§34. Fourier Series: Theorems by Dirichlet and Parseval

To get from Cesaro summability, as in Fejer's THM, to convergence, as in Dirichlet's THM below, we may use Hardy's Tauberian THM (§33).

To get from uniform Cesaro summability to uniform convergence, we may use a uniform version of Hardy's THM.

THM: For each sequence of complex functions \( a_n : X \to \mathbb{C} \) (with \( X \) any set), put

\[
S_n(x) = a_1(x) + \cdots + a_n(x)
\]

\[
\sigma_N(x) = \frac{1}{N} \sum_{n=1}^{N} S_n(x).
\]

If

\[
\sigma_N(x) \to s(x) \quad \text{uniformly on } X
\]

and if \( n a_n \) is uniformly bounded, i.e. for some \( B > 0 \),

\[
|a_n(x)| \leq B \quad \forall n \in \mathbb{N}, \forall x \in X,
\]

then

\[
S_n(x) \to s(x) \quad \text{uniformly on } X.
\]

Proof. Exercise, using the following hints: In the proof of Hardy's Tauberian THM on 29.4.4.5, let \( a_n = a_n(X) \), \( s_n = s_n(X) \), \( \sigma_n = \sigma_n(X) \) and add \( \forall x \) at every step, until you get to \( s_n \) which is now defined by

\[
\varepsilon_n = \max \{ |\sigma_n(x) - s_n(x)| : \forall x \in X \}.
\]
Then \( \lim_{N \to \infty} 29.5 \) implies

\[
(1) \quad |S(x) - S_N(x)| \leq \left( \frac{N}{H} + 1 \right) \varepsilon_N + \frac{B h}{N}.
\]

Since \( S_N \to s \) uniformly, \( \varepsilon_N \to 0 \). The rest of the argument proceeds as on 29.5.

**THEOREM (Dirichlet, 1850).** For \( f \in C([0,1]) \) with period 1, the Fourier series of \( f \) converges to \( f \) uniformly.

**Proof.** We first show that \( \|c_n\| \) is bounded. In fact, for \( n \neq 0 \),

\[
e_n = \int_0^1 d e_n = \int_0^1 \left( \frac{e_n}{2\pi n} \right) = \int_0^1 \frac{d' e_n}{2\pi n}
\]

by integration by parts and periodicity so \( n|c_n| \) bounded.

This follows from the boundedness of the continuous function \( f \).

In the Uniform Hardy Theorem, put \( a_0 = c_0 \) and \( a_n = c_n - \frac{c_{n-1}}{2\pi i n} \) for \( n > 1 \). Then \( n|c_n| \) bounded implies \( n|c_n| \) uniformly bounded.

**THEOREM (Poisson).** For \( f \in C([0,1]) \) with period 1,

\[
\sum_{n=-\infty}^{\infty} |c_n|^2 = \int_0^1 |f|^2.
\]

**DEF.** For complex-valued continuous functions \( f, g \) on \([0,1]\), we put

\[
\langle f, g \rangle = \int_0^1 \overline{f} g, \quad \|f\|_2 = \sqrt{\langle f, f \rangle}.
\]
and observe that

- \( \langle f, g \rangle \) is linear in \( f \)
- \( \langle f, g \rangle = \overline{\langle g, f \rangle} \) "conjugate symmetry"
- \( \langle f, f \rangle > 0 \) with equality \( \iff f = 0 \)

It follows from the first two that

- \( \langle f, g \rangle \) is \underline{conjugate linear} in \( g \), i.e.
  \[ \langle f, g_1 + g_2 \rangle = \langle f, g_1 \rangle + \langle f, g_2 \rangle \] and \( \langle f, cg \rangle = \overline{c} \langle f, g \rangle \).

We put \( \|f\|_2 = \sqrt{\langle f, f \rangle} \).

**Lemma 1.** For all (complex-valued) \( f, g \in C[0, 1] \),

1. \( \|f + g\|_2^2 = \|f\|_2^2 + \|g\|_2^2 + 2\Re \langle f, g \rangle + \|g\|_2^2 \)
2. \( |\langle f, g \rangle| \leq \|f\|_2 \|g\|_2 \)
3. \( \|f + g\|_2 \leq \|f\|_2 + \|g\|_2 \).

**Proof.**
1. This follows from \( |w_1 + w_2|^2 = |w_1|^2 + 2\Re w_1 \overline{w_2} + |w_2|^2 \).
2. This follows from the Schwarz inequality for real functions:
   \[ |\langle f, g \rangle| = \left| \int_0^1 f \overline{g} \right| \leq \int_0^1 |f \overline{g}| = (\int_0^1 |f|^2)^{\frac{1}{2}} (\int_0^1 |g|^2)^{\frac{1}{2}} \leq (\int_0^1 f \overline{g} \overline{f} \overline{g} \overline{f})^{\frac{1}{2}} \]
3. This follows from (1) and (2):
   \[ \|f + g\|_2^2 \leq \|f\|_2^2 + \|g\|_2^2 \leq \|f\|_2^2 + \|g\|_2^2 \]
DEF. Complex valued \( \varphi_1, \ldots, \varphi_n \in \mathcal{C}[0,1] \) are orthonormal if

\[ \langle \varphi_l, \varphi_m \rangle = \begin{cases} 1 & \text{if } l = m, \\ 0 & \text{if } l \neq m. \end{cases} \]

EXAMPLE. The functions \( e_m \) for \( m \in \mathbb{Z} \) are orthonormal.

Proof. Since \( |e_m(x)| = 1 \), \( \|e_m\| = 1 \). For \( l \neq m \),

\[ \langle e_l, e_m \rangle = \int_0^1 e_l(x) \overline{e_m(x)} \, dx = \int_0^1 e_{-m}(x) \, dx = \frac{e_{-m}(0)}{2\pi i m(m)} = 0, \]

by periodicity of \( e_{-m} \).

LEMMA 2. For orthonormal \( \varphi_1, \ldots, \varphi_n \) in \( \mathcal{C}[0,1] \) and \( f \in \mathcal{C}[0,1] \), let

\[ c_m = \langle f, \varphi_m \rangle \quad \text{for } m = 1, \ldots, n \]

be the corresponding "Fourier coefficients." Then

\[ \begin{align*}
(1) & \quad \| f - \sum_{m=1}^n c_m \varphi_m \|_2^2 = \| f \|_2^2 - \sum_{m=1}^n |c_m|^2, \\
(2) & \quad \| f - \sum_{m=1}^n c_m \varphi_m \|_2^2 \leq \| f \|_2^2 - \sum_{m=1}^n b_m c_m \overline{c_m} \|_2^2
\end{align*} \]

for all choices of coefficients \( b_1, \ldots, b_n \) with \( \iff b_m = c_m \) for \( m = 1, \ldots, n \).

Proof. By (1) in Lemma 1,

\[ \| f - \sum_{i=1}^n c_i \varphi_i \|_2^2 = \| f \|_2^2 - 2 \text{Re} \langle f, \sum_{i=1}^n c_i \varphi_i \rangle + \| \sum_{i=1}^n c_i \varphi_i \|_2^2 \]

\[ = \| f \|_2^2 - 2 \sum_{i=1}^n |c_i|^2 + \sum_{i, m=1}^n \overline{c_i} c_m \langle e_i, e_m \rangle, \]

which reduces to \( \| f \|_2^2 - \sum_{i=1}^n |c_i|^2 \) by orthonormality.

For (2), we write

\[ f - \sum_{i=1}^n b_i \varphi_i = f - \sum_{i=1}^n c_i \varphi_i + \sum_{i=1}^n (c_i - b_i) \varphi_i, \]
\[ \langle f - \sum_{m=1}^{n} c_m e_m, e_l \rangle = \langle f, e_l \rangle - \sum_{m=1}^{n} \langle c_m e_m, e_l \rangle = 0 \]

by orthonormality, so

\[ \langle f - \sum_{m=1}^{n} \epsilon_m e_m, \sum_{m=1}^{n} (\epsilon_m - \delta_m) e_m \rangle = 0. \]

It follows from Lemma 1, (1) that

\[
\| f - \sum_{m=1}^{n} \epsilon_m e_m \|_2^2 = \| f - \sum_{m=1}^{n} \epsilon_m e_m \|_2^2 + \left\| \sum_{m=1}^{n} (\epsilon_m - \delta_m) e_m \right\|_2^2
\]

\[
= \| f - \sum_{m=1}^{n} \epsilon_m e_m \|_2^2 + \sum_{m=1}^{n} (\epsilon_m - \delta_m)^2
\]

again by orthonormality, from which (2) follows.

Proof of Parseval's theorem. By Fejer's Theorem, \( \sigma_k \to f \)

unifromly, so

\[ \| f - \sigma_k \|_2 \to 0. \]

By Lemma 2, (2),

\[ \| f - s_n \|_2 \leq \| f - \sigma_n \|_2. \]

and

\[ \| f - s_n \|_2 = \| f \|_2^2 - \sum_{|m| < n} |k_m|^2. \]

It follows from \( \check{\check{\bullet}} \) and \( \check{\check{\bullet}} \) that \( \| f - s_n \|_2 \to 0 \),

which, with \( \check{\check{\bullet}} \) gives

\[ \sum_{|m| < n} |k_m|^2 \to \| f \|_2^2. \]