta} \), then \( b_1 \) is covered, as \( b_1 \in I_2 = (a_2, b_2) \) (after a further reindexing). If \( b_2 > \tilde{\beta} \), stop. If \( b_2 < \tilde{\beta} \), continue \( \ldots \). We get at least
\[
    a_1 < \tilde{\alpha} < b_1 < a_2 < b_2 < \ldots < a_m < \tilde{\beta} < b_m
\]

for some \( M \leq L \) (after a lot of reindexing):

\[
    a_1 \tilde{\alpha} b_1 a_2 b_2 \ldots a_m \tilde{\beta} b_m
\]

from which
\[
    mJ - 2\varepsilon = \tilde{\beta} - \tilde{\alpha} \leq \sum_{k=1}^M m(I_k) \leq \sum_k m(I_k).
\]
We may suppose the series on the right converges, since if it diverges the lemma is true. Since \( \varepsilon \) may be taken arbitrarily small we conclude that \( \forall \) holds.

In the general case of (3), \( U = U U_k = \sum I_n \), so

\[
\forall \quad mU = \sum I_n
\]

Since \( I_n = I_n \cap (U U_k) = U (I_n \cap U_k) \), the usual case \( \forall \) gives

\[
\forall \quad m I_n \leq \sum m(I_n \cap U_k).
\]

Combining \( \forall \) and \( \forall \) gives

\[
\forall \quad mU \leq \sum \sum m(I_n \cap U_k)
\]

\[
= \sum \sum m(I_n \cap U_k),
\]

the justification for the switch in summation signs being that the terms are nonnegative so either both double series converge to the same number or both diverge. Since \( U_k = \sum (I_n \cap U_k) \),

\[
\forall \quad m(U) = \sum m(I_n \cap U_k),
\]

by (2). Combining \( \forall \) and \( \forall \) gives (3).

For the proof of (4), we use the following next formula:

**Lemma.** Given \( \varepsilon > 0 \), put \( \Delta = \{ 0, \pm \varepsilon, \pm 2\varepsilon, \pm 3\varepsilon, \ldots \} \). Then \( U \) a any finite union of disjoint open intervals,

\[
mU = \lim_{\varepsilon \to 0} \delta \#(U \cap \Delta).
\]
Proof. It suffices to do the case \( U = (a, b) \). Here

\[
\#(U \cap A) = \# \{ y \in \mathbb{Z} \text{ with } a < n \delta < b \} = \# \{ y \in \mathbb{Z} \text{ with } \frac{a}{\delta} < n < \frac{b}{\delta} \} = \delta (b - a) + O(1),
\]

where \( O(1) \) stands for a term which is bounded when \( \delta \to 0 \).

It follows that

\[
\delta \#(U \cap A) \to b - a = m(U) \text{ for } \delta \to 0.
\]

EXERCISE. If \( U \) and \( V \) are each finite unions of disjoint open intervals then both \( U \cup V \) and \( U \cap V \) are finite unions of disjoint open intervals.

Proof of (4). Suppose first that \( U \) and \( V \) are each finite unions of disjoint open intervals. Then the EXERCISE above shows that the real formula in the LEMMA applies to all four of \( U, V, U \cup V \) and \( U \cap V \). For \( \delta > 0 \), the four sets \( U \cap \Delta, V \cap \Delta, (U \cup V) \cap \Delta = (U \cap \Delta) \cup (V \cap \Delta) \) and \( (U \cap V) \cap \Delta = (U \cap \Delta) \cap (V \cap \Delta) \) are all finite so (by \( \delta > 1 / \delta \) !)

\[
\#((U \cup V) \cap \Delta) + \#((U \cap V) \cap \Delta) = \#(U \cap \Delta) + \#(V \cap \Delta).
\]

Multiplying by \( \delta \) and letting \( \delta \to 0 \) gives (4) in this case.
In the general case of (4),

\[ U = \sum I_k, \quad V = \sum J_k \]

For \( \varepsilon > 0 \), choose \( K \) and \( L \) in \( \mathbb{N} \) so that

\[ R_U = \sum_{k > K} I_k \quad \text{and} \quad R_V = \sum_{k > L} J_k \]

each have measure \(< \varepsilon \). Put

\[ S_U = \sum_{k = 1}^{K} I_k, \quad S_V = \sum_{k = 1}^{L} J_k, \]

so that

\[ U = S_U + R_U, \quad V = S_V + R_V. \]

Then

\[ U \cup V = S_U \cup S_V \cup R_U \cup R_V \]

so by (1) and (3)

\[(*) \quad m(S_U \cup S_V) \leq m(U \cup V) \leq m(S_U \cup S_V) + 2\varepsilon \]

Also,

\[ S_U \cap S_V \subset U \cap V \subset (S_U \cap S_V) \cup R_U \cup R_V \]

so

\[(***) \quad m(S_U \cap S_V) \leq m(U \cap V) \leq m(S_U \cap S_V) + 2\varepsilon. \]

From (*) and (***) it follows that \( m(U \cup V) + m(U \cap V) \) is between

\[ m(S_U \cup S_V) + m(S_U \cap S_V) = m(S_U) + m(S_V) \geq m(U) + m(V) - 2\varepsilon \]

and

\[ m(S_U \cup S_V) + m(S_U \cap S_V) + 4\varepsilon = m(S_U) + m(S_V) + 4\varepsilon \leq m(U) + m(V) + 4\varepsilon. \]

Finally, let \( \varepsilon \to 0 \) to conclude (4).