§35 Fourier Series: Dirichlet's Proof of Jump Discontinuities.

Dirichlet proved his theorem in 1850, working directly with $D_n$. We ignore the uniformity statement and show how he proved that $S_n(x) \to f(x)$ pointwise for $f \in C^1(\mathbb{R})$ of period 1.

Just as in the proof of Fejer's Theorem, we start with

\[
(1) \quad f(x) - S_n(x) = \int_{-\frac{1}{2}}^{\frac{1}{2}} (G(x) - G(x-y)) D_n(y) dy
\]

which follows from $\int_0^1 D_n = 1$ and the period one ness of $D_n$.

But, unlike $F_n$, $D_n$ is not nonnegative, nor does $D_n(y) \to 0$ for $\delta \leq y \leq 1 - \delta$. Instead, using

\[
D_n(y) = \frac{\sin((\pi n-1)n\pi y)}{\sin n\pi y},
\]

Dirichlet writes (1) as

\[
(2) \quad f(x) - S_n(x) = \int_{-\frac{1}{2}}^{\frac{1}{2}} g(y) \sin((\pi n-1)n\pi y) dy
\]

with

\[
g(y) = g_n(y) = \frac{f(x) - f(x-y)}{\sin y}
\]

for $y \neq 0$.

Not so fast! What about $y = 0$, where $g$ is not defined?

For $y \neq 0$ and $y \to 0$ we have

\[
\frac{f(x) - f(x-y)}{y} \to f'(x) \quad \text{and} \quad \frac{y}{\sin y} \to \frac{1}{n}
\]
so \( g(y) \to t(x) \) for \( y \to 0, g \neq 0 \). Therefore, if we define
\[ g(x) = \frac{t(x)}{n}, \]
then \( g \) is defined and continuous for all \( y \),
and
\[ (\frac{d}{dx} - t(x-y)) L_n(y) = g(y) \sin((2\pi - 1)y \text{ for all } y, \text{ giving } (2) \).

In view of \( (2) \), to get \( s_n(x) \to t(x) \) it suffices
in view of \( \sin \theta = \frac{1}{2i} (e^{i\theta} - e^{-i\theta}) \) to prove that

**Lemma (Dirichlet–Riemann)** For each \( g \in C[a, b] \),
\[ (3) \quad \int_{a}^{b} g(y) e^{it} dy \to 0 \text{ for } |t| \to \infty. \]

**Proof.** First suppose \( g \in C'(a, b) \). By integration by parts,
\[ \int_{a}^{b} g(y) e^{it} dy = g(y) \frac{e^{it}}{it} \int_{a}^{b} - \int_{a}^{b} g(y) \frac{e^{it}}{it} dy. \]

Using \( |e^{i\theta}| = 1 \), it follows that
\[ \left| \int_{a}^{b} g(y) e^{it} dy \right| \leq \left( 2 \max |g| + (b-a) \max |g'| \right) \frac{1}{|t|} \]
from which (3) follows.

In general, for \( g \in C[a, b] \), given \( \varepsilon > 0 \) there is an \( h \in C'(a, b) \) for which \( \max |g-h| < \frac{\varepsilon/2}{b-a} \).

For example, there is a polynomial \( h \) with this property,
by Weierstrass' Theorem. It follows that
\[ \left| \int_{a}^{b} g(y) e^{it} dy - \int_{a}^{b} h(y) e^{it} dy \right| = \left| \int_{a}^{b} (g(y)-h(y)) e^{it} dy \right| < \varepsilon/2, \]

Since \( h \in C'[a, b] \), the previous case gives \( \left| \int_{a}^{b} h(y) e^{it} dy \right| < \varepsilon/2 \)
for all \( |t| \geq T \), so for each \( |t| \geq T ', \)
\[ \left| \int_{a}^{b} g(y) e^{it} dy \right| < \varepsilon. \]
REMARK. The integral in this LEMMA is small for large $k$ not because $g(y)$ is small — we aren’t assuming that, nor because $e^{i ty}$ is small — it isn’t, but because, somehow, for large $t$, the oscillations of $e^{i ty}$ must produce a kind of cancellation, something analogous to the way $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$ converges (because of the $(-1)^{y}$ factor) even though $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$ diverges.

DEF. A function $f$ is piecewise continuous (or piecewise smooth) on $[a, b]$ if there is a partition

$$a = x_0 < x_1 < \ldots < x_n = b$$

and continuous (or $C^1$) functions $f_1, \ldots, f_r$ on the closed intervals $[x_0, x_1], \ldots, [x_{r-1}, x_r]$ respectively, so that $f(x) = f_j(x)$ on the open interval $(x_{j-1}, x_j)$, for $j = 1, \ldots, r$.

EXAMPLE. Let $f(x) = 1$ or $-1$ according as $0 < x \leq \frac{1}{2}$ or $\frac{1}{2} < x < 1$ and define $f(0), f(\frac{1}{2}), f(1)$ arbitrarily. Then $f$ is piecewise smooth on $[0, 1]$.

Here $f_1 = 1$ on $[0, \frac{1}{2}]$ and $f_2 = -1$ on $[\frac{1}{2}, 1]$.

REMARK. There is no requirement that $f_j(x_j) = f_{j+1}(x_j)$ for $j = 0, \ldots, r-1$ and no hypothesis about the values $f(x_j)$ for $j = 0, 1, \ldots, r$.

DEF. Let $f$ be piecewise continuous (or piecewise smooth) on $[a, b]$. The (possible) discontinuities at the $x_j$ are called jump discontinuities because although the left and right limits
\[ \lim_{x \to x_j^-} f(x) = f_j(x_j^-) \quad \text{for } j = 1, \ldots, r \]

and

\[ \lim_{x \to x_j^+} f(x) = f_{j+1}(x_j) \quad \text{for } j = 0, \ldots, r-1 \]

exist, they need not agree for \( j = 1, \ldots, r-1 \).

DEF Let \( f \) have period 1 and be piecewise continuous on each finite interval, or, equivalently, let \( f \) be piecewise continuous on \([0, 1] \) with \( f(0) = f(1) \) and be defined on all of \( \mathbb{R} \) by periodicity. We say \( f \) is a normalized if at each discontinuity \( x_j \),

\[ f(x_j) = \frac{1}{2} (f(x_j^-) + f(x_j^+)) \]

EXAMPLE In the EXAMPLE above we must have \( f(\frac{1}{2}) = 0 \) and also \( f(0) = f(1) = 0 \) to have a normalized function.
DEF The Riemann integral of a piecewise continuous function $f$ on $[a, b]$ is defined by

$$\int_a^b f = \int_{x_0}^{x_1} f + \cdots + \int_{x_{n-1}}^{x_n} f.$$ 

We skip the simple proof that this integral, thus extended, is still linear in $f$, additive in $[a, b]$, and satisfies $\left| \int_a^b f \right| \leq (b-a) M$ with

$$M = \max_{x \in [a, b]} \max_{y \in [a, b]} |f(y)|.$$ 

DEF For periodic piecewise continuous functions $f$, we define the Fourier coefficients $c_n$ and the sums $s_n$ and $\sigma_n$ as before.

THEOREM (Fejer) For periodic piecewise continuous normalized $f$,

$$\sigma_n(x) \to f(x) \quad \text{for} \quad N \to \infty, \quad \text{for each} \ x.$$ 

THEOREM (Dirichlet) For periodic piecewise $C^1$ normalized $f$, 

$$s_n(x) \to f(x) \quad \text{for} \quad N \to \infty, \quad \text{for each} \ x.$$ 

THEOREM (Riesz) For periodic piecewise continuous $f$, 

$$\sum_{n=1}^{\infty} |c_n|^2 = \int_0^1 |f(x)|^2.$$ 

REMARK: We cannot claim uniform convergence of piecewise Fejer and piecewise Dirichlet since a uniform limit of continuous functions is continuous. However (proof omitted), the convergence is uniform.
in those Theorems, not on \( \mathbb{R} \), but on \( \mathbb{R} - \bigcup_{i=0}^{n} \lambda (x_i - \delta, x_i + \delta) \) where the \( x_i \) are the points of discontinuity and \( \delta > 0 \) is arbitrary.

Proof of Fejer's Theorem: Starting from

\[
\sigma_N(x) = \int_{0}^{1} f(x-y) F_N(y) \, dy
\]

\[
= \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x-y) F_N(y) \, dy
\]

\[
= \int_{0}^{\frac{1}{2}} (f(x-y) + f(x+y)) F_N(y) \, dy
\]

(by last step by the evenness of \( F_N \)), and

\[
f(x) = \frac{1}{2} (f(x^-) + f(x^+))
\]

\[
= \int_{0}^{\frac{1}{2}} (f(x^-) + f(x^+)) F_N(y) \, dy
\]

(by the normalization assumption, the evenness of \( F_N \) and \( \int_{0}^{1} F_N = 1 \)), we get

\[
\sigma_N(x) - f(x) = \int_{0}^{\frac{1}{2}} [f(x-y) - f(x^-)] + (f(x+y) - f(x^+)) F_N(y) \, dy.
\]

From this point the proof proceeds as before: For each \( x \), the expression \[ \int_{0}^{\frac{1}{2}} \] is a bounded function of \( y \) which \( \to 0 \) for \( y \to 0 \), so \( \sigma_N(x) \to f(x) \) just as in the proof of continuous Fejer.

Proof of Poisson-Dirichlet: To get \( S_n(x) \to f(x) \) from \( \sigma_N(x) \to f(x) \)

it suffices as in the proof of continuous Dirichlet to show that \( |\lambda(n)| \) is bounded. For piecewise smooth \( f \), this may be proved as follows. Let \( 0 = x_0 < x_1 < \ldots < x_r = 1 \) include all points
By discontinuity of \( f \), and let \( f_1, \ldots, f_r \) be the \( C^1 \) functions on \([x_0, x_1], \ldots, [x_r, x_{r+1}]\) which agree with \( f \) on \((x_0, x_1), \ldots, (x_r, x_{r+1})\) respectively. Then

\[
C_n = \int_0^1 f(x) \overline{e_n}(x) \, dx = \int_{x_0}^{x_1} f_1(x) \overline{e_n}(x) \, dx + \cdots + \int_{x_r}^{x_{r+1}} f_r(x) \overline{e_n}(x) \, dx
\]

Just as in the proof that \( |C_n| \) is bounded for \( C^1 \) functions, we may integrate by parts: For \( s = 1, \ldots, r \),

\[
\int_{x_{s-1}}^{x_s} f_s(x) \overline{e_n}(x) \, dx = \frac{f_s(x)}{-2\pi i n} \overline{e_n}(x) \bigg|_{x_{s-1}}^{x_s} - \int_{x_{s-1}}^{x_s} f'_s(x) \overline{e_n}(x) \, dx
\]

giving

\[
|\int_{x_{s-1}}^{x_s} f_s(x) \overline{e_n}(x) \, dx| \leq \left( \max |f_s| + (x_s - x_{s-1}) \max \|f'_s\| \right) / 2\pi \sqrt{n}
\]

Combining these \( r \) bounds gives \( |C_n| \) bounded.

Any \( f \) becomes periodic: Given periodic piecewise continuous \( f \), "repair" its discontinuities: Define periodic continuous \( \tilde{f} \) to agree with \( f \) except on neighborhoods \((x_j - \delta, x_j + \delta)\) of the discontinuities of \( f \), where \( \delta > 0 \) is chosen so small that these neighborhoods don't overlap and on \([x_j - \delta, x_j + \delta]\), \( \tilde{f} \) interpolates linearly between the values \( f(x_j - \delta) \) and \( f(x_j + \delta) \), i.e., \( \tilde{f}(x_j \pm \delta) = f(x_j \pm \delta) \) and \( \tilde{f}' \) is a constant (depending on \( j \)) on each of these neighborhoods \((x_j - \delta, x_j + \delta)\).

Draw a picture illustrating the case in which \( f \) is real-valued.
denote by $\tilde{\sigma}_n$ the $n$th Cesàro sum for $f$. By continuous Fejér,
$\tilde{\sigma}_n \to \tilde{f}$ uniformly, so
\[
\| \tilde{\sigma}_n - \tilde{f} \| \to 0 \quad \text{as } n \to \infty, \quad \text{for each small } \delta.
\]

We show next that
\[
\forall \quad \| \tilde{f} - f \| \to 0 \quad \text{as } \varepsilon \to 0.
\]

In fact,
\[
\| \tilde{f} - f \|^2 = \int_0^1 \left| \tilde{f}(x) - f(x) \right|^2 \, dx
\]
\[
= \int_{x_0}^{x_0 + \delta} \left| \tilde{f}(x) - f(x) \right|^2 \, dx + \sum_{i=1}^{r} \int_{x_i - \delta}^{x_i + \delta} \left| \tilde{f}(x) - f(x) \right|^2 \, dx
\]
\[
\leq 2r\delta \max_{\varepsilon} \max_{\varepsilon} \left| \tilde{f} - f \right|^2.
\]

Clearly $|f|$ is bounded and so by our construction, $|f|$ is bounded by the same bound, so there is a constant $M$ independent of $\delta$ so that
\[
\| \tilde{f} - f \|^2 \leq 2r\delta M^2.
\]

which implies $\varepsilon$.

From $\forall$ and $\forall \epsilon > 0$ we can at choose $\delta$
so that $\| \tilde{f} - f \| < \epsilon/2$, then choose $n$ so that $\| \tilde{\sigma}_n - f \| < \epsilon/2$
thus finding $\tilde{\sigma}_n$ so that $\| \tilde{\sigma}_n - f \| < \epsilon$. It follows, as in
the proof of continuous Fejér, that $\| \tilde{\sigma}_n - f \| < \epsilon$.

Thus $\| \tilde{\sigma}_n - f \| \to 0$, which with $\| f \|$ on 35, 6 gives p.c. Fejér.